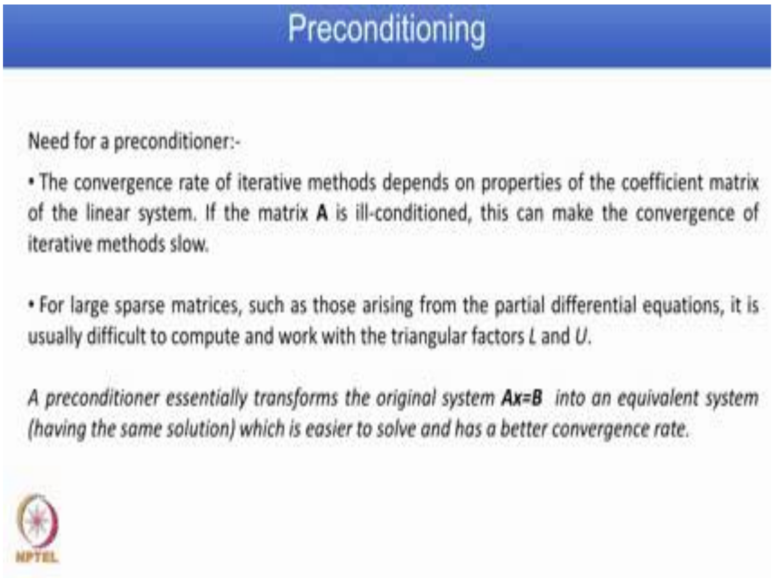
Foundation of Computational Fluid Dynamics
Dr. S. Vengadesan
Department of Applied Mechanics
Indian Institute of Technology, Madras

Lecture – 37

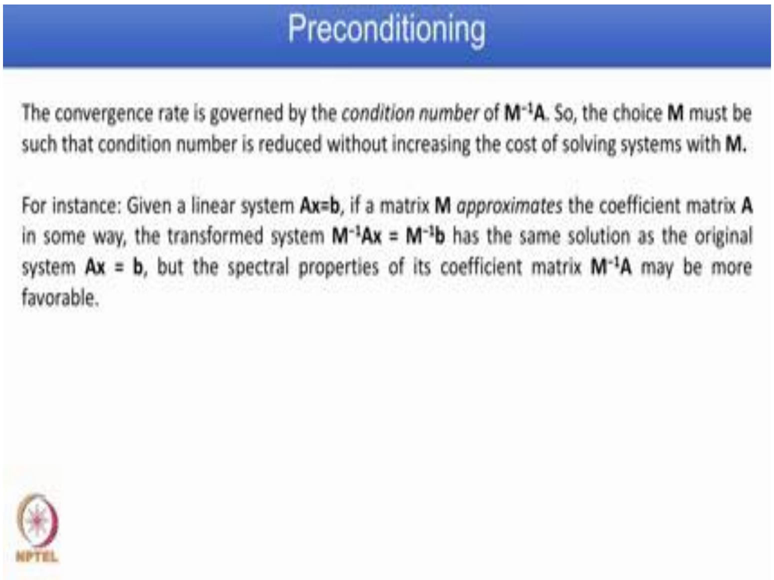
It is my pleasure to welcome you again to this course on CFD. Today is the last class for this week; previously we have done detail procedure for direct methods as well as iterative methods. There are some difficulties while employing iterative methods; and today's class, we particularly going to see what difficulties will be there, and how to overcome those difficulties. There is a procedure called preconditioning and today's class we are going to talk about different preconditioning and how to check what kind of preconditioning to be done.

(Refer Slide Time: 00:57)



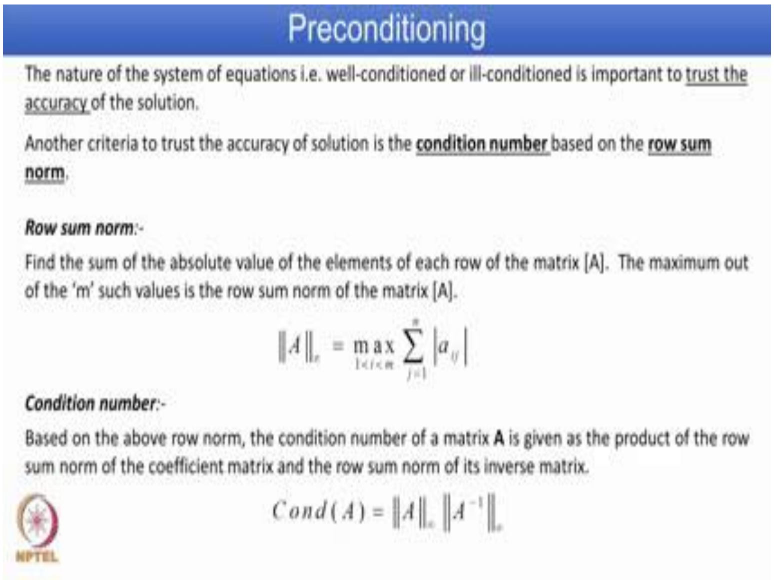
Need for free conditioner the conversion rate of iterative methods depends on properties of the coefficient matrix of the linear system. For example, if the coefficient matrix A is ill-conditioned then convergence of the iterative methods becomes very, very slow. And for large sparse matrices, which usually happens when you deal with PDE for fluid dynamics or heat transfer. It is usually difficult to compute or solve such situation with the help of LU decomposition. In tough situation, we follow a procedure called pre conditioner; a pre conditioner essentially transformers the original system A x equal to B into an equivalent system which has almost the same solution as what it be for the original system and that equivalent system is easier to solve and it has better convergence rate.

(Refer Slide Time: 02:06)



Let us get some more idea above this preconditioning. A is the original coefficient matrix and x use letter symbol M for pre conditioner then we find inverse of the pre conditioner as M inverse then you find product of M inverse A and condition number for that product matrix, then the convergence rate is govern by the condition number of the product matrix that is M inverse A. The choice of the M should be such that the condition number is reduced without increasing the cost of solving the system with M. For instance, if A equal to b then matrix M is as close as A, in such a way the transformed system M inverse A x equal to M inverse b has a same solution as the original system A x equal to b, but the spectral properties of its coefficient matrix M inverse A may be more favourable for transforming the system and getting the solution.

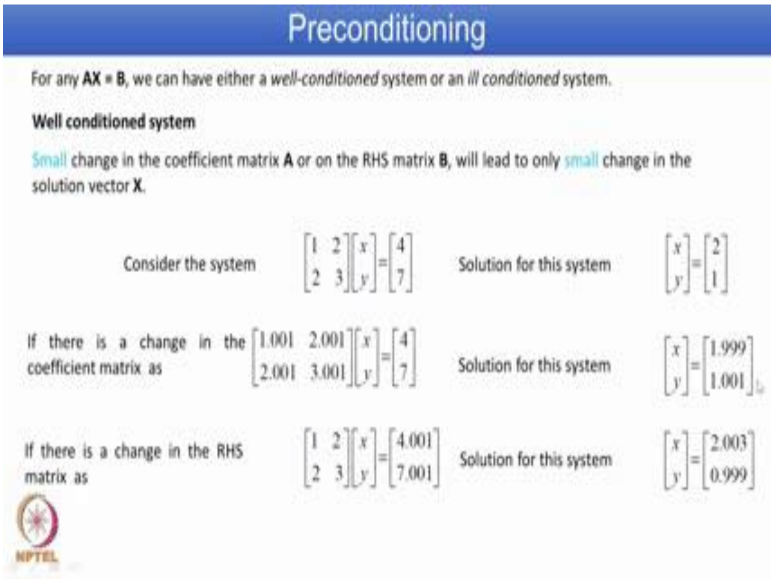
(Refer Slide Time: 03:29)



The nature of the system of equation that is whether it is well condition or ill-conditioned is important to trust the accuracy of the solution; another criteria to trust accuracy the solution is what is known as a condition number and that is based on what is known as a row sum norm. Now, let us explain what is row sum norm and how to obtain condition number. To find row sum norm, sum of absolute values of the elements of the each row of the matrix A, the maximum out of such values is what is known as a row sum norm of the particular matrix A; in terms of expression it is shown here. So, you find for that coefficient matrix A corresponding elements, you sum them up for j going from one to n and you take the maximum of all the rows and that becomes the row sum norm of that coefficient matrix A and it is mathematically given here.

From that row sum norm, it is possible to find or define condition number as product of row sum norm of coefficient matrix and row sum norm of its inverse. So, for the given matrix A, you have one row sum norm. If you happen in to inverse of it that is A inverse, you have another row sum norm for that particular inversed matrix, and you take a product of it that will result in what is known as condition number. In the last slide, I mentioned about condition number row sum norm, we will now explain or get idea of these definitions with the help of a matrix. Before going to the detail let us first understand what is ill-conditioned and well-conditioned system of matrix.

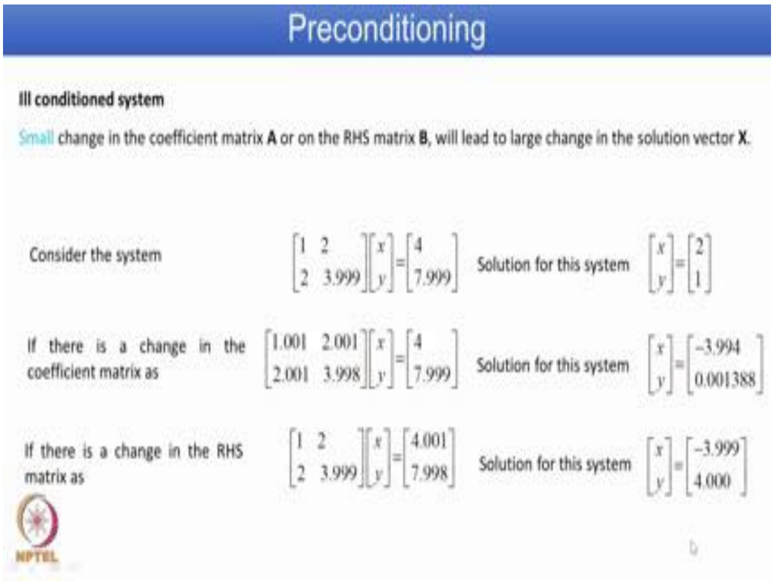
(Refer Slide Time: 05:34)



For example, you consider a system A X equal to B, this system can be either well-conditioned or ill conditioned. Let us get an idea of what is well conditioned or ill conditioned with the help of the example. Well-conditioned system definition any small change in the coefficient matrix a or the right hand side matrix B will lead to only a small change in the solution vector x. Consider the system A x equal to B, where coefficient matrix A is defined as with the elements as 1 2 2 3, and solution vector x as x and y, and then right and side victor B as 4 7. If you solve this, you get answer as x y equal to 2 1. Now, what we do we make a small change in the coefficient matrix as shown here instead of one it is now 1.001; similarly, other elements as 2.001 2.0013.001; the solution vector remains the same, the matrix on the right also remains the same

For such system where there is only a small change in the coefficient matrix solution will look like this; x and y equal to 1.999 and 1.001. And you can observe that these values are very close to the original value of 2 and 1. Let us take the other case where you make a small change the right hand side matrix and that is what is shown here. So, coefficient matrix is the same, the right hand side matrix now it is 4.001 and 7.001, for such system solution appears to be as shown here; x and y equal to 2.003 and 0.999. So, you can observe, for this example, if you make a small change either in the coefficient matrix or on right on side matrix, it results only a small change in the solution matrix.

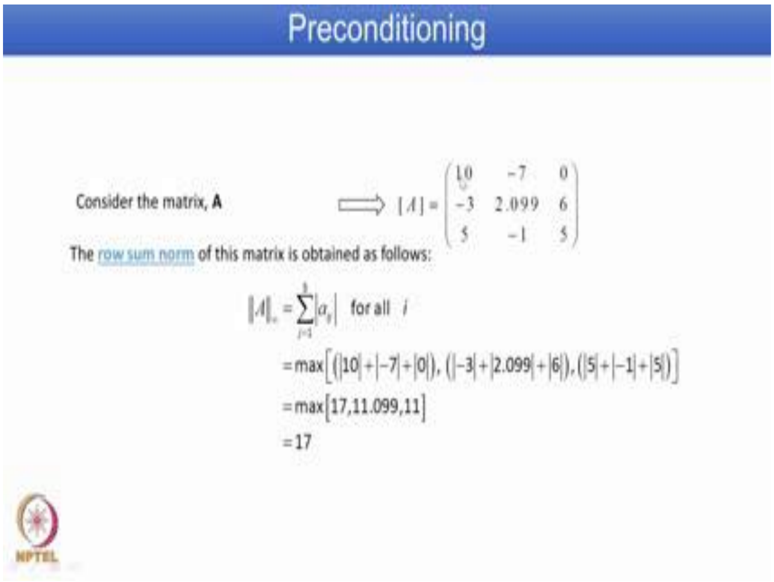
(Refer Slide Time: 07:56)



Let us take the other example to understand what is ill conditioned system. In the case of in condition system, any small change in the coefficient matrix or the right on side matrix will lead to a large chain in the solution vector. Consider again example matrix as shown here 1 2 2 3.999 and x y as the solution vector equal to right hand side as shown here. Solution of this system is as shown here x and y equal to 2 and 1.

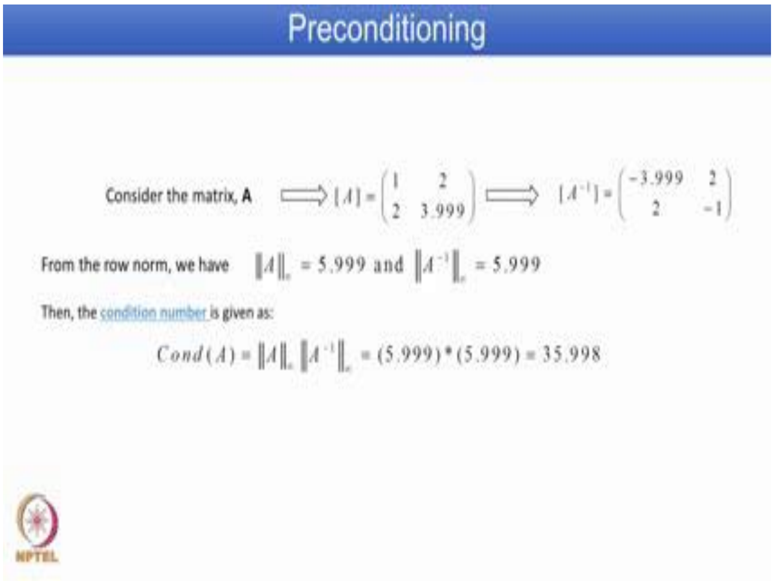
We take the first case, we change the coefficient matrix slightly, so instead of 1 2 2 3.99, we get 1.001 2.001 2.001 3.998, right hand side matrix is same. For this system solution appears to be as shown here x and y equal to minus 3.994 and 0.01388. You can observe here, this value is very much different from the original solution value. Let us take the other case where we change right hand side matrix and it is shown here B as 4.001 7.998 the coefficient matrix is the same. Now, for this system, solution matrix appears to be as shown here; x equal to minus 3.999 and y equal to 4.0. Once again you can observe by comparing solution matrix for this system with original system, there is a very big difference and this kind of a system is what is called ill-conditioned system.

(Refer Slide Time: 09:49)



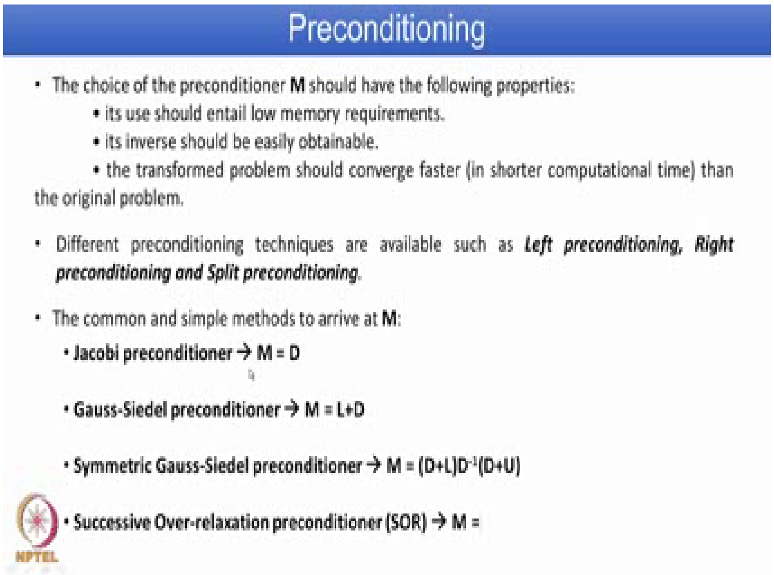
Now, let us go to the definition of rows sum norm and try to explain. Consider the matrix A as shown here with elements 10 minus 7 0 minus 3 2.099 6 5 minus 1 5. We define row sum norm as a maximum of summation of rows and we take the magnitude of the elements to calculate row sum norm. And formula wise, it is shown here, and this is infinity norm equal to summation sign. And in this case, we have three rows, so j is equal to 1 to 3, and we consider each element and we take the magnitude for each element for all i. Now, for this example matrix, we try to apply this definition, so max of first row 10 minus 7 0 and that is what is shown here; second row minus 3 2.099 6 and that is what is shown in second bracket terms; third row is shown in the third bracket term. Then you find the summation and you find maximum is 17. So, row sum norm infinity norm for this matrix is 17.

(Refer Slide Time: 11:09)



Let us find out how to do condition number. For this, we take another example matrix, simplified matrix A as 1 2 2 3.999, inverse of this minus 3.99 2 2 and minus 1. So, we find row sum norm of the original matrix 5.999, and row sum norm of the inverse matrix as 5.999. Now, condition number is a product of row sum norm of both the matrix. So, as defined here condition number of matrix A equal to row sum norm of original matrix multiplying row sum norm. So, we substitute the values and we get value as 35.998.

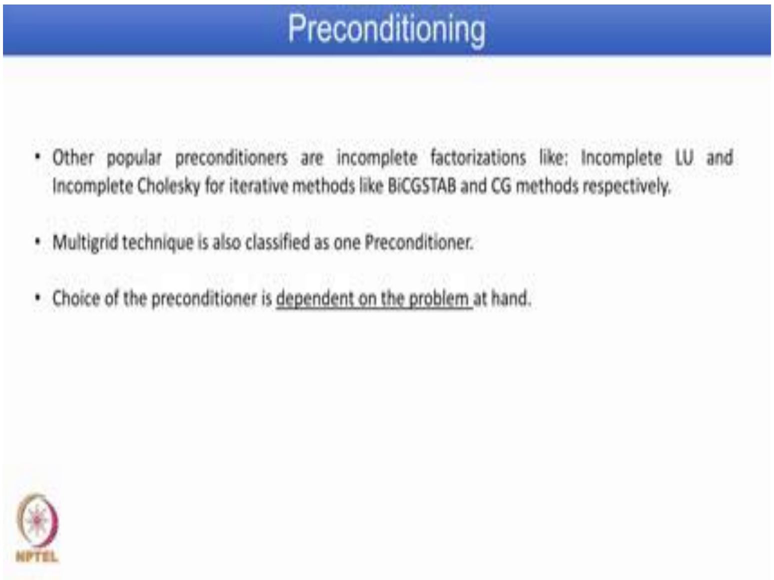
(Refer Slide Time: 11:57)



So, we explain with the help of example coefficient matrix row sum norm and how to get condition number. Now, we proceed to know about different preconditioning available. The choice of pre conditioner M should have the following properties; it use should entail low memory requirements, it should not add to additional memory; its inverse should be easily obtainable. Please remember we want to do preconditioning, because original coefficient matrix, there was difficulty in getting the inverse. We multiply their original coefficient matrix A with preconditioner – M, the pre conditioner M should be easily inverse able. Then the transformed problem should converge faster than the original problem.

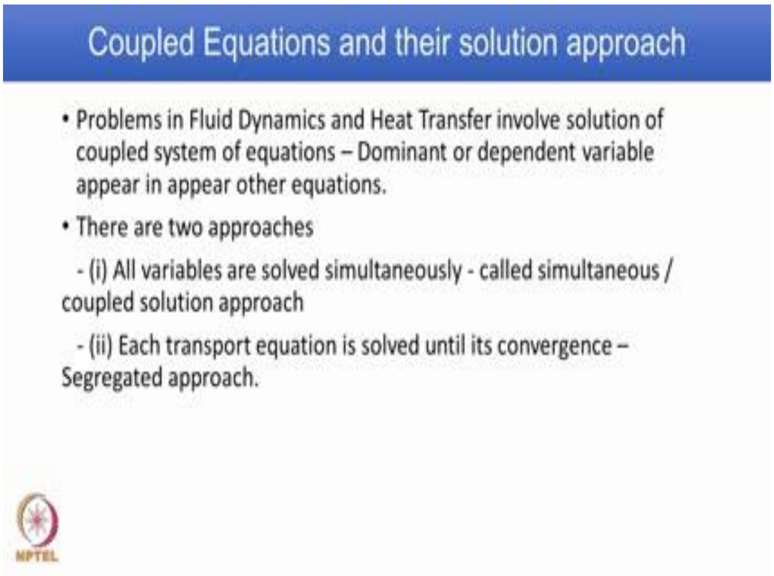
Different preconditioning techniques are available such as left preconditioning, right preconditioning and split preconditioning. Common and simple methods to arrive at M; first one Jacobi preconditioner, where the diagonal elements of the coefficient matrix a itself is a preconditioner. So, M is a symbol we are using to identify it is a pre conditioner. So, it is possible to use the diagonal elements of the original coefficient matrix as a preconditioner and such method is called Jacobi preconditioning. Second one Gauss-Siedel pre conditioner, where the lower plus the diagonal is used as a preconditioner M. Third option available symmetric Gauss-Siedel preconditioner and M in such case is given by this expression where D plus L multiplying D inverse and then D plus U. Then last one successive over relaxation preconditioner – SOR, where M is defined with the product of 1 by omega multiplying D plus omega in to L, where omega is a relaxation parameter.

(Refer Slide Time: 14:24)



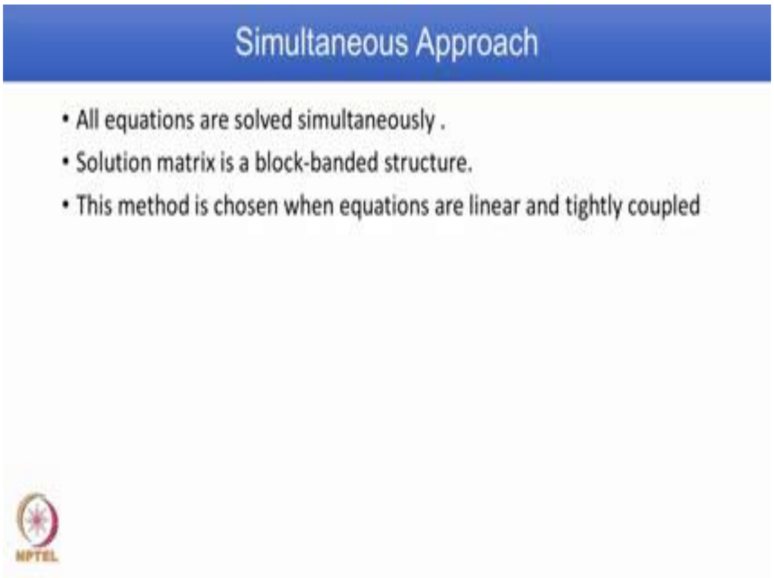
There are other popular preconditioners are available for example, in complete L U and incomplete Cholesky for iterative methods like BiCGSTAB and conjugate gradient method respectively. There is also another technique called multigrid technique, which is also classified under preconditioner. Choice of pre conditioner is very important and it varies from problem to problem. So, far we have seen matrix inversion procedure, direct methods, iterative methods and then preconditioning methods which is helpful in the case of iterative methods.

(Refer Slide Time: 15:19)



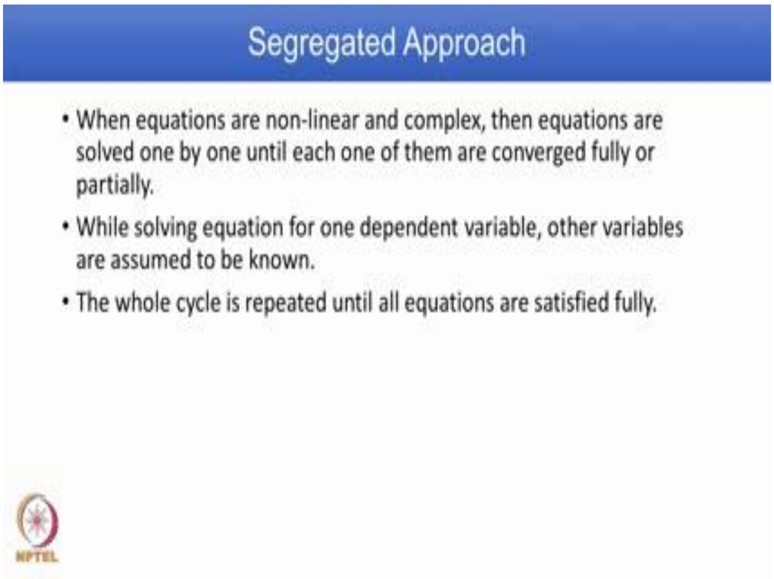
Over all, unique system of equations, there is the procedure to get the solution. Problems in Fluid Dynamics and Heat Transfer involve solution of coupled system of equations, where we have dominant or dependent variable appearing in every other equation. There are two approaches; in the first approach, all the variables are solved simultaneously and this is called simultaneous coupled solution approach; in the second approach, every transport equation is solved until its convergence is reached then the next transport equation is trigged to get the solution, such approach is called segregated approach. We will have some more explanation about these two methods.

(Refer Slide Time: 16:16)



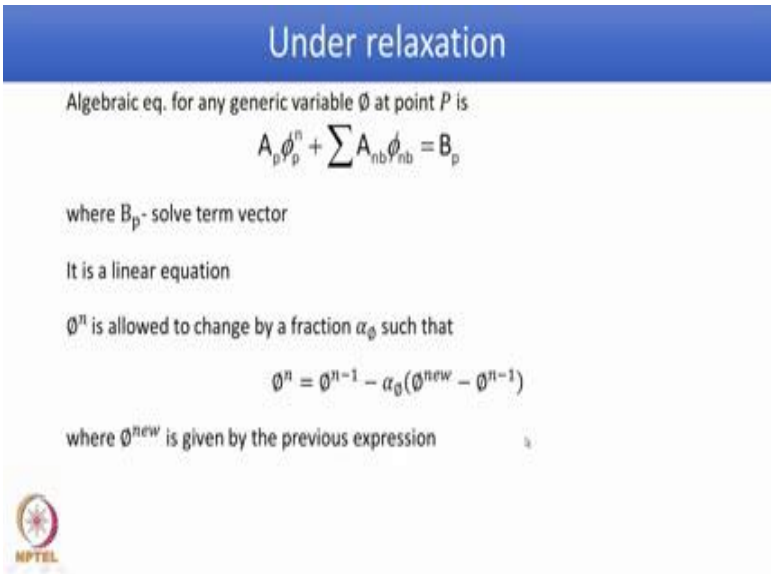
In the case of simultaneous approach, all the transport equations are solved simultaneously. For example, you have force convection problem where you have a where Navier-Stokes equation and there is the energy equation. Both Navier-Stokes of equation and energy equation are called handled simultaneously. Solution of one immediately affects the solution of the other. In solution matrix obtained for such simultaneous approaches is the block-banded structure, and this method is chosen where equation are linear and very tightly coupled. I gave an example of force convection heat transfer problems.

(Refer Slide Time: 16:57)



In the case of segregated approach, when equations are non-linear and complex, then it is advise able to go by solving one equation at a time get a converged solution then go to the next equation. So, equations are solved one by one until each one of them are converged fully or partially. While solving equation for one dependent variable, other variables appearing in the particular equation or assume to be known either from previous iteration values or pervious time level value. The whole cycle is repeated until all equations are satisfied to the convergence criteria fully.

(Refer Slide Time: 17:47)

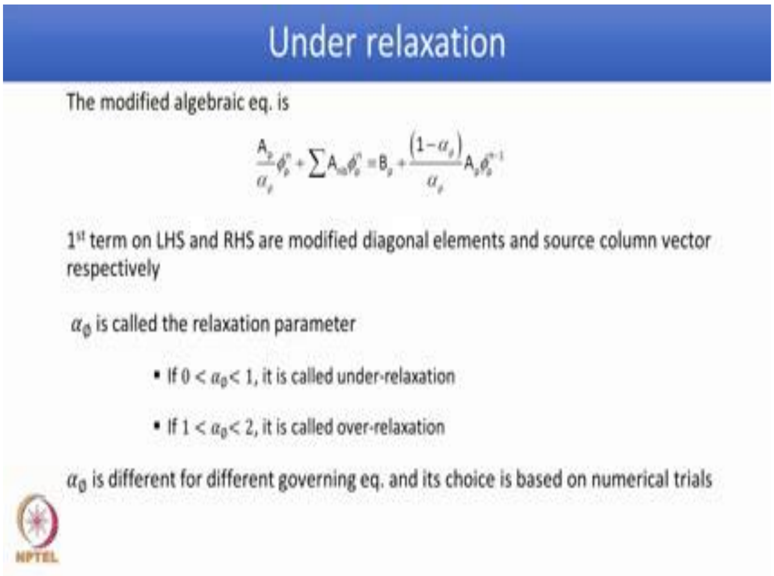


.

We also want to learn another procedure called under relaxation. Algebraic equation for any generic variable phi, whether you follow finite difference or finite volume method, we finally, will have a form as shown here A p phi p n plus neighbouring node coefficients multiplying the variable equal to known value on the right side. In this p - the subscript p is used for node of interest. If you solve this equation then you get phi at the node of interest p at the current time level n. Now, the question is whether to use that value as it is or whether you can relax little bit and then use it for the next iteration, such procedure it is what known as under relaxation procedure.

Say this equation linear equation phi n is allowed to change by a fraction alpha phi in such a way as soon here phi n is equal to phi n minus 1 that is the previously obtain value minus alpha times new value with the new old value. So, this difference phi new minus phi n minus 1 will tell how much the difference has happened between last iteration or last time value with the current iteration or current time value. You take a fraction of it that is what is given by the factor alpha, and you can add or subtract the old value, where phi new is given by previous expression as shown here.

(Refer Slide Time: 19:41)



Now, if you employ that under relaxation procedure in the linear system of equation, they get rewritten in the form as shown here including relaxation factor. So, in programming it the easy all that you have to do is explain alpha the beginning of the calculation then the values are adjusted according to the expression as shown here. First term on the left side, and term on the right side are modified diagonal elements and source vector respectively. Alpha phi is called relaxation parameter. So, while solving equation, you can appropriately define relaxation parameter for different variable. For example, velocity can have one value, pressure can have another value, and if you are solving for turbulent flow k epsilon variables can have another value. So, alpha is the relaxation parameter which varies for different variable. If alpha is less than one such a procedure is called under relaxation; if alpha is greater than one then the procedure is called over relaxation. It is different for different governing equation and the value is decided based on different numerical trails and experience.

So, in this week, we have seen in matrix inversion procedure; there are two methods - direct methods and iterative methods. We went in detail for direct method, Gauss elimination, tridiagonal matrix algorithm, L U decomposition method. We also listed five methods under iterative methods and we learned a technique called preconditioning technique to be used for iterative methods. We also learned couple of other things what is known as how to proceed to get solution segregated solver and simultaneous solver. There is also one more technique called under relaxation technique, which helps in fast convergence. With this, we come to the end of this week class; next week, we are going to have a demonstration of a code, description of the problem, how the code is written and how to obtain different results.

Thank you.