Foundation of Computational Fluid Dynamics
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Lecture – 30

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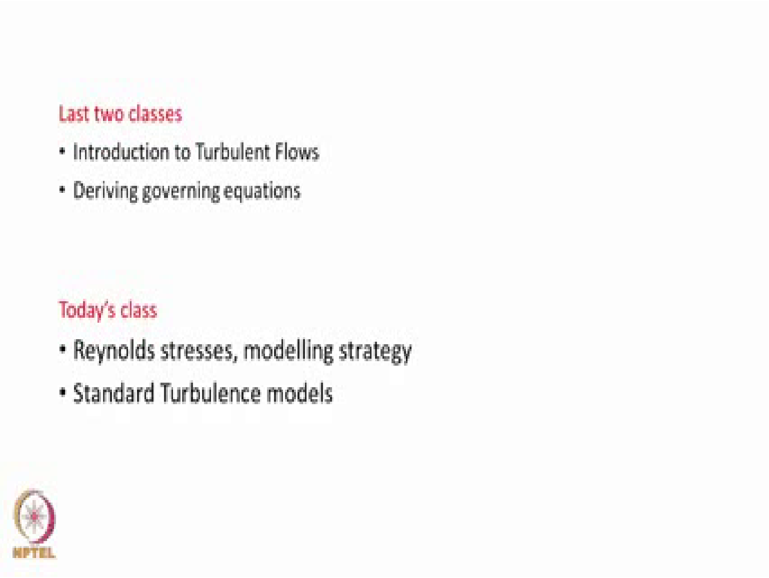

Greetings, welcome, and welcome again to this course. Last two classes, we have talked about turbulent flows characteristics, then we learnt how to derive governing equations for turbulent flows. In today's class we will continue that exercise, we try to understand more about Reynolds stresses how to model Reynolds stresses, and different approaches available to model these Reynolds stresses. Then we will see whether we can do something about introducing standard turbulence models.

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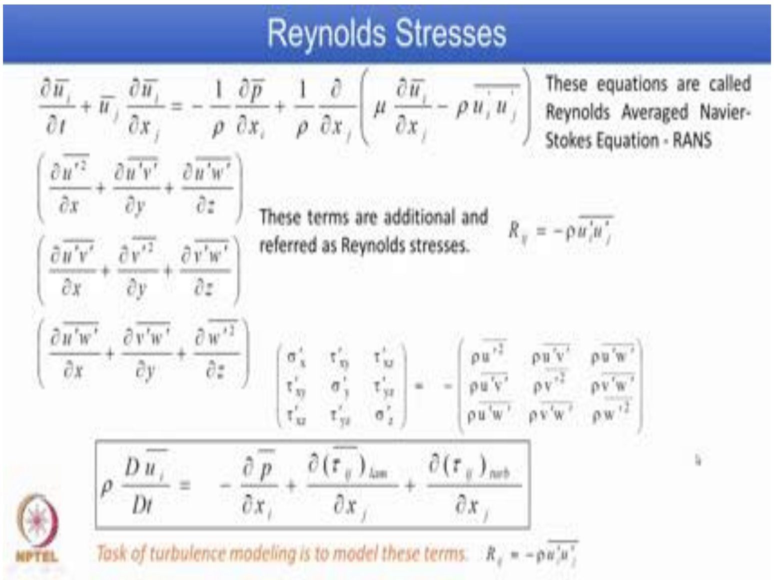

I rewrite here the governing equation for turbulent flow, which is specially called Reynolds Average Navier-Stokes equation RANS. We understand terms in the equation; two terms on the left hand side - time derivative term, convection term; on the right hand - time and pressure radiant term then the additional term which is appearing, because of time averaging procedure is rho u i prime u j prime bar, and this term appears from the convection term. For the sake of convenience, we move it from the left hand side to the right hand side and it is grouped along with viscous stress term. You can also check the unit wise row u i prime u j prime, it has the same unit as stress term. In the expanded form that is by substituting values for index i and j, and following that guide line for index notation, we are able to write detailed of each terms as shown here. And these terms are additional in a Navier-Stokes equation and they are referred as Reynolds stress term, we use the symbol either tau or r i j equal to minus rho u i prime u j prime bar.

Now we put only the stress term in the form of the matrix as shown here. So, we learned that along the diagonal, we have the normal stress term which represented as sigma and half diagonal elements we have shear stress term which is represented as tau. Now if we look at the shear stress term, for example, the first row we have rho u prime v prime bar and rho u prime w prime bar and we look at the first column second row, so rho u prime v prime bar first column third row u prime w prime bar.

So, if you look at first row second column that is rho u prime v prime; first column second row element rho u prime v prime, they are same. Similarly the other component u prime w prime that is the last element in the first row, and first column last element u prime w prime they are same. Hence, originally we have nine unknowns 3 by 3 - nine unknowns, three diagonal elements stress term and six half diagonal element shear stress term. Among the six half diagonal elements shear stress term only three appears that they are symmetric with the step two diagonal. So, those three terms which appears on one side are same as three term that appear on other side. Hence the total number of unknowns though it appears 9, it is actually only six that is three along the diagonal and three half diagonal, total we have six unknown, six additional stress or six Reynolds stresses unknown stresses.

In the standard notation form the same equation can be written using the time derivative plus local plus convection term put together as the total derivative on the left hand side, then you have pressure radiant term on the right hand side then we have laminar stress term and turbulent stress term. As I mentioned in the last class, laminar and turbulent stress, they are brought in the same way to compare the ordered on the role played by each term appropriately to apply for turbulence flows. Now the job is to model these unknown terms, the six additional stress term and procedure to model these six additional stress term is what is known as the turbulence modeling.

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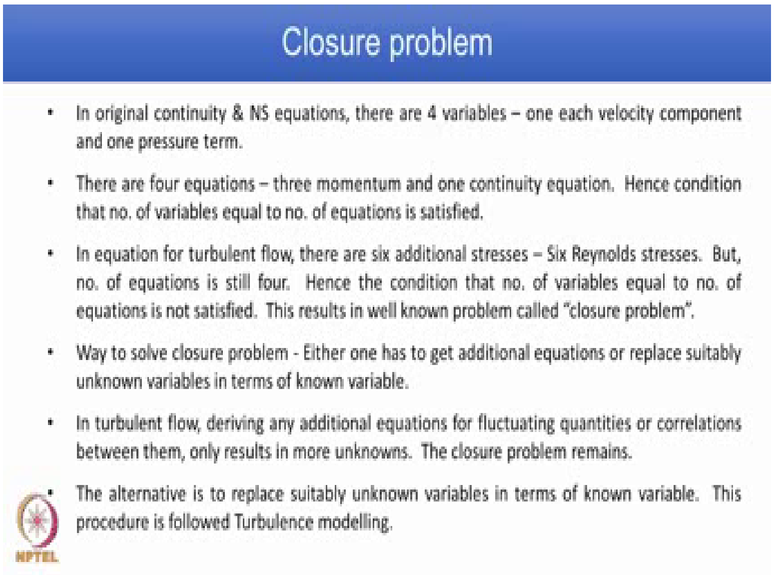

We learnt about the closure problem what is the meaning of closure problem. We revisit that again originally we have continuity equation, and three equations as a Navier-Stokes equation. So, they are four equation and they are four variables, three velocity component and one pressure term. The number of equations equal to number of variables is satisfied. In turbulent flow, we have additionally six stress term whereas number of equation remains as four, hence we have total ten variables where that number of equation as four, hence it becomes a closure problem.

There are two ways of solving this closure problem either in somewhere rather you get additional transport equation or PDE; or suitably replaced unknown variable in terms of known variable that is what is known as turbulence model. In turbulent flow, if you try derives additional equation, it is possible for each fluctuating quantity separately, also for stresses. And if you happened to loop then you will notice at the end it only results in additional unknowns, hence the closure problem remains. In the next few slides, I am going to explain how this actually will happen. So, the alternative is to replace suitably unknown variables in terms of known variable, and this procedure is what is known as the turbulence modeling.

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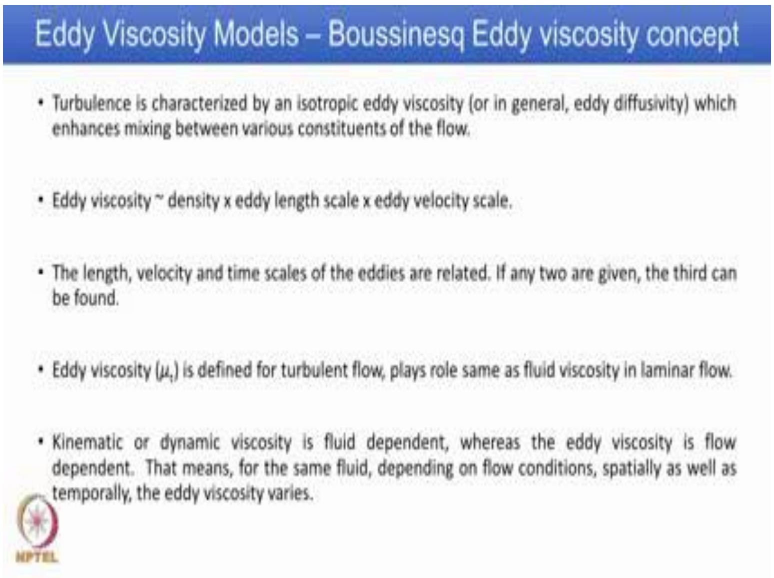

We use the concept called eddy viscosity concept proposed by Boussinesq. And this eddy viscosity is something like how viscosity is playing a role in laminar flow. The turbulence is characterized by an isotropic eddy viscosity in more generic form it is called eddy diffusivity, which helps in enhancing mixing between various constituents of the flow, and based on dimensional argument eddy viscosity is proportional to density multiplying by eddy length scale multiplying by eddy velocity scale. So, in a flow, you have all the three scales length velocity and time scales, and we know they are related to each other based on dimensions. So, if any quantities are known it is possible to find the third one.

We use same symbol mu with the subscript t to represent eddy viscosity. So, eddy viscosity u t is defined for the turbulent flow, and it plays same role as the fluid viscosity we already know mu and nu defined for laminar flow. The kinematic or dynamic viscosity is fluid dependent that mean you may have a air, you may have water as a fluid media kinematic or dynamic viscosity is a fluid dependent, whereas the eddy viscosity is a flow dependent that mean for the same fluid, for example, the water you are dealing depending on the flow condition it may be varying spatially as well as temporarily, whereas the dynamic viscosity or kinematic viscosity is fixed for the same fluid for the same flow problem.

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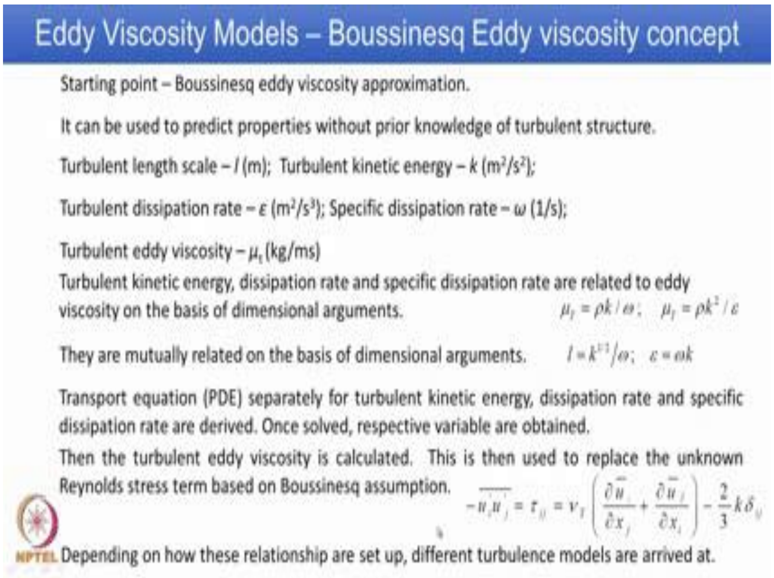

The starting point for all the turbulence modeling is a Boussinesq eddy viscosity assumption. It is used to predict properties without having to know any prior knowledge about turbulent structure. Now let me define some more quantity, what is known as the turbulent length scale we use letter l, and it has the unit as m-meter, turbulent kinetic energy symbol k which is square of the velocity, so meter square plus second square; turbulent dissipation rate epsilon which is rate of dissipation of energy, so it becomes meter square plus second cube then specific dissipation rate omega, which is one upon s. We also define turbulent eddy viscosity mu t and it has the same unit as laminar viscosity k g per meter second. Turbulent kinetic energy, dissipation rate and specific dissipation rate, they are all related to eddy viscosity based on dimensional arguments as shown here. So, mu t equal to rho k over omega or mu t equal to rho k square over epsilon.

It is also possible to relate these three coordinative based on again dimensional arguments, for example, l equal to k to the power of half by omega and epsilon equal to omega into k. It is possible to write separate PDE, other words transport equation for each one this quantity; turbulent kinetic energy, dissipation rate and specific dissipation rate or length scale. And you solve them, you get respective variables obtained. Then the turbulent viscosity is calculated based on the relation that we have just now shown. This is then used to replace the unknown Reynolds stress based on Boussinesq assumption. So, unknown Reynolds stress term is u i prime u j prime bar, either we can use r i j or tau i j is related to mu t if you use rho on one side, otherwise it is mu t because mu by rho is mu and t stands for turbulent flow then we have velocity gradient terms minus 2 by 3 k delta i j. We already learnt delta i j is Kronecker delta and that takes the value of 1, if i equal to j equal to 0, if i not equal to j. For this was a proposed by Boussinesq, and this is the starting point foundation point for all the turbulence modeling, depending on how these relationships are set up, you get different turbulence model.

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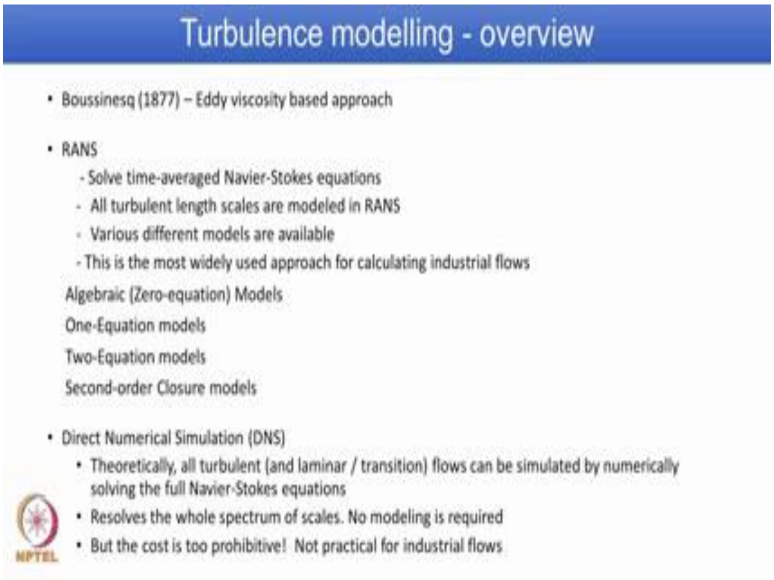

The turbulence model have been in existence since identification of laminar flow and turbulent flow in a flow region, there has been development on the modeling side, there have been many attempts on improving the models are approached to solve the turbulent flow itself. So, overall the turbulence modeling is classified as RANS approach, where we solve time averaged Navier-Stokes equations. All turbulent length scales are modeled in the RANS and we have depending on how these quantities are variables are related, how this quantities mean you have turbulent kinetic energy, specific dissipation rate, dissipation rate, length scale. The equation for length scale depending on how this are related, we get different turbulent model. And this is one of the successful or most widely used approach for calculating industrial turbulent flows.

We also have under this different classification what is known as the algebraic model or zero equation model, one equation model, two equation model, and second order closure model. So, this zero equation, one equation, and two equations represent number of additional PDE you used to solve or to represent the turbulent flow second approach is what is known as the direct numerical simulation. As the name indicates you obtain solution without following any modeling strategy. So, theoretically all turbulent that is laminar and transition flows can be simulated by numerically solving the full Navier-Stokes equation as I mentioned before the structure of the turbulent flow equation is a same as the original Navier-Stokes tress equation. If you interrupted the variable as instantaneous then it is for turbulent flow, otherwise it is for laminar flow.

So, it is possible to use the same equation here for turbulent flow, and it to solve or if you resolve all the scales corresponding to velocity, length and time then you get what is known as the direct numerical simulation approach. So, it resolves whole spectrum of scales, absolutely no modeling is required; and other hand, though it looks very good. There is a great disadvantage the cost is exhaustibly prohibitive it is very high, because in order to resolve all the scales, you need to have mesh resolution as well as delta t is very fine enough that is very expensive and it is tried only as a academic level not attempted for any industrial flows problem.

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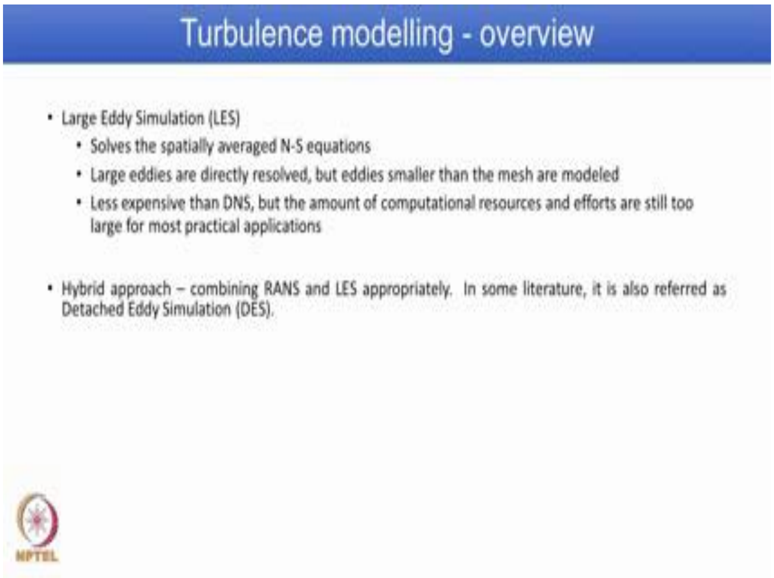

The other approach is what is known as the large eddy simulation in short form it is called LES. It solves spatially averaged Naviers-Stokes equation where large eddies are directly resolved, but small eddies smaller than the mesh that is used are modeled. It is less expensive compared to DNS, but slightly more compare to RANS. The amount of computational resources and efforts are still too large in recent there are attempt to use LES for industrial problems as well there is alternative approach what is known as the hybrid approach were we can combine RANS and LES appropriately. So, depending on how there are combine you get a solution and this is called hybrid approach some literature, they will call with special name what is known as the detached eddy simulation, otherwise DES. We have three approaches RANS, LES and DNS and in between there is a strategy hybrid approach.

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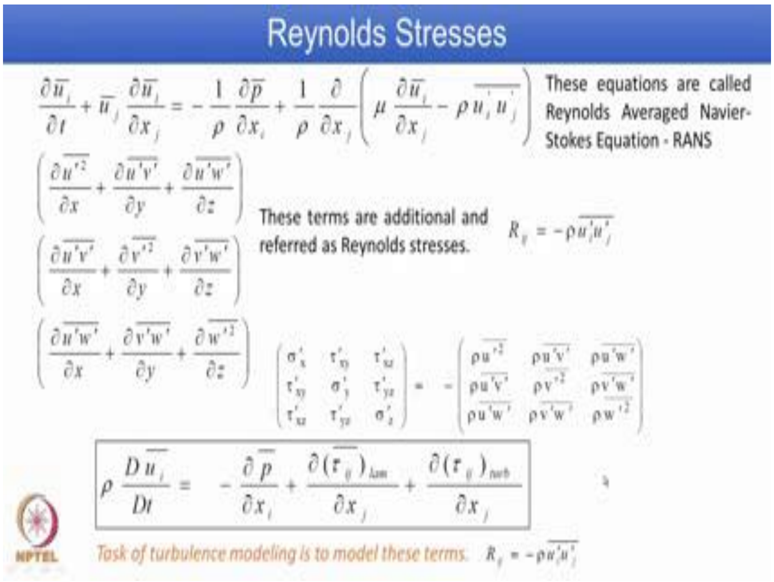

Now go into the equation once again, I am displaying here the RANS equation, time averaged Navier-Stokes equation, and we know this stress term and we identify normal stress as well as shear stress. We also identify half diagonal elements shear stress or symmetric with respect to the diagonal. Hence, total number of unknowns coming through Reynolds stress is only six - 3 diagonal normal stress term and 3 half diagonal shear stress term. In the conventional form, it is written with the time with material derivative or total derivative on the left hand side; on the right hand side, we have pressure derivative term, the second term based on laminar viscosity is written as the laminar stress, and third term based on turbulent stress is written as here as turbulent stress.

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Equation for turbulent stresses

$$\frac{\partial u_i'}{\partial t} + \overline{u}_k \frac{\partial u_i'}{\partial x_k} + u_k' \frac{\partial \overline{u}_i}{\partial x_k} + \frac{\partial (u_i' u_k' - \overline{u_i' u_k'})}{\partial x_k} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} + \nu \frac{\partial^2 u_i'}{\partial x_k \partial x_k}$$

Multiply above eqn. with u_j' and the eqn. below with u_i' and then average

$$\frac{\partial u_j'}{\partial t} + \overline{u}_k \frac{\partial u_j'}{\partial x_k} + u_k' \frac{\partial \overline{u}_j}{\partial x_k} + \frac{\partial (u_j' u_k' - \overline{u_j' u_k'})}{\partial x_k} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_j} + \nu \frac{\partial^2 u_j'}{\partial x_k \partial x_k}$$

$$\frac{\partial (\overline{u_i' u_j'})}{\partial t} + \overline{u}_k \frac{\partial (\overline{u_i' u_j'})}{\partial x_k} = -\overline{u_i' u_k'} \frac{\partial \overline{u}_j}{\partial x_k} - \overline{u_j' u_k'} \frac{\partial \overline{u}_i}{\partial x_k}$$
$$- \frac{\partial (\overline{u_i' u_j' u_k'})}{\partial x_k} - \frac{1}{\rho}\left\{ \overline{u_i' \frac{\partial p'}{\partial x_j}} + \overline{u_j' \frac{\partial p'}{\partial x_i}} \right\} + \nu \left\{ \overline{u_i' \frac{\partial^2 u_j'}{\partial x_k \partial x_k}} + \overline{u_j' \frac{\partial^2 u_i'}{\partial x_k \partial x_k}} \right\}$$

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Now, the question is whether it is possible to get equation for stresses, because we mention the closure problem that we have ten unknowns, now we have only 4 equations. The two approaches either we get additional transport equation for those additional stresses or new unknowns or replaced the unknowns by a known. In the first approach, that is trying to derive equation for additional unknowns. We will end up getting more unknowns, we are going to derive equation for those unknown, and at the end we will observe that you get only more unknowns rather than closing the problems. So, this is the equation that it is obtained for fluctuation. In this equation we use i and k, and these are obtained very systematically following all the arithmetic that is we have a instantaneous decompose into summation of mean and fluctuation. You apply this decomposition then you get equation for fluctuations.

Now this is the representation, in the sense, the index notation i is very generic. So, you can replace the index i by j and get the equation for another fluctuation. In other words, we have one equation for one fluctuation, we have another equation for the second fluctuation. So, if you multiply the above equation by u j prime and the second equation that the u i prime then take the average. So, for each term that operation is performed each term we multiply by u j prime in this then similarly in this each term is multiply by u i prime then you do average then you get finally, an expression as shown here. It is purely arithmetic and it is possible to obtain the final expression as shown here.

Now if you look at this expression, this expression, the term variable term is actually u i prime u j prime bar that is a variable in this equation. Hence our desire the starting desire is to get additional PDE for this unknown is obtained. However, when you look at this equation in close, you will observe that you are getting a new term, the third order correlation term u i prime u j prime and u k prime bar.

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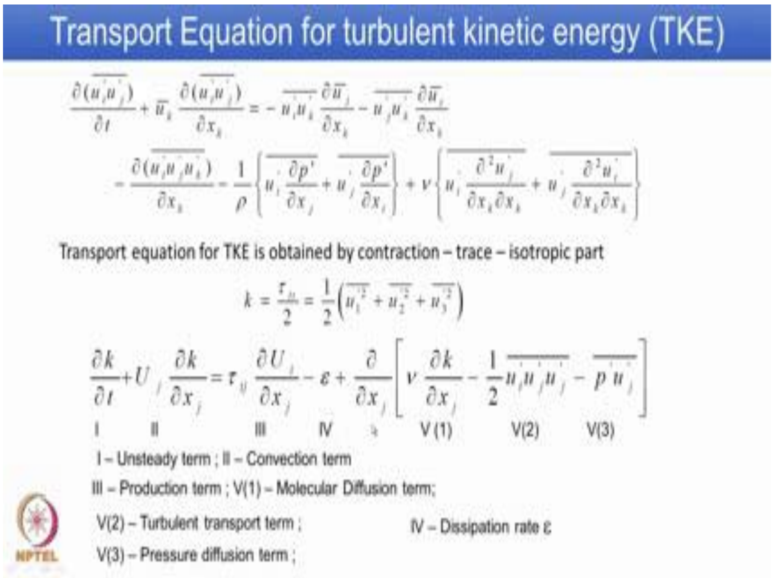

So, this u i prime, u j prime, u k prime bar is a new unknown and you can identify by substituting value for i, j and k we get so many unknowns, actually you get 27 new unknowns. Now we understand the closure problem can never been achieved. So, there is alternative way that we have to plan that is what we going to see in turbulence modeling. So, in this equation, again we follow one step in the tensor algebra what is known as the contraction. The contraction is nothing, but taking a trace or taking the isotropic part. So, if you substitute i is equal to j then you will get all diagonal elements in the Reynolds stress tensor. So, we get one equation for each normal stress term.

For example, if you substitute i equal to 1, and j equal to 1, it becomes first normal stress; i is equal to 2, j is equal to 2, we get second normal stress; i is equal to 3, j equal to 3, we get third normal stress. And you follow that for all the terms then you get equation for each normal stress. And in trace procedure, the contraction because the indexes is repeating, you do summation and that is written here k which is the symbol used for denoting turbulent kinetic energy is tau i i by 2, and tau i i is nothing but normal stress because we are doing trace into summation u 1 prime square bar plus u 2 prime square bar plus u 2 prime square bar. And we follow this then on this equation that is the transport equation for Reynolds stress, if you apply, the trace procedure for each term and then after following some step, you get finally a equation for turbulent kinetic energy.

So, in this equation k is the variable, and you get equation for turbulent kinetic energy. Now in this equation again we have multiple terms; first term is an unsteady term, second term is a convection term, third term which is the first term on the right hand side is the production term, fourth term is a dissipation term. And there are three sub terms in for a fifth term, the first term for the fifth term is a molecular diffusion term, second term for the fifth term is the turbulent diffusion term, and the third term for the first term is the pressure diffusion term. Let me give some more explanation the second term on the left hand side is the convection term, this has u j as the convection velocity and dou k by dou x j is the variable and its corresponding spatial derivative.

Now, when you look at this term and again try to compare convectional term in the momentum equation. Convection term in the momentum equation is non-linear whereas is straightly, this is linear. Now first term on the right hand side, which is marked here as a third term this what is called production term tau i j tau i j is a Reynolds stress term and dou u i by dou x j is the mean velocity gradient, and this is the product of Reynolds stress with the mean velocity gradient that will give you what is known as the production of turbulent kinetic energy.

Then fifth term, there are three terms; first term is because of the molecular viscosity, hence it is called molecular diffusion term. The second term we have turbulent quantities, turbulent fluctuation quantity, and we can interpret this u j prime one quantity u j prime is a agent responsible for transporting this turbulence stress. We know u i prime, u j prime is a turbulent stress, and you have the additional term u j prime. So, we can interpret this term as a turbulent diffusion term, and third term is the pressure diffusion term. Now the question is all are supposed to be known expect for second term as well as third term and dissipation term.

So, in today’s class, we have learnt the first stage in the turbulence modeling, starting from transport equation for each Reynolds stress term, and how to get the trace of it to obtain what is known as the transport equation for turbulent kinetic energy. In the next class, we will start from here and do actual modeling for those additional terms in the transport question for the turbulent kinetic energy. And we go into standard k epsilon and k omega model expression and explanation.

Thank you.