Foundation of Computational Fluid Dynamics
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Lecture – 23

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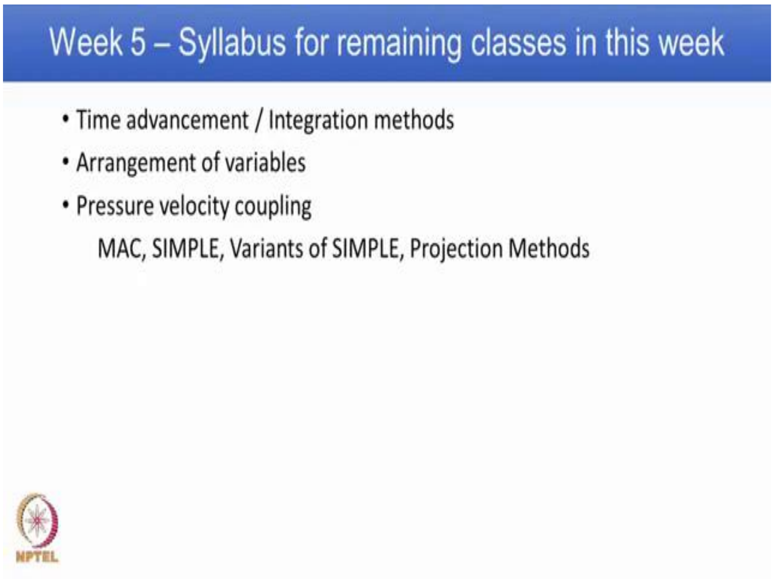

Greetings and welcome again to this course on CFD. Today is our third lecture for this week that for the remaining week we are going to see in detail for time integration or advance method, arrangement of variables. Another important topic what is known as a pressure velocity coupling, there are different methods available under pressure velocity coupling we are going to see them in detail.

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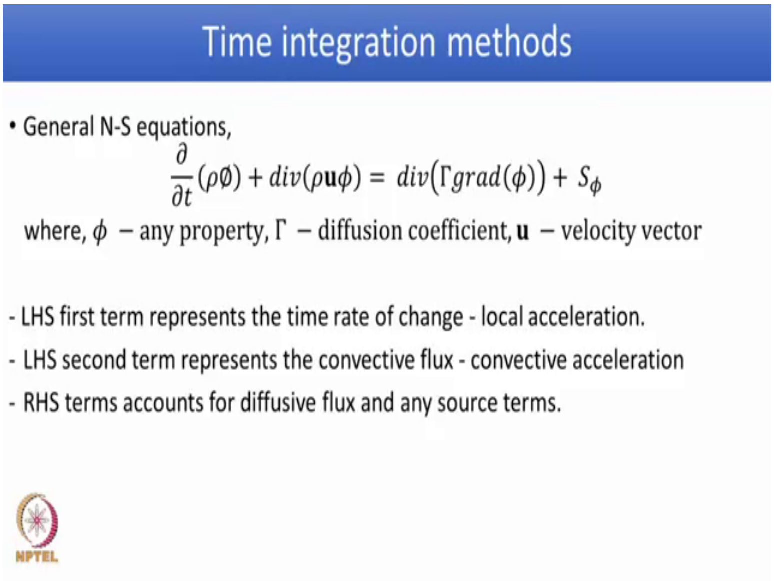

So, time integration methods, in the general Navier stroke equation, we have considered diffusion term separately; convection diffusion term together, and there is a one more important term that is a time derivative term. So we have to know how to do that in finite volume strategy as well. The general Navier Stoke equation is written as shown here dou by dou t of rho phi divergence of rho u phi equal to divergence of gamma grad phi plus S phi, where phi is any property and gamma is a diffusion coefficient and u is a velocity vector. To recognise the terms on the L.H side first term represent time derivative change, otherwise it is a local acceleration; second term is a convictive flux; on the right side, the diffusion term or any other source term. So we perform the time integration on the control volume.

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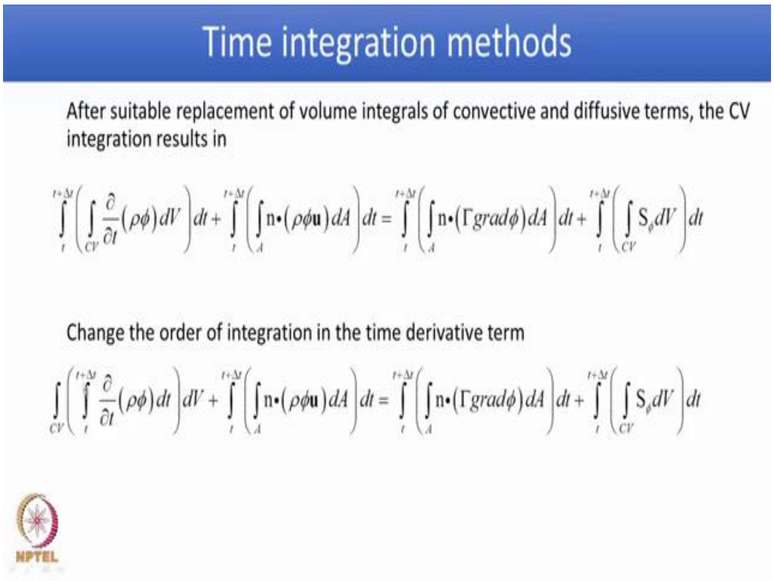

After suitable replacement of volume integrals of convective and diffusive terms, the CV integration results as shown here. So you have a time derivative term, so you go from present time level t by increment in time delta t so the integration limit is from t to t plus delta t. And then first term time derivative term integrated with the control volume that is a new term we have seen for the first time; remaining all known before the convection term, diffusion term or any source term. We can replace the time derivative term integration over the control volume and integration over the derivative so change the order of integration the time derivative term that will result in the formulation as shown here. So this is from time t to t plus delta t.

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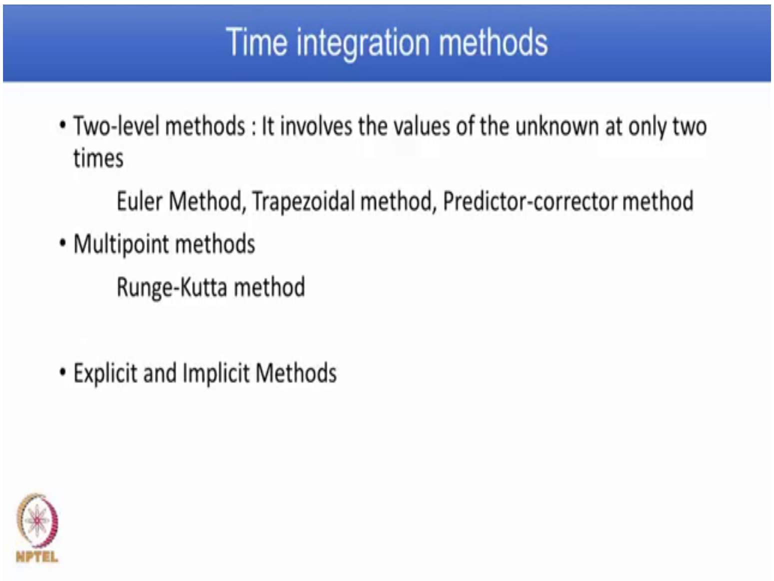

So there are many methods available; first we will see two level methods. Two level corresponds to it involves values of the unknown at two time levels; under the two level methods, we have Euler method, trapezoidal method, predictor corrector method. And there is a multi-point methods basically two level methods you can have a maximum of second order accuracy. If you want to have higher order accuracy for the time derivative integration also, we have to add a some other procedure and that is multi point methods one of that we are already familiar is a Runge Kutta method. We also have a classification what is known as a explicit method as well as implicit method.

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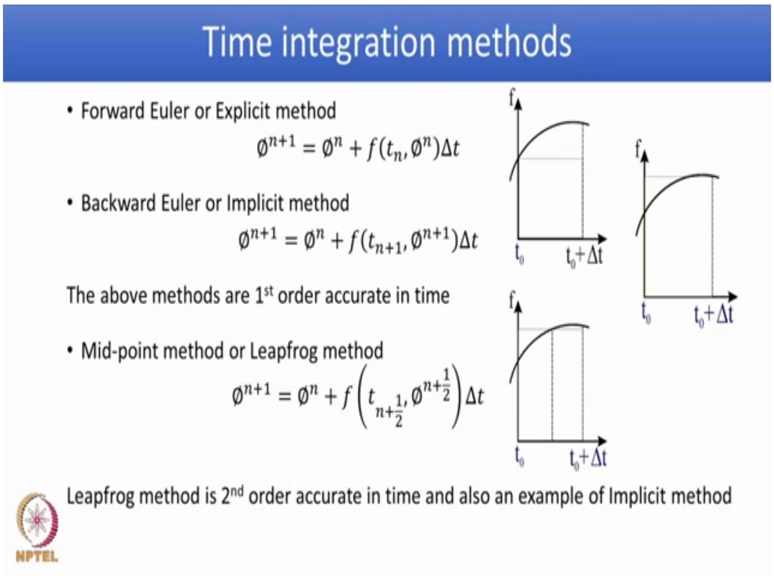

First we will see in detail Euler method, so in Euler method again you have forward Euler or explicit method and that is shown here phi at n plus 1. Here superscript n plus 1 corresponds to time, so phi at n plus 1 means phi at new time level which is defined as n plus 1 is equated to phi at n plus that is the old time level with some derivative. And graphically it is shown here t 0 and t 0 plus delta t and new value that is a function here at t 0 plus delta t is evaluated from phi at 0 time 0 that is what is shown by this. This is supposed to be the functional variable function, this curve represent function variation. And backward Euler or implicit method which takes the value at the function itself and that is shown graphically here. So t 0 plus delta t value is use as at n plus one value also. both these methods that is Euler method is usually or it is you can prove, there it is first order accurate in time.

Why do you have to worry about order of accuracy in time, so if you solve the Navier Stoke equation in full, we already learned convection term, diffusion term discretization of it and it is possible to increase the order of accuracy for convection term and diffusion term. It is necessary as far as possible the time integration term also of the same order as the spatial derivative terms, hence when you solve by that Navier-Stokes equation a particular problem you can confidently say for example, it is second order accurate in time; second order accuracy in space; otherwise, depending on the order of the accuracy you have to make very clear statement. For example, if you use Euler's scheme for time integration, and second order accurate any second order accurate scheme for convection as well as diffusion scheme then you have to say it is first order accurate in time and second order accurate in space.

There is another procedure what is known as midpoint method, it is also called as leapfrog method where you evaluate the function at half time level between previous time level and current time level that is given here as n plus half and graphically it is shown here. So value at n plus half is in middle and phi at n plus 1 using phi at n plus half is used leapfrog method you are to increase the order of accuracy from first order to second order accurate, and it is also implicit method.

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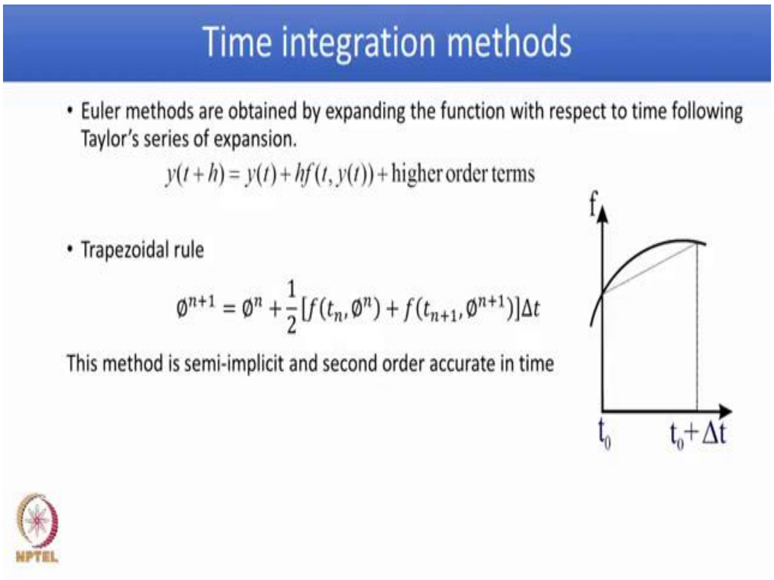

How do you obtain all this Euler's methods? We have already learn Taylor series of expansion we had use the Taylor series of expansion to get any order of accuracy difference formula and we explain only for spatial they had not actually restricted only for spatial there also used for time integration and that is what you have seen here. So in this case, y of t plus h by Taylor series expansion is as shown here. Y of t plus h f of t coma y of t plus higher order term when you know that h square two factorial second derivative and so on you are only interested in first derivative for time. So we suitably replace then you get different Euler formula. Next is the trapezoidal rule where this is combination of both implicit as well as explicit, and you have a weightage of half fifty percentage weightage for the explicit part and fifty percentage weightage for implicit part, and this is also known as semi implicit scheme or Crank-Nicholson scheme. Graphically it is shown here. So we have a trapezoidal rule, so t 0 at the point shown here and t 0 plus delta t it is shown here, and you fit a trapezoidal curve you have a trapezoidal fit between the two then you get this formula derive. It is explicit fifty percent, implicit fifty percent so this is called semi implicit and it is a second order accurate in time.

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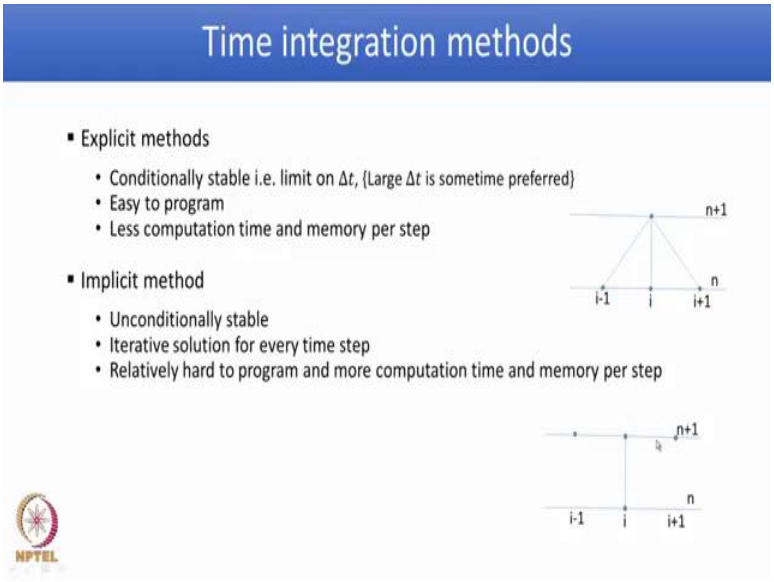

Let us look at some details about explicit method. They are conditionally stable, you already learned about stability von-Neumann stability analysis. Hence all the explicit schemes are conditionally stable. There is a criteria called CFL criteria that is either you have a limit on delta t or you have a limit on delta x. You have dealing with a time integration method, so there is a limit on delta t; large delta t is sometime preferred for a problem you want to capture different scales when you want to have large delta t. It easy to program because current level values are not required, all the previous values are only required and you would not have to work hard to write a very difficult program. It is less computational time; for every step it takes only less computational time and less memory per step. If you look at graphically the explicit method, so value at n plus 1 is evaluated based on variable value at nth level from i minus 1, i and i plus 1.

Let us look at implicit method. All the implicit methods are unconditionally stable; that means, you can have affords to use any delta t. It usually follows iterative solution for every time step explicit method one invention is enough, whereas implicit method requires iterative procedure because it accounts for the variable value at the current time level also. It is relatively hard to program as compared to explicit method and it takes more computational time as well as memory for every step; graphically it is shown here. So value at n plus one is evaluated based on value at n plus one based on, but at i, i minus 1 as well as i plus 1.

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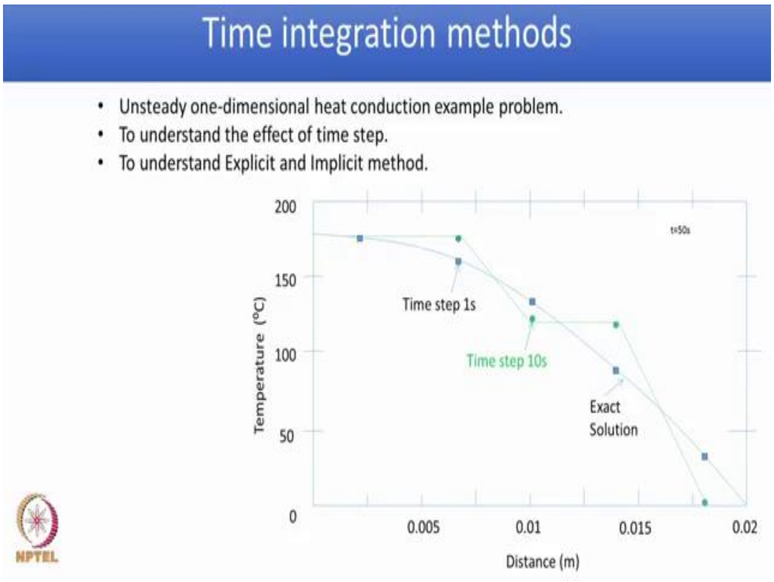

Let us look at the performance of this explicit scheme, implicit scheme, and the role of delta t in the explicit scheme. Already there is a problem considered that is unsteady one-dimensional heat conduction example problem. We take this problem that is already solved it is only a demonstration to show the effect of time step as well as to understand explicit implicit method performance. So what is shown here is a time is a distance and temperature for one particular calculation results obtain after 50 seconds. If you use different delta t, performance by explicit scheme and what is shown here is if you use time step of 10 seconds that is delta t is 10 seconds is slightly larger. When you have a explicit scheme delta t matrix, because it is conditionally stable.

And in this example we show two different delta t 1 is delta is equal to 10 seconds and delta t equal to 1 second, so which is a smaller time step. And exact solution is already known for this problem that is already plotted. And if you use delta t 10 seconds after the calculation is proceeded 50 seconds you see performance there is a vigil it does not reach exact solution, it goes up, it goes down it never reach at the exact solution. If you delta t 1 second that is finer delta t then you see performance it reaches almost same as exact solution and performance is better so in next plot for the same problem if you happen to imp if you happen to incorporate implicit scheme and you see the performance.

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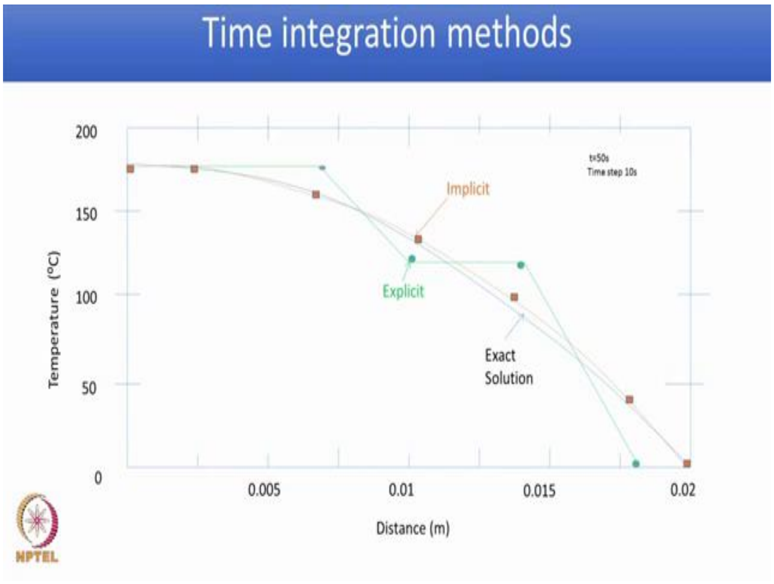

For the same final times 50 seconds and then 1 delta t that is 10 seconds, we already noticed explicit scheme, it shows step wise solution ,whereas implicit scheme it captures almost same as exact solution. So you understand effect of implicit scheme as well as explicit scheme on the results. You also have to see the time required to obtain the solution as well as memory required to use explicit scheme or implicit scheme.

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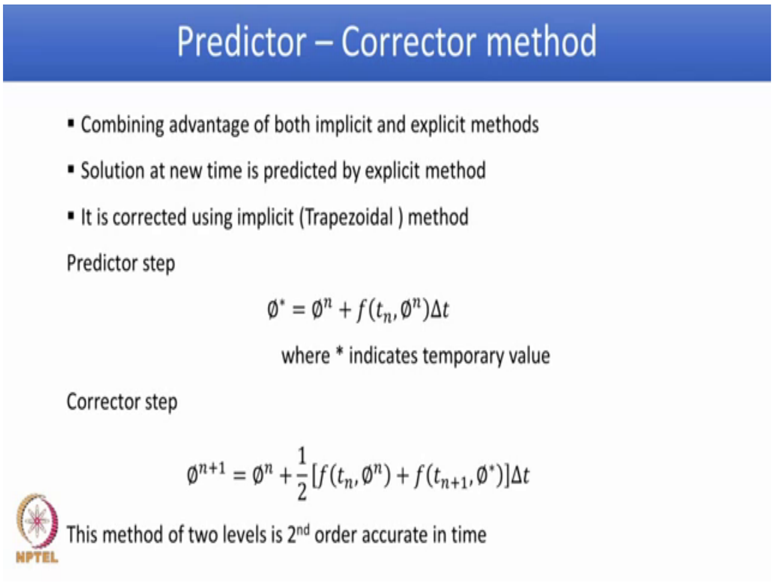

So the question is whether you can combine both implicit scheme, explicit scheme, because each of them supposed to have some advantage in solution at u time step is predicted by explicit scheme. And then the next step it is corrected using the implicit scheme so such so such kind of two times procedure is called predictor corrector method so one proposal is given here u time level is predicted by explicit scheme and it is corrected by implicit scheme.

For example, predictor step because it is only predicted it is not a exact value we use a superscript star so phi star is phi of n phi star equal to phi of n plus f of t n phi n delta t. And this we already know, it is forward Euler or explicit scheme and star super script star indicates temporary value or predicted value, and this will be corrected using a trapezoidal rule and that is what is shown here. So when you use a trapezoidal rule, the predicted value becomes a known value and that is what suitably used here. When you write down the explicit part and when you write down the implicit part this what is known as a corrector step. So this method of two levels and it is second order accurate in time.

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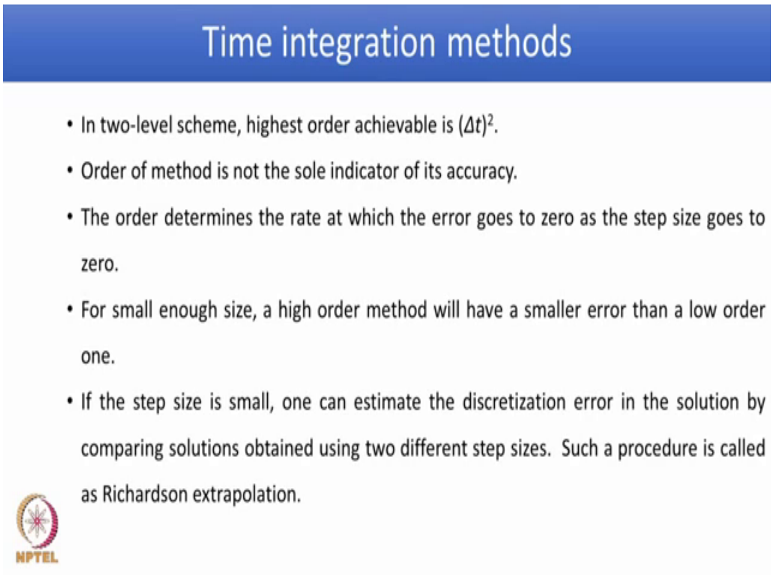

In two level schemes highest order achievable is delta t square order of method is not the sole indicator of its accuracy. The order only determines the rate at which error goes to zero as the step size goes to zero. So the order, it is not just indicator of accuracy, even if you increase the order determine rate at which error goes to zero. So if you decrease the delta t when you already seen in the results. If you have a delta t very high delta t the performance by explicit scheme is poor, but the same explicit scheme performance better if you reduce the delta t, so for smaller f size higher order method will have a smaller error than a lower order one. If the step size is small then one can estimate the discretization error in the solution by comparing solution obtain by two different step size so it is possible to get idea of one what is error in the new delta t and that procedure what is known as a Richardson extrapolation procedure.

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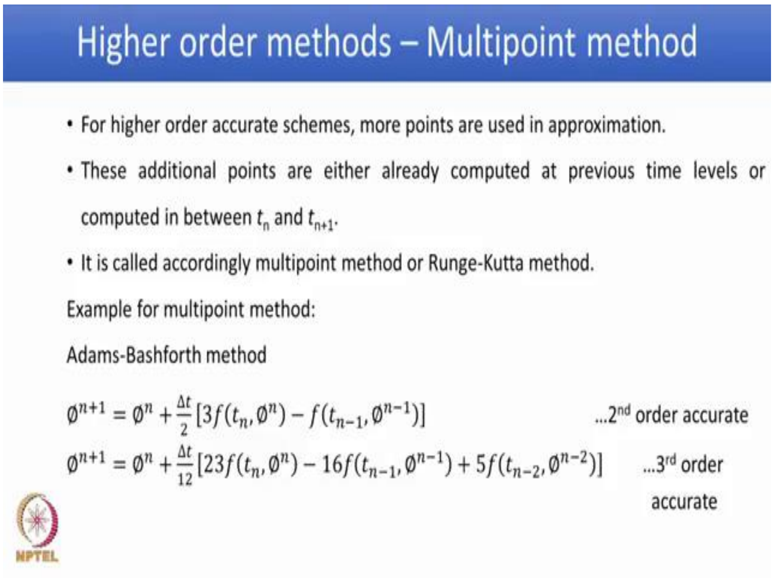

So we have seen we have a two level method, the two level method highest possible accuracy is second order accurate. The question is now whether it is possible to devise the scheme to have the higher order accuracy even for time derivative term. I already mention when you solve a Navier-Stokes equation full, you have a convection term, diffusion term, pressure term or any other source term. And we know very well how to obtain different higher order accuracy for all this term which are spatially derivatives. We have to have a corresponding higher order accuracy possible even for the time derivative term, so overall solution can be called of the same order of accuracy both in time as well as in space. So we are going to look at procedure how to obtain higher order methods even for time derivative and such a scheme is also known as a multi-point scheme.

In multi-point scheme as a name indicated we use more points and how you obtain this more points this additional points are either already computed at previous time level. So for example, you are at correct time level n plus one we have a values at n just immediate previous time level you also have a value at previous time level n minus 1 or n minus 2. So this one possibility or the difference between time n and time n plus 1 can be divided into number of points, and values can be computed at this intermediate points, such a scheme is also multi point method. It is called accordingly multi point methods and there are multiple options available, you already know a method called Runge Kutta method which is a multi-point method.

One example of multi point method is shown here what is known as Adams-Bashforth method. Here again we have a one predictor as well as one corrector. So phi at n plus 1 is phi of n plus delta t by 2 3 times function evaluated at time level n and one time function evaluated at n minus one level then this is actually one step, which is second order accurate Adams-Bashforth method. It is also possible to increase even in Adams Bashforth method next level this is third order accurate by considering one more additional points at much previous time level. So the formula is finally shown here phi at n plus one is regulated to phi at time level n then phi at time n minus 1 then phi at time n minus 2, so n is a just immediate previous time level n minus 1 is one before n minus 2 is even one before that.

So by considering more points in the previous time level you are able to construct the method this is what is known as Adams-Bashforth method. And how do you obtain this coefficient 3, 23, 16, 5 I have already mentioned by using a Taylor series expansion even for time then you are able to construct these methods. So first one uses only immediate previous time level, it becomes second order accurate; the second one uses two time levels before it becomes third order accurate. One more observation all this uses previous time level, hence it becomes explicit scheme, so Adams-Bashforth scheme you are able to obtain explicit second order as well as third order.

The next one is what is known as Adams-Moulton method. In this the second the third order accurate scheme is shown here which consider values at time level n and time at n minus one and then one more point which is at the present level itself that is time at n plus 1. So time at n plus 1 is present time level and variable value at that present time level is used because that is there on the n left hand side. So this scheme is implicit scheme. So you can actually construct like this to get any order of accuracy multi point level, explicit, implicit or combination of both a common procedure is used n minus 1 is order of Adams-Bashforth method as a predictor, and mth order of Adams Moulton method has also corrector.

So we have already seen Adams-Bashforth method two scheme second order accurate, third order accurate; Adams Moulton method one scheme third order accurate. So you can combine both in the form of predictor as well as corrector method. You can use these Adams-Bashforth, Adams Moulton method independently; you can also combine both of them in the form of the predictor as well as corrector method, and that what is described here. You can use one order lower Adams-Bashforth method as a predictor and the one order the decided order as a corrector. It is easy to construct any order of accuracy by this Adams method and only one evaluation of derivative per time is possible.

Te disadvantages if you start your calculation say time is equal to zero then you do not have values as a previous time level n minus 1, n minus 2, so any of this method Adams-Bashforth method or Adams Moulton method are not self-starting. So, one has to restart to any other previous completely explicit method, to get first value either at n minus 1, or n minus 2 before switching it to Adams-Bashforth method or Adams Moulton method. So your initial calculations are not accurate enough.

So in this class, we have seen different time integration procedures, explicit method, implicit method, influence of explicit method, influence of implicit method for a illustration problem, influence of delta t on the performance by explicit method. Then we also learned how to get higher order accuracy for time derivative by predictor, corrector method and we showed Adams Bashforth method and Adams Moulton method and how to combine them also. Next class, we will see another interesting topic until then have fun.

Thank you.