Foundation of Computational Fluid Dynamics

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Lecture – 03

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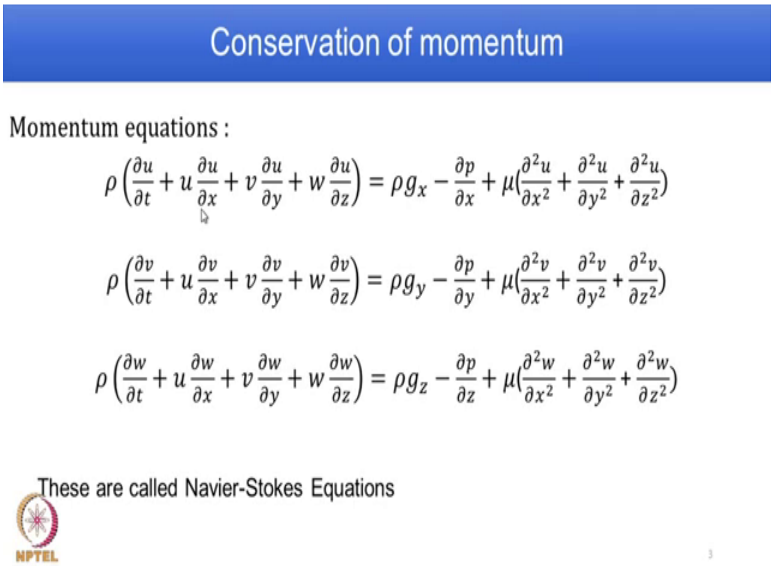

Welcome you all, we are now on to module three of this MOOC course. Last two classes, we have done some review on mathematical formulation, operators, important properties, description of the flow, and then we started doing equations. So, we first did conservation of mass or continuity equation then we went on to momentum equation, which is called conservation of momentum and we have written a momentum equation in both vector form and as well as scalar form, and that is repeated here. All the three components of a momentum equation and we know this also called Navier-Stokes equations. We recognize here again, first term is unsteady term or local acceleration, the remaining three terms are convective acceleration.

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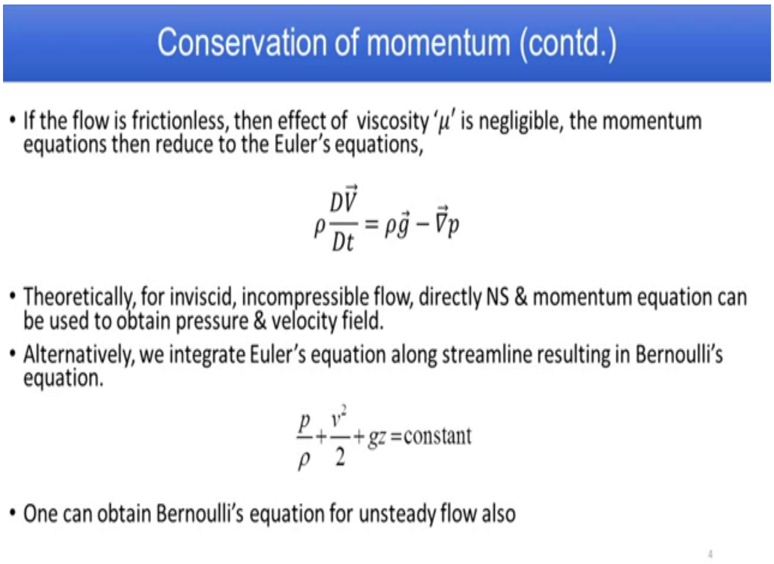

Like we did in conservation of mass, if you simplify for different situation, then the original the generic continuity equation get reduced as two-dimensional only with spatial derivatives. Similarly, conservation of momentum equation also can be reduced for different simplified situation. For example, if you say the flow is frictionless, then effect of viscosity mu term is negligible. So, in the Navier-Stokes equation mu term appearing as viscous term on the right hand side. So, that term is removed and you get resultant equation or Euler's equations. So, left hand side remains the same, D by Dt – total derivative of the velocity, there is one term that is removed, because of the viscous effect is not consider. The external force and pressure term remains same. So, this equation is derived or deduced by scientist Euler, hence the name is given as Euler's equations..

So, theoretically, for any inviscid, where viscous effect is not there in incompressible flow, you can directly either Navier-Stokes equation and momentum equation can be used to obtain pressure as well as velocity field. So, you can also integrate this Euler's equation along a streamline and after some simplification, it results in what is known as Bernoulli's equation and that is given here p by rho plus v square by two plus g z equal constant. We know Bernoulli's equation is a energy equation. So, you get a pressure in a p, kinetic energy or velocity energy and potential energy. So, Euler's equation if you integrate, along a streamline then you get a Bernoulli's equation. So, Euler's equation is derived or deduced for frictionless flow. So, Bernoulli's equation is the restriction for only frictionless flow. You can also obtain Bernoulli's equation for unsteady flow also.

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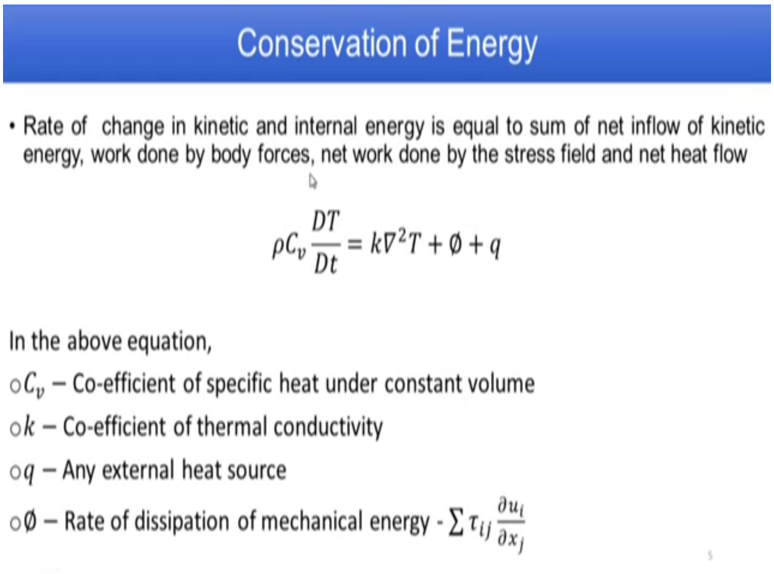

So, next third equation, which is conservation of energy equation; once again we are not going to the details of the derivation steps. We first state the energy equation itself which is given here as rate of change in kinetic and internal energy is equal to sum of net inflow of the kinetic energy, work done by the body force and net work done by the stress field as well as net heat flow. And this definition is applied then there are many arithmetic operation, finally, the form that is given here is given dou rho C v D by Dt of temperature on left hand side, and there are three other terms on the right hand side. So, in this equation, C v is the coefficient of specific heat in constant volume; and k is the coefficient of thermal conductivity; and q is any external heat source, and phi is a rate of dissipation of mechanical energy and that is given by this expression.

So, in this expression, it is the summation and tau i j dou u i by dou x j, where i and j are indices it is representing direction 1, 2, 3. So, if you substitute i is equal to one, j is equal to 1, 2, 3, i is equal to 2, j is equal to 1, 2, 3; i is equal to three, j is equal to 1, 2, 3, you get so many components in this equation. This index notation is also otherwise called tensor notation, we are not going to the details of tensor notation here, only this term is expressed in this form. So, as I mentioned before from the definition after for simple condition, the equation is derived and it is given here. There are other forms of energy equation available in other references.

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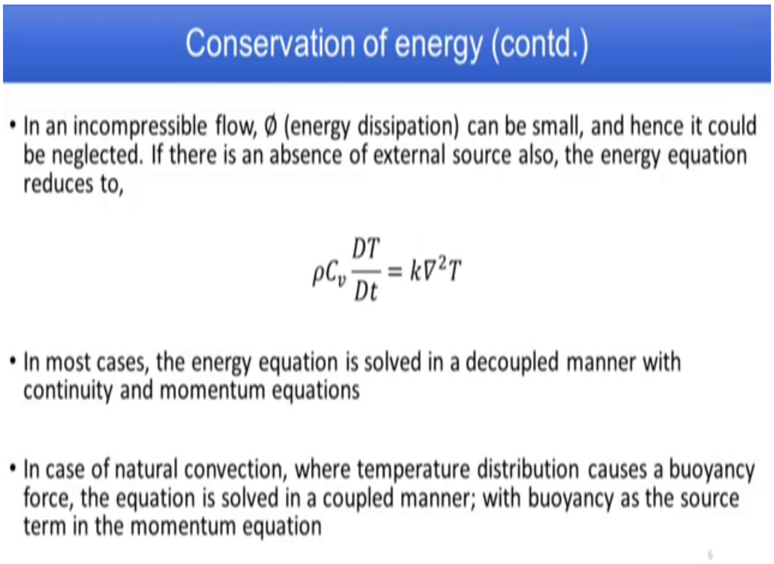

As we did exercise, how to simplify conservation of mass, conservation of momentum for different situation; conservation of energy equation is also simplified. So, in the case of incompressible flow phi which represent the energy dissipation will be very, very small, and hence the effect can be neglected from that equation. Similarly, if there is absence of external source, then energy equation gets reduced to only one term on the right hand side, so in most equation. So, we have conservation of mass and conservation of momentum; conservation of momentum of course, there are three explicit equations for each direction. So, in essence there are four equations, one for conservation of mass, and three equations for conservation of momentum – x component, y component, and z component.

So, if you look at number of variables, you have mu, v, w – three, and pressure – four. So, there are four unknowns and there are four equations. So, mathematically it is a closed system of equation along with number of boundary condition you specify then you will able to solve those four equations without any problem. If you are looking for only simple basic fluid mechanics problem then those four equations are enough. And if you have a problem where energy transfer is also important then you activate or include energy equations also. So, whether you solve energy equation, in line with momentum equation or not is a question. If for most of the situation, the energy equation is solved in a decoupled manner with the continuity and momentum equations. What is the meaning of decoupled, you solve mass equation and momentum equation get a converged solution based on mass equation and momentum equations then you trigger or activate the energy equation in the code, get a converged solution for energy equation, then moved to the next iteration, the cycle repeats. So, this is known as a decoupled way of solving equations.

But in problem where there is a natural convection, where temperature distribution causes a buoyancy force, then this procedure is not applicable; we need to solve energy equations coupled manner with the momentum equation, because in natural convection problem, buoyancy force is an external acting as a external source term in the momentum equation. Hence, we have to solve energy equations, as and when you are solving momentum equation, which is going again as an energy, as a source term the momentum equation.

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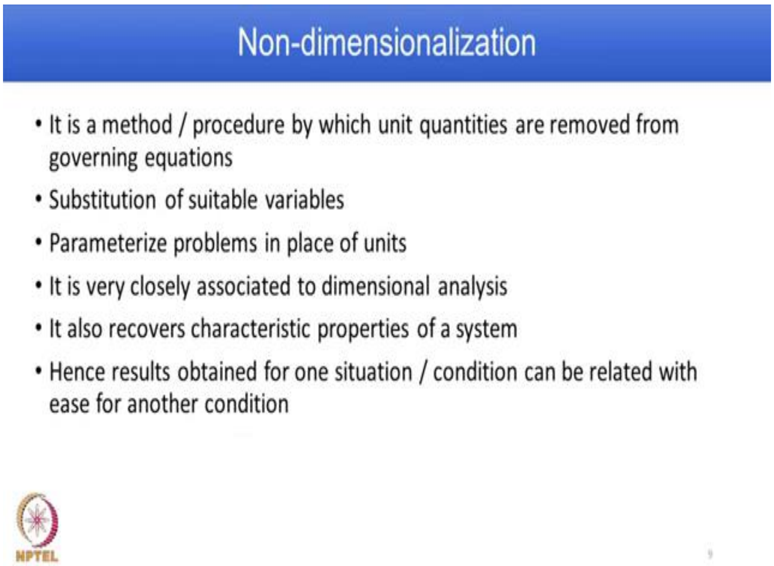

Next, important topic is non-dimensionalization. What do we meaning of non-dimensionalization. It is a method or a procedure by which unit quantities are replaced. So, we have seen three major equations, mass, momentum and energy. In all the three equations, we also observed the variables, velocity, density, gravity, viscosity then temperature reserve. So, all these variables have units. The reserve a way by which you can remove the units and solve the equations in non-dimensional form, and there is a procedure and that is known as a non-dimensionalization procedure. How do we do, we identify suitable variables and replace actual variable or non-dimensionalized actual variable by the corresponding suitable variable which is acceptable. Now, what is the advantage, it parameterize the problem in place on units. So, this reduces number of parameters in the problem. And you can also identify by this procedure characteristic properties of the system, which mean there is a particular scale exist in a flow either velocity scale or a lens scale or time scale, you can identify corresponding scale by the non-dimensionalization and the scale information is very helpful either to control the flow or manipulate the flow or to design a system based on the scale.

Now, the advantage is suppose, you are running or doing experiment, running a simulation for one particular geometrical condition or flow condition, and there is a need to repeat for some other condition, if you do non-dimensionalization then you do not have to do this repetition of cases for different situation; results obtained one situation or condition can be obtained with ease for another condition.

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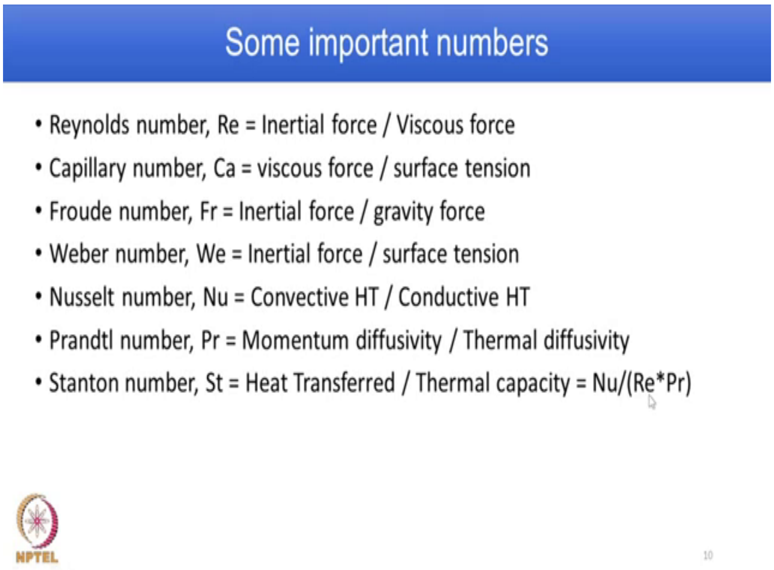

So, in connection with that, we need to also know many non-dimensional numbers and there are also many other important numbers. So, I am listing here only few of them, they may be more depending on the problem that you are studying. First one is the Reynolds number, usual symbol is given as Re, and which is the ratio of inertial force to the viscous force. Next is the capillary number, which is given a symbol Ca which is the ratio of viscous force to the surface tension. Froude number, which is used for which is generally used flows via gravity Fr consider. So, it is defined as ratio of inertial force to the gravity force. Next non-dimensional number is a Weber number, again given a symbol We, which defined as ratio of inertial force to the surface tension force. So, it basically means either inertia is dominating or surface tension force is dominating depending on the value of the number.

And heat transfer context, we have Nusselt number, which is given a symbol Nu and it is the ratio of convective heat transfer to conductive heat transfer. Then there is Prandtl number – Pr, which is ratio of momentum diffusivity to thermal diffusivity. The one more number what is known as a Stanton number. So, which is defined based on heat transferred to thermal capacity, and it is also related to Nusselt number and Prandtl number as given here Nu denominator Re multiplied by Pr. As I mentioned before, there are many more non-dimensional numbers for different problems.

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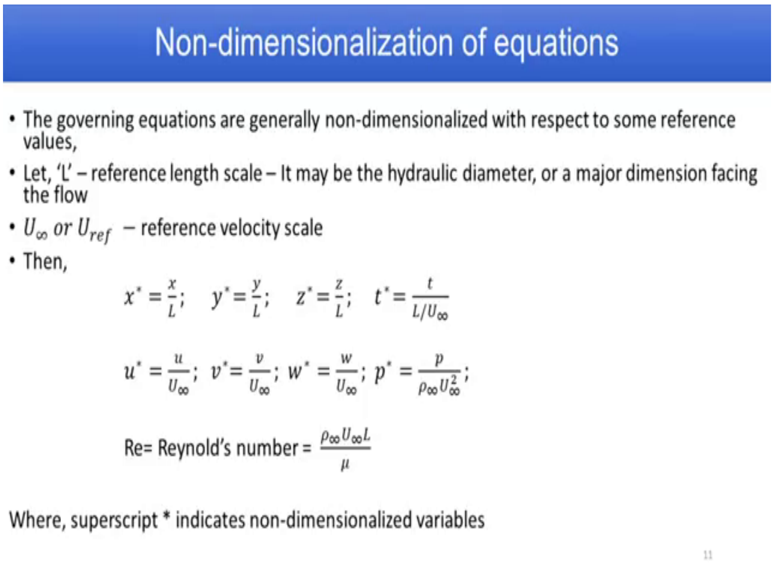

So, we have seen important of non-dimensionalization, and some important non-dimensional numbers. Now, we are actually going to do non-dimensionalization of governing equation. We will do a detail derivation for one equation that is conservation of mass, and you can follow the procedure to get non-dimensionalization equation for other equations. So, to do that we need to define what is known as reference scale. So, in a problem, you need to identify reference length scale, reference velocity scale. If you consider for example, flow through a circular cross-section pipe, diameter of the pipe can be a reference length scale. If you consider flow through a rectangular cross-section pipe, then hydraulic diameter, which is defined as four a by p, where a is the area, and p is the perimeter that can be a length scale. If you consider flow over an aerofoil, the cord length can be a length scale.

So, one needs to identify, which is the major influencing length scale in a problem. It may vary from problem to problem. Then one also identify reference velocity scale. So, again if you consider the flow through circular cross-section pipe, if the inlet is uniform velocity then uniform velocity can be a reference velocity scale. The inlet is parabolic profile either the bulk velocity can be a velocity scale or the average the middle maximum velocity can be a reference velocity scale. If you consider a flow over a flat plate, the free stream velocity U infinity can be a reference velocity scale. So, one needs to identify reference length scale and reference velocity scale. Once we do that in the governing equation all the length scale related quantities are non-dimensionalized with the reference scale. Similarly, all the velocity related quantities are non-dimensionalized by the reference velocity scale.

And we also know you can get a time from velocity as well as length. So, we will see how to do, let say for example, all the length scale in all the length related information, so in this case, x, y and z. So, x is non-dimensionalized with L, and this non-dimensionalized quantity is referred with the symbol superscript star. So, x star is non-dimensionalized length and x is the length, actual length, L is the reference length, and we know both have the same unit. So, x star has no units. You can define similarly for other two dimensions length dimension, we use the length scale in all. So, it is y over L and z over L. And t-time is non-dimensionalized from these two L over U infinity will give you the time. So, any time related quantity for example, the first term time derivative term is divided by L over U infinity to get non-dimensionalized time t star.

Similar manner, you can also do for velocity. So, it is u star is non-dimensionalized velocity component in x-direction, which is u is the velocity dimensional quantity and U infinity is the reference velocity. So, u over U infinity will give the non-dimensionalized velocity quantity u star. You can extend this to get v star and w star. And pressure, is also another term appearing; the pressure is related to U infinity square and you define reference density. So, if you take the unit of density, and velocity square then you find the product of them has the same unit as pressure, you are able to non-dimensionalized pressure term p over rho infinity U infinity square will result in p star which is non-dimensionalized pressure.

And we have just now seen different non-dimensionalized number. So, Reynolds number is appearing in momentum equation, when you non-dimensionalized and we know the definition of Reynolds number is ratio of inertial force to the viscous force. And if you use this non-dimensionalized quantity, then finally, you end up in expression for Reynolds number as rho infinity U infinity L by mu. Of course, you can define mu over rho infinity also as a mu and Reynolds number becomes U infinity L over nu. So, all the star quantities are non-dimensionalized variables, now that we are done the first exercise, we can repeat this, we can substitute this term in continuity equation to get a non-dimensionalized continuity equation.

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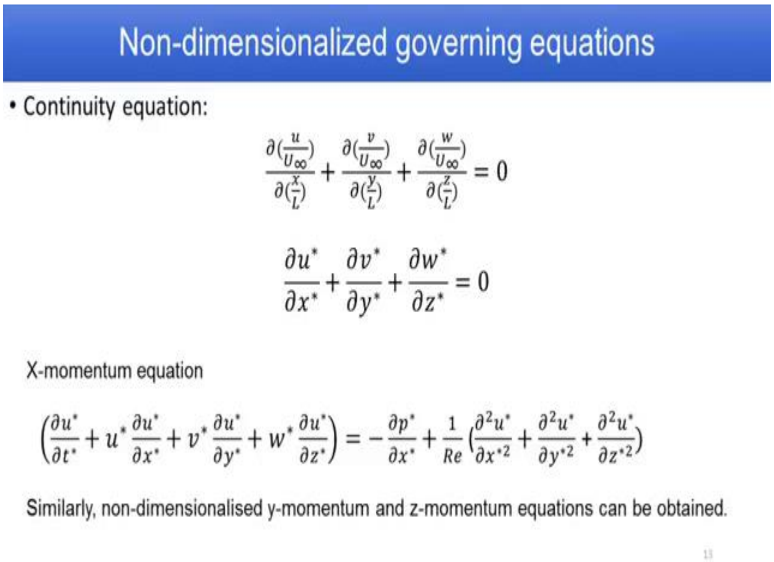

Now, we will try to do non-dimensionalization of the continuity equation. So, we know dou u by dou x and dou v by duo y and dou w by dou z equal to zero. And we just now learned how to represent them by length scale, velocity scale and that is written here. And we also defined u by U infinity as u star and x over L as x star and so on. So, we get dou u star by dou x star dou v star by dou y star and dou w star by dou z equal to zero; so this equation is actually non-dimensionalized continuity equation. You can immediately observe none of the quantity have units. So, take any of these, first term for example, u star is already non-dimensionalized there is no unit; x star is non-dimensionalized, there is no unit. Similarly, other two terms so no term has any unit. So, this entire equation has no unit.

You can extend this procedure to get non-dimensionalized momentum equation, and that is given here. First term dou u star by dou t star, similarly u star dou u star by dou x star and so on up to last term also the viscous term. So, we see here, as I explained continuity equation, here also no term has unit. So, dou u star, of course, you know already u star has no unit, t star has no unit. So, the entire term has no unit, similarly, other terms so the entire equation is without any particular unit. And one can extend this procedure to get non-dimensionalized y-momentum and z-momentum equations. So, if you solve for example, this equation for one problem then you see the advantage, if there is a change in any design parameter, for that problem then you need not repeat this calculation just for the sake of changed dimensions or changed design condition.

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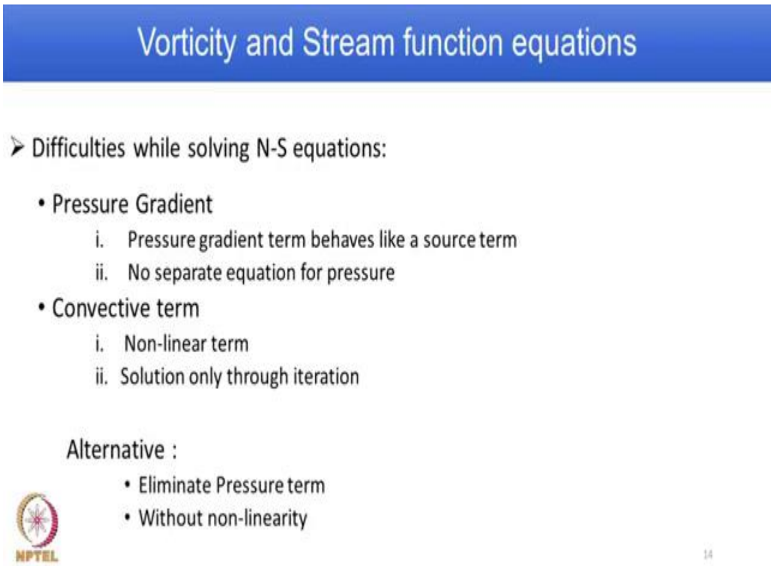

We go to the next important topic, what is known as a vorticity stream function equation. There are certain difficulties while solving Navier-Stokes equation. What are the difficulties, one is the pressure gradient, pressure gradient terms appears on the right hand side dou p by dou x, dou p by dou y and dou p by dou z. So, this pressure gradient term behaves like a source term and if you observe we have a continuity equation and we have three momentum equations. Primary variables u, v, w are there in these three four equations, but there is no separate equation for pressure. Wherever the pressure is appearing in all the three equations derivative dou u by dou x, this is the term first term in the momentum equation, x-momentum equation, convective acceleration first term. And function u multiplying its own derivative dou u by dou x is non-linear in nature. Hence treating this non-linear term is again very trick.

Now, you may have to do many iteration to get a solution so, whether we can have an alternative to overcome these two main difficulties. So, if you can eliminate pressure term or and if you can get a equation where such a non-linearity behavior is not appearing in that equation. Now, such alternative approach is what is known as a vorticity stream function formulation.

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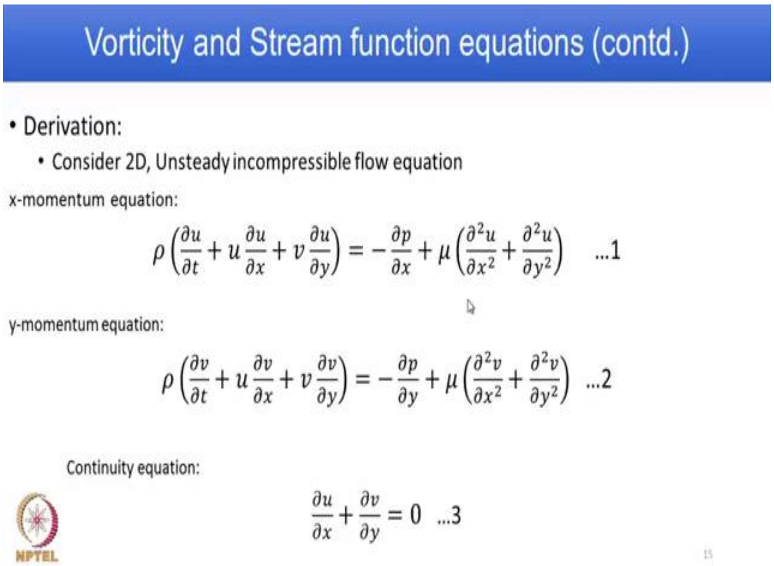

So, we will do a detailed derivation vorticity stream function formulation. Starting point is always again momentum equation. So, take a momentum equation and consider the two D situation, which means you can consider x and y momentum equation, unsteady term. So, they are repeated here, x-momentum equation and y-momentum equation, and continuity equation is also written here for the sake of completeness. Now, what we do this is x-momentum equation referred a equation one, and y- momentum equation is referred here as equation two. So, we are going to do some manipulation from these two equations. And we are going to see that.

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Vorticity and Stream function equations (contd.)

- Differentiate Eq.1 w.r.t 'y' and Eq.2 w.r.t 'x', then subtract second eqn. from the first

$$\rho\frac{\partial}{\partial y}\left\{\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)\right\}$$

$$\rho\frac{\partial}{\partial x}\left\{\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)\right\}$$

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So, differentiate equation one which is x-momentum equation with respect to y. So, all the terms in the x- momentum equation are differentiated with respect to y. We follow similar exercise equation two, which is the y-momentum equation differentiate all of them with respect to x and subtract the second equation from the first equation. So, mathematically it is given here; this is the first equation in this bracket, first equation, which is x-momentum equation, differentiate with respect to y. Similarly, the second momentum equation differentiate with respect to x, whatever you obtained, subtract one from the other. So, what we do will take term-by-term, and see how this is done.

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Vorticity and Stream function equations (contd.)

- Let's take the first term,

$$\left(\frac{\partial}{\partial x}\frac{\partial v}{\partial t}-\frac{\partial}{\partial y}\frac{\partial u}{\partial t}\right)=\frac{\partial}{\partial t}\left\{\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\right\}$$

- Define vorticity ω as, $\omega=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}$

$$\text{Then }\frac{\partial}{\partial t}\left\{\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\right\}=\frac{\partial \omega}{\partial t}$$

- Let's do the second term, $u\left\{\left(\frac{\partial}{\partial x}\frac{\partial v}{\partial x}-\frac{\partial}{\partial y}\frac{\partial u}{\partial x}\right)\right\}=u\frac{\partial}{\partial x}\left\{\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\right\}=u\,\frac{\partial \omega}{\partial x}$

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Now, we will do that exercise term-by-term. Let us take the first term; first term is the time derivative term. And you see here, dou v by dou t which is coming from the y-momentum equation, and that is taken derivative with respect to x, dou u by dou t is coming from x-momentum equation that is taken derivative with respect to y. So, subtract and you are able to get this. Now, this partial derivative can be exchanged with this; similarly this partial derivative can be exchanged with this. So, common term dou by dou t is taken out and dou v by dou x and dou u by dou y term appearing inside the bracket. You can immediately recognize the dou v by dou x minus dou u by dou y, we already defined, which is from the vorticity vector in the first class.

So, this is one component of that vorticity vector omega- z component and that is given by dou v by dou x minus dou u by dou y. So, if you substitute omega into this expression, then you get that equation that expression is written here as dou by dou t of omega. So, this is the first term in the voracity transport equation, the time derivative term, first term in the vorticity transport equation. We will extend this procedure for the second term, which is a convective term, again you can recognize dou v by dou x, which is coming from the second term in the y-momentum equation, take a partial derivative with respect to x then dou u by dou x which is the second term in x-momentum equation take a partial derivative with respect to y. Subtract one from the other then take a common term dou by dou x outside, you get dou v by dou x minus dou u by dou y inside the bracket and this term is same as what is shown here dou v by dou x minus dou u by dou y, and we define that as vorticity.

So, you get u into dou omega by dou x. So, as you can see here, this is the second term in the vorticity transport equation. We are going to repeat this procedure for term-by-term to get a complete vorticity transport equation.

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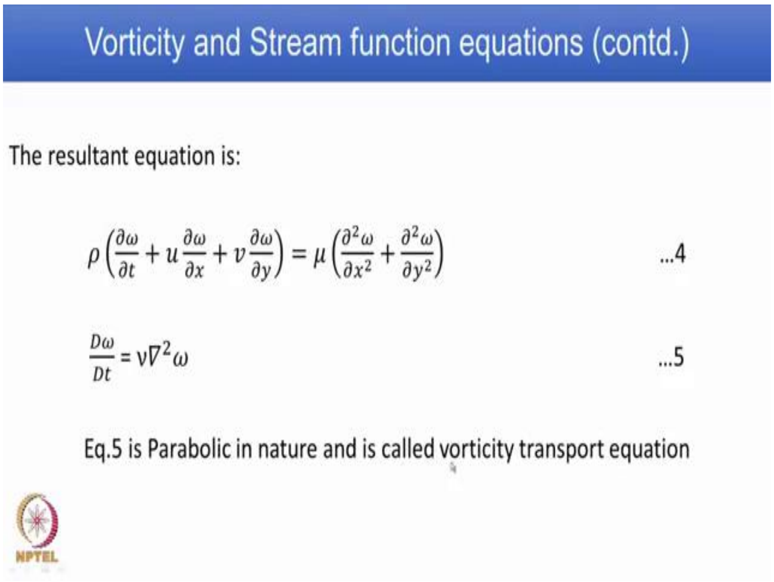

So, finally, the resultant equation will look like this. First term dou omega by dou t, we have seen in detail, u into dou omega by dou x we have seen and v into dou omega by dou y, you can derive. Similarly, you can repeat that exercise on the right hand side and you get the equation like this. So, if you look at this equation, the form the way the equation looks all similar. We had a similar format for u-momentum equation, except that the variable is now vorticity omega; otherwise you are again able to recognize this is the local and these two are convective kind of term. And if you put them together, you get total derivative of expression in terms of omega. The difference here is you see the function use u, but it is not multiplying its own derivative, it is the different function. Hence the non-linearity associated with the convective term in the original momentum equation is simplified in this equation.

So, if I express in terms of total derivative D by D t of omega representing all the three terms here on the left hand side; and then right hand side, is mu into del square omega. So, this equation five is actually what is known as a parabolic in nature. We are going to talk about equation nature next week. So, we will come to know what is parabolic in nature; right now we just accept, this is parabolic and that equation five is what is known as vorticity transport equation.

So, in this class, we have started momentum equation, went over the momentum equation once again, simplification of the momentum equation, obtain Euler’s equation, Bernoulli's equation then non-dimensionalization, some important non-dimensionalized numbers, we did a detailed work in on how to obtain non-dimensionalized continuity equation, it helps to avoid repeating calculation for different situation. Then we went on to do another important topic what is known as a vorticity stream function formulation; in that we did the first part how to obtain a transport equation for vorticity. We will close here for this class. So, next class, we will start from here obtain governing equation for stream function and relating vorticity and stream function and few more interesting topics.

Thank you.