# Foundation of Computational Fluid Dynamics

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**Lecture – 04**

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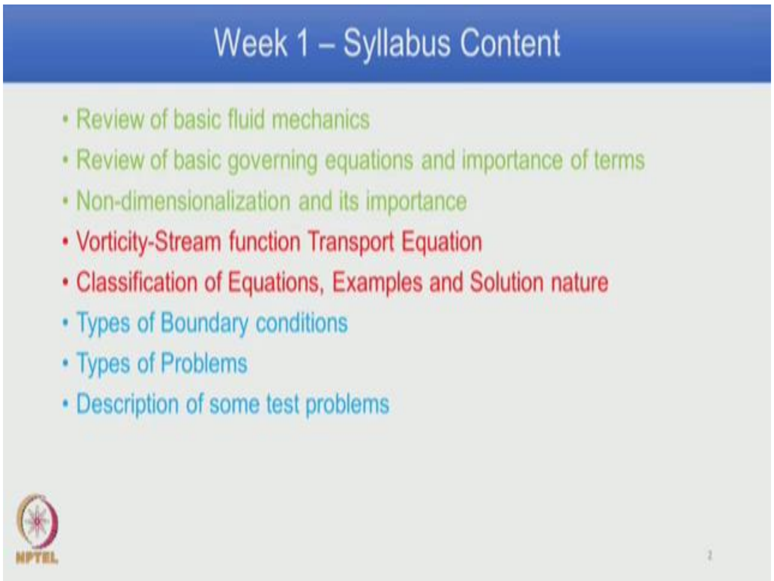
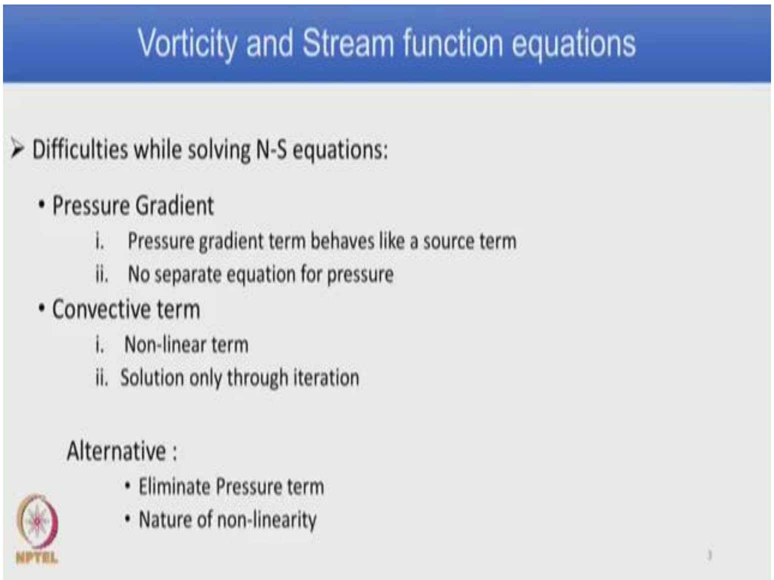



I welcome you all once again to this course on CFD. We are now on to module four of week one. Last class, we have seen some review on basic fluid mechanics, governing equations, importance terms, non-dimensionalization and why you want to do non-dimensionalization. Today's class we particularly see vorticity-stream function formulation, classification of equations, examples for each classifications, solution nature.




After that we will quickly review what we did towards end of last class. We observed while solving Navier-Stoke equations, we have four equations, three momentum equations and one continuity equation; there are four variables, primary velocity variables u, v, w and pressure; though we have separately equations for velocity. There is no separate equation for pressure; and pressure acts like a source term in all the three equations. Another major difficulty is convective term; we noticed that in convective term, velocity multiplying its own derivative, which behaves like a non-linear term, and solution is obtained only through iteration. There is a question, is there a way to overcome. So, alternatively, we have to devise the equation, in such a way there is no pressure term as well as whether we can remove or rewrite non-linearity in the convection term.

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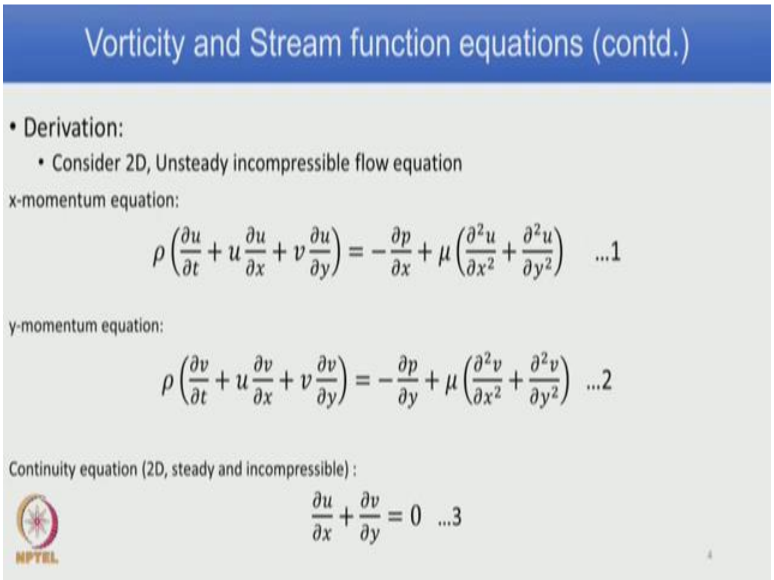


So, towards that we take first 2D unsteady incompressible flow situation two-dimensional Navier-Stokes equations. So, we rewrite here full equations, we also write continuity equations for 2D steady incompressible situation.

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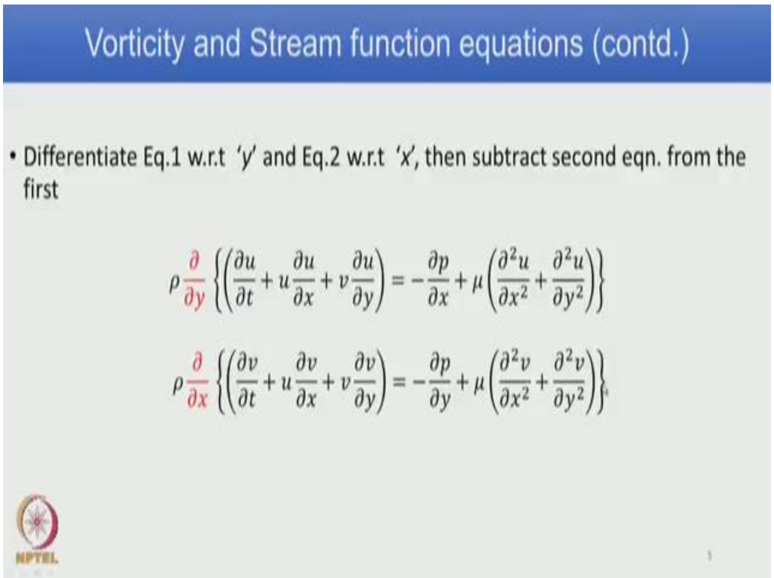


So differentiate equation one that is the x-momentum equation with respect to y, and differentiate the second momentum equation for v with respect to x, and subtract one from the other. This is the algebraic steps that we are going to follow, so that is written here as dou by dou y of the entire x-momentum equation and dou by dou x of entire y-momentum equation. this operator dou by dou y needs to be operated upon each term in this equation; similarly for the other equation. So, what we do, we will take first term explain how to do, also for the second term, then the procedure is repeated for the remaining term.

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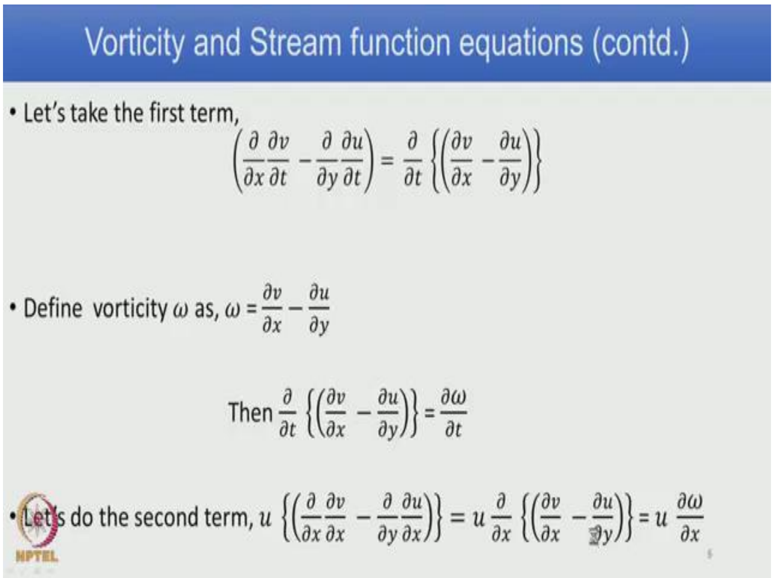


So, we will first we will take the first term that is unsteady term, so dou by dou x of dou v by dou t which is coming from the second momentum equation, dou by dou y of dou y by dou t, which is coming from the first momentum equation, subtract one from the other. And we know this partial derivative can be interchanged, so dou by dou t common is taken out, dou v by dou x minus dou u by dou y is inside of that. we also know the definition of vorticity omega, we learned in the first class, so which is dou v by dou x minus dou u by dou y.

So if you substitute definition of vorticity in this expression then we get first term of the vorticity transport equation, which is dou by dou t of omega, you can repeat this exercise for each term, so here I am showing you for the second term, which is dou v by dou x dou by dou x of dou v by dou x, which is the second term in the y-momentum equation. Similarly, dou u by dou x, which is the second term in the first momentum equation, and you take a dou by dou y of dou u by dou x and velocity is outside. So, this if you do then we get u into dou by dou x and dou v by dou x minus dou u by dou y, this is same way that common term, and here it is exchanged, so the common term is taken out. Once again we recognize the term in the bracket as vorticity, so u into dou by dou x of omega. So, we get second term which is like convection term of the vorticity transport equation.

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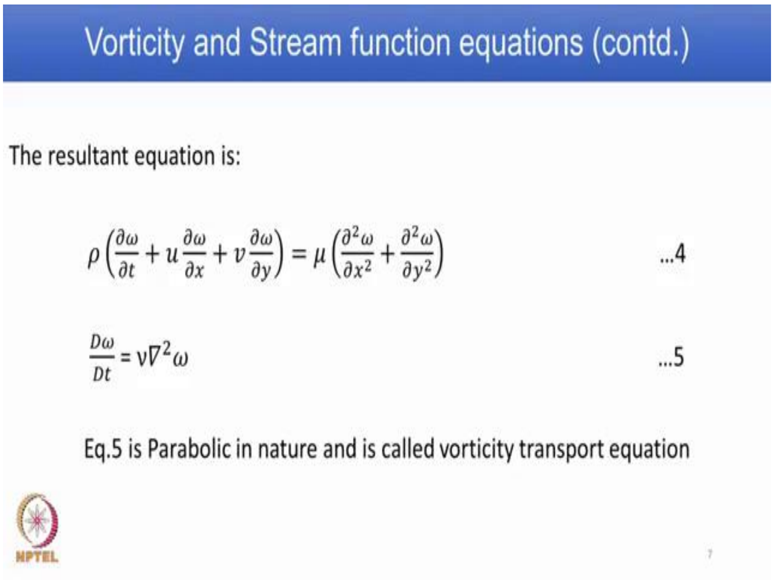


We repeat this exercise for each term, and we get a complete equation as this. So, rho into dou omega by dou t plus u into dou omega by dou x plus v into dou omega by dou y on the left hand side; and on the right hand side, mu into dou square omega by dou x square plus dou square omega by dou y square. if you look at this term each of these term, first term is the unsteady term, which is the local a acceleration, similar to local acceleration term in momentum equation and next two terms are convection terms, similar to convection term in momentum equation. we do replace them, rewrite this equation, taking the definition of total derivative so which is capital D by Dt of omega; and on the right hand, mu into del square omega. So, equation five that is this equation, it is actually parabolic in nature. We are going to talk about solution or equation classification in few slides down the line. So, right now we just take that this is the parabolic equation and equation five is called vorticity transport equation. So, last class, we had seen up to this point.

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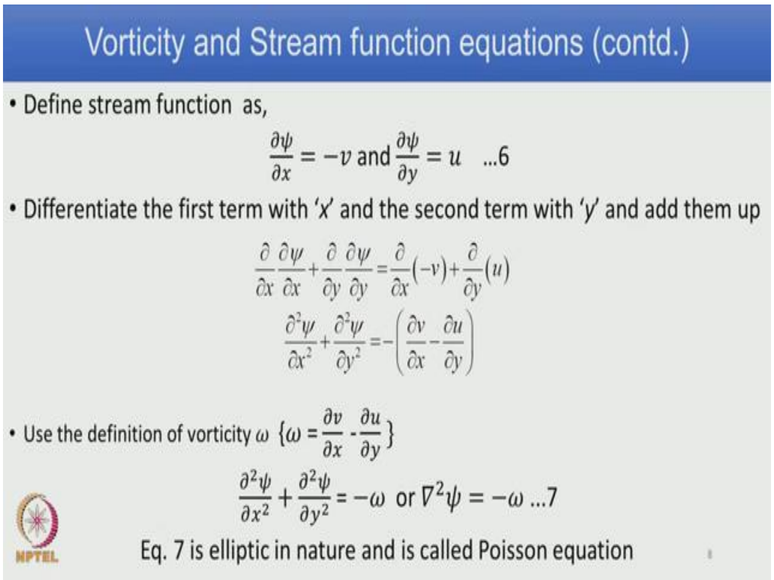


We just wanted to review and then we move onto the next part that is the stream part. We know the stream function definition relating to the velocity dou psi by dou x equal to minus v and dou psi by dou y to u. Here also we start doing some arithmetic and calculus operation. So, take the first term, differentiate with respect to x; take the second term, differentiate with respect to y and add them up dou by dou x of dou psi by dou x plus dou by dou y of dou psi by dou y equal to dou by dou x of minus v plus dou by dou y of u. So, on the left hand side, you get dou square psi by dou x square plus dou square psi by dou y square; and on the right hand side, you can rearrange a little bit, so minus sign dou v by dou x minus dou u by dou y. Again, we are able to observe the term in the bracket on the right hand side, which is dou v by dou x minus dou u by dou y, which is the vorticity.

Hence, we use the definition of the vorticity to rewrite that equation as dou square pis by dou x square plus dou square psi by dou y square equal to minus omega; in other words, del square psi equal to minus omega. We numbered this equation as seven. And as I said before we are going to talk about classification of equation, two slides down the line, and this equation seven is elliptic in nature. And del square psi, if it is del square psi equal to zero, such a equation is called Laplace equation, whereas, if it is del square psi equal to some source term on the right hand side, which is minus omega this is called Poisson equation.

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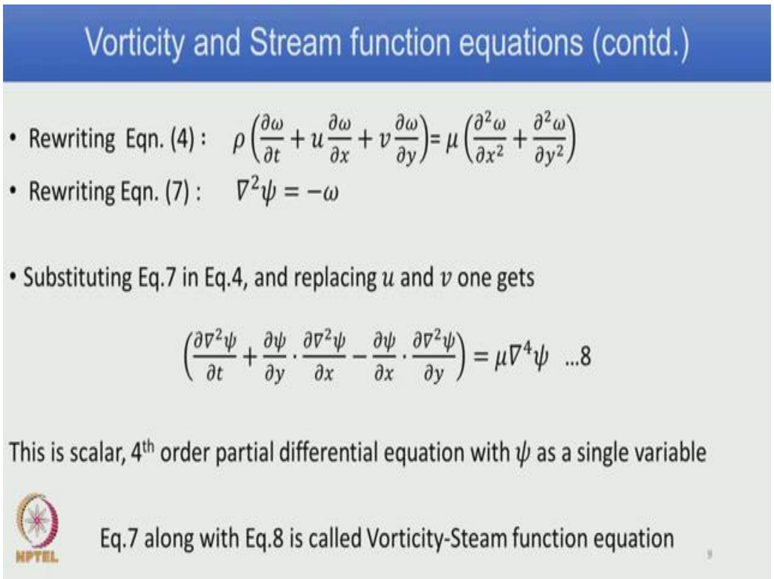


We rewrite this equation once again, that is writing equation four which is the vorticity transport equation, and writing equation seven, which is in terms of stream function relating to vorticity del square psi equal to minus omega. what we do, we substitute this relationship that is relating omega with psi into this equation, so each of these term omega are now written in terms of psi, we will see how it is. We also replace velocity variable u and v in terms of psi, we get a equation like this dou by dou t and then omega is replaced with del square psi, similarly u is replaced with dou psi by dou y and so on. the common term, common sign, minus sign is the common, which is removed, cancelled from the entire equation, so you get a finally, you get equation like this. if you look at this equation, first this is the fourth order equation, because you have a fourth order term here. And then in this equation only psi is the variable, there are no other variables, so it is a complete transport equation only with variable psi.

And this is the four scalar equation, because psi does not have a direction. equation seven, that is del square psi by del square psi equal to minus omega, and equation eight, which is a transport equation written only with the psi, these two together are called vorticity-stream function equation. So if you solve with corresponding boundary condition then you get psi from equation eight, which is related to equation seven with omega that is vorticity. And from stream function psi and vorticity omega, one can recover the velocity field u and v.

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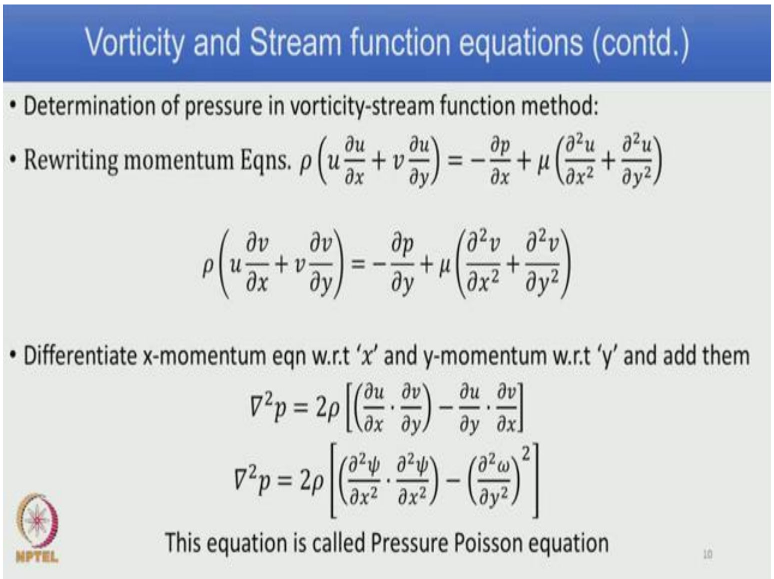


So, now the question is we started the vorticity-stream function equation with a node that there is no pressure term, but we are also interested to know the pressure in the flow field. So if one solves using a vorticity-stream function equation, then how to get a pressure is a question. So, we will setup a procedure to get pressure from vorticity-stream function formulation also. Once again we rewrite the momentum equation for the sake of immediate reference, which is given here, what is given here is the u-momentum equation, again rewrite v-momentum equation. And then differentiate first momentum that is x-momentum equation with respect to x, differentiate the second equation, which is y-momentum equation with respect to y; add them together. I am not showing you the full all the steps involved in the derivation, we get the term finally, like this, which is del square p on the left hand side, and remaining terms on the right hand side.

One can also replace u and v in terms of psi, so we get another equation, del square p equal to two rho into remaining term. So, you can observe that once you have solve the previous equation eight, which is at transport equation written with psi as a single variable then omega, which is the transport equation for vorticity from these two variables after solution obtained one can get pressure field. Another point observe, as you can see on the left hand side del square p and then there is a source term on the right hand side, so this equation is actually pressure Poisson equation. So, instead of solving momentum equation, three momentum equation and one continuity equation, it is also possible to approach by solving vorticity transport equation and get pressure field, velocity field as well.

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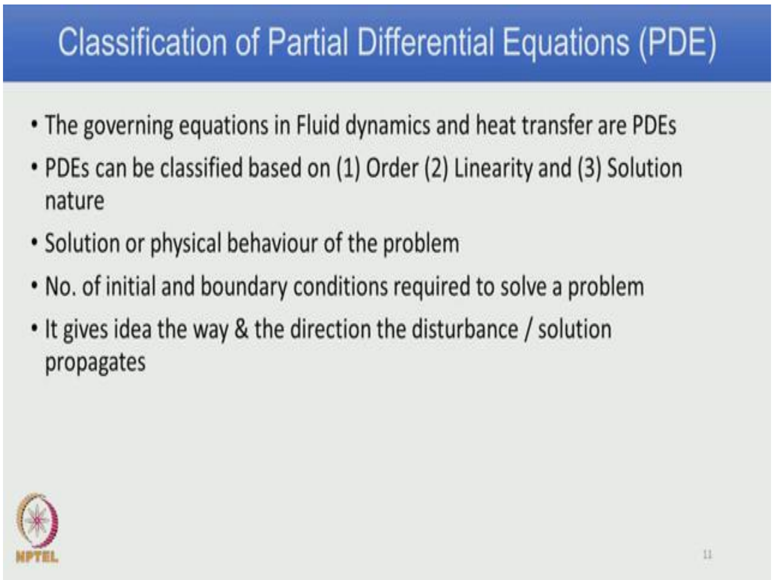


So, we move onto the next topic that is classification of PDE. We know we have already seen also here; in Fluid Dynamics as well as in Heat Transfer, most of the transport equations, they are all PDE. In general, PDE can be classified based on order, we have already seen first order, second order, fourth order based on linearity, similarly based on the solution nature. Particularly here, we are going to talk about classification of PDE based on how the solution behaves like that is necessary to know, because that will also decide what kind of initial boundary conditions required, what kind of final boundary conditions required, before get into the solution of the equations. And it gives idea the way or the direction in which the disturbance or solution propagates.

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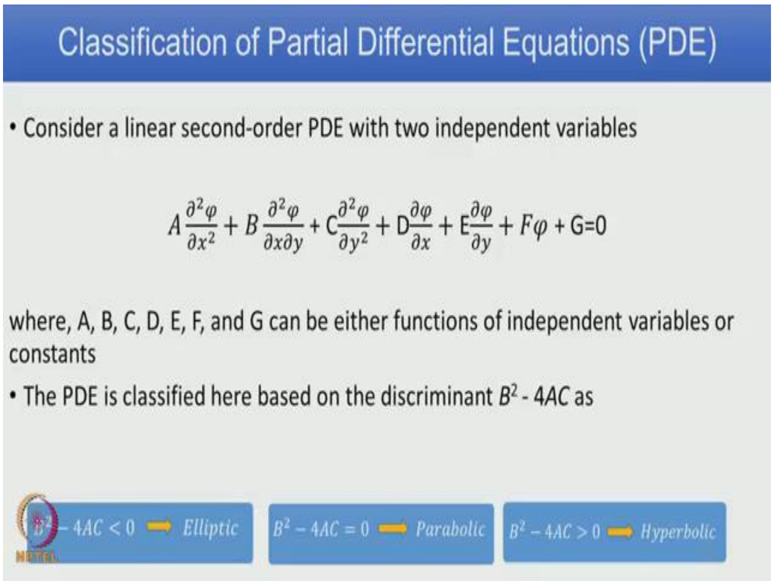


There is a general procedure, consider a linear second order PDE with two independent variables, which is usually given here. And it is possible to fit any linear second order PDE into this generic form A dou square phi by dou x square B dou square phi by dou x and dou y and so on. So, it is possible to fit or rewrite any governing equation into this generic form. this A, B, C are and so on, they are function of independent variables or constants. we are going to make a classification based on the discriminant that is B square minus 4 AC; in other words, if B square minus 4 AC is less than 0, that values happens to be less than zero then equation is or the solution associated with the equation is classified as elliptic PDE. Similarly, if B square minus 4 AC is equal to zero then it is parabolic; and B square minus 4 AC is greater than 0, then it is hyperbolic.

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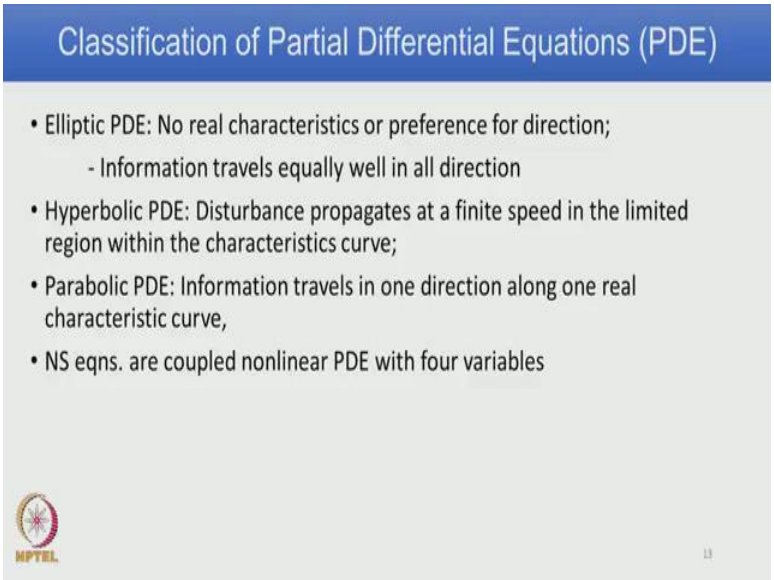


We are going to see details of this. So, in elliptic PDE, there is no real characteristics, there is no specific direction preference. So, for example, if through a stone into a pond or well, there is ripple generated or a wave generated, and you can observe that ripple or wave propagates like a circular fashion in all the in all the direction. So, there is no real direction for this solution. So, information travels equally well in all direction up to the boundary. In the case of hyperbolic PDE, disturbance propagates at a finite speed in a limited region within the characteristics curve. There are two characteristics curves in hyperbolic PDE, compressible flow falls under hyperbolic PDE. And parabolic PDE, here information travels in only one direction along one characteristics curve.

Having defined the classification elliptic parabolic and hyperbolic one tends to ask the question what about the Navier-Stokes equation itself. The complete Navier-Stoke equation that is unsteady term, convection term, three-dimensional along with the pressure term on the right hand side, does not fall under any one such category specifically. They are coupled non-linear PDE, and as we have seen before with four variables.

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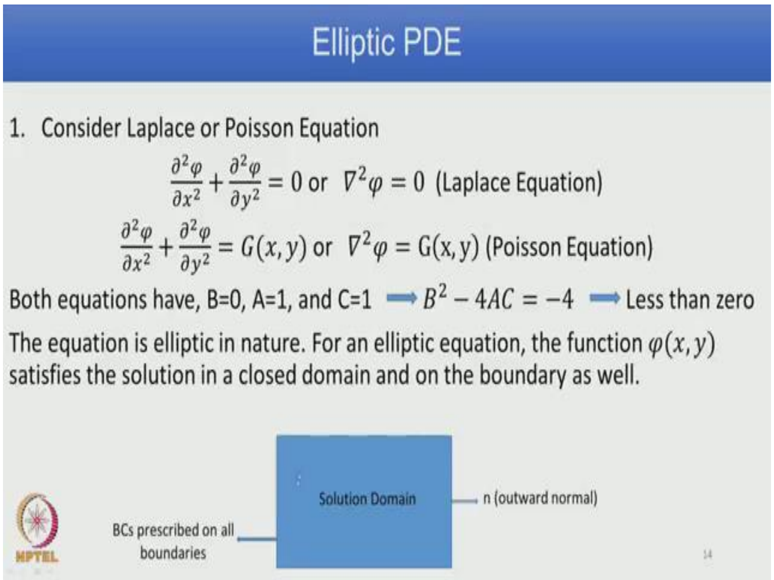


We will now specifically see what is elliptic PDE, so one of the example that is given here is Laplace equation or Poisson equation, del square phi equal to 0, this is called Laplace equation. This equation is generic form, it has for pressure or for temperature distribution, the phi can be any variable. If on the right hand side, if it is not 0, but it is with our source term, which is given here as G of x and y then that particular equation is called Poisson equation. If you follow the discriminant procedure, taking the coefficient values, so B is 0, A is 1, and C is 1, so if you rewrite B square minus 4 AC then you get minus four, which happens to be less than 0, then you can also conform that this is the elliptic PDE. So in elliptic PDE, as we have seen before, the solution is not having any particular direction it moves in all direction equally without any bias up to the boundary point. This is also given in this figure. So, you have to specify boundary condition on all the four directions, and solution is restricted full domain.

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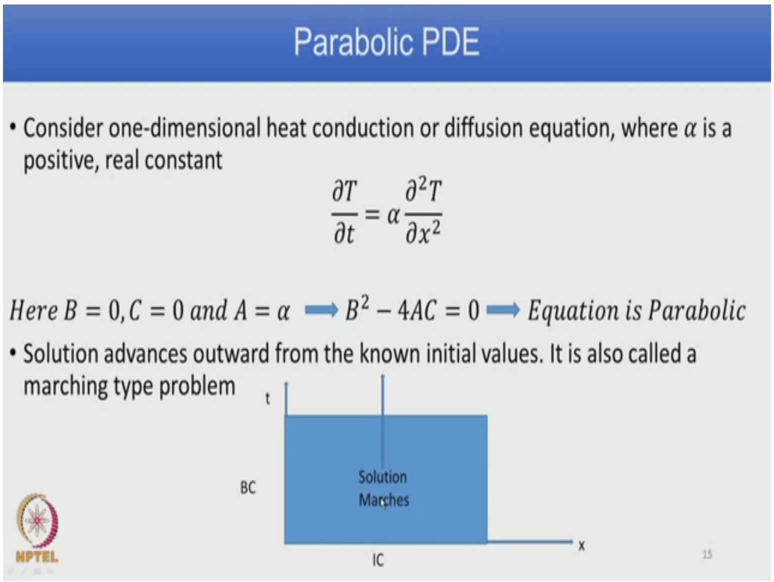


So, next is a parabolic PDE, again I am giving one example, it is one-dimensional heat conduction or diffusion equation and it is given for alpha positive, real constant as dou T by dou t equal to alpha dou square T by dou x square, it is a one-dimensional equation and capital T, which is to represent temperature and this small t is for time. And follow the procedure of discriminant evaluation B equal to 0, and C is equal to 0, and A is equal to alpha, for this equation fitting with the generic equation. So, B square minus 4 AC equal to 0. So, we know for that value equation is parabolic in nature.

So, as you can see here, the time derivative component appears and we have observed that Navier-Stoke equation, first term on the left hand side is the time derivative term. Solution advances outward from the known initial values; that means, it only marches in one direction and you need to specify initial values. So such a problem is also called marching type problem. And sketch wise, it is given here, so this is the starting, and you specify initial condition and there is a boundary condition on for different time value, and solution marches in one direction. You do not come back and try to do boundary condition implementation at time is equal to 0.

In today's class, we have done a complete derivation on vorticity transport equation, understood the advantage vorticity transport equation, how to obtain pressure field as well as velocity field, once you solve the vorticity transport equation; the next topic, we took classification of partial differential equation pertaining to solution obtain in Fluid

Dynamics as well as in Heat Transfer. We have understood what is elliptic PDE with an example; we also described, what is parabolic PDE and how does the solution look like. So we will take another interesting topic in next class.

Thank you.