Foundation of Computational Fluid Dynamics
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Lecture – 21

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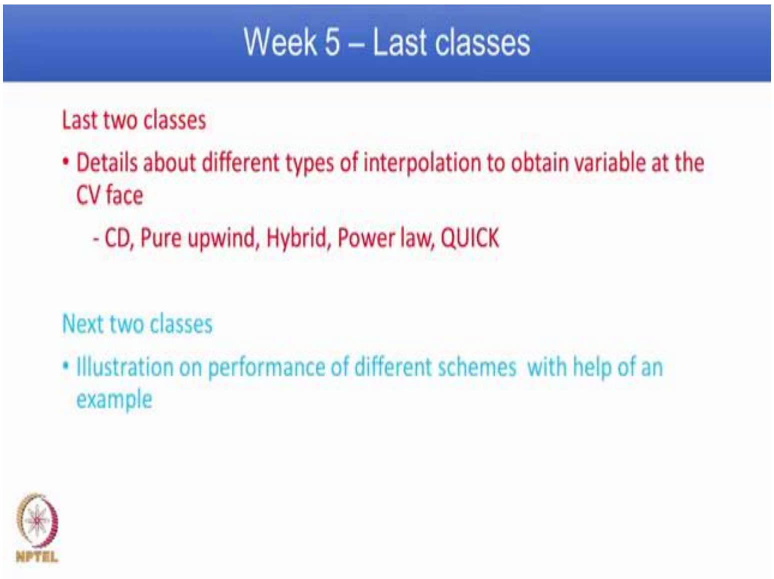

Greetings and welcome again to this course. Last two classes, we had seen in details about different types of interpolation to obtain variable value on face. Why do you want to do that, because in convection term, we have the variable value appearing, and they are to be evaluated on control surfaces. Accuracy of this scheme depends on what types of approximation you are using. And we had seen five approximation procedure – central differencing type approximation, pure upwinding, hybrid scheme, power law scheme and Quick scheme. We had seen all these different approximation in detail, mathematical formulation for different situation, and how to put along with diffusion equation discretization, a generic formula finite volume difference formula, and how these coefficients are related to each other. So, node of interest coefficient is related to neighboring nodes.

And in the case of pure upwinding, we had a term called alpha, which decides the direction of the flow, according to the direction of the flow the switching happens. Similarly, in the case of QUICK scheme, we had a factor by name beta, which again depending on the direction of the flow, it takes appropriate weightage or the correct direction and switches. In today's class and followed by next class, we will see with an example how each of this scheme behave for different mesh size for the same problem get a field for performance of each of these scheme.

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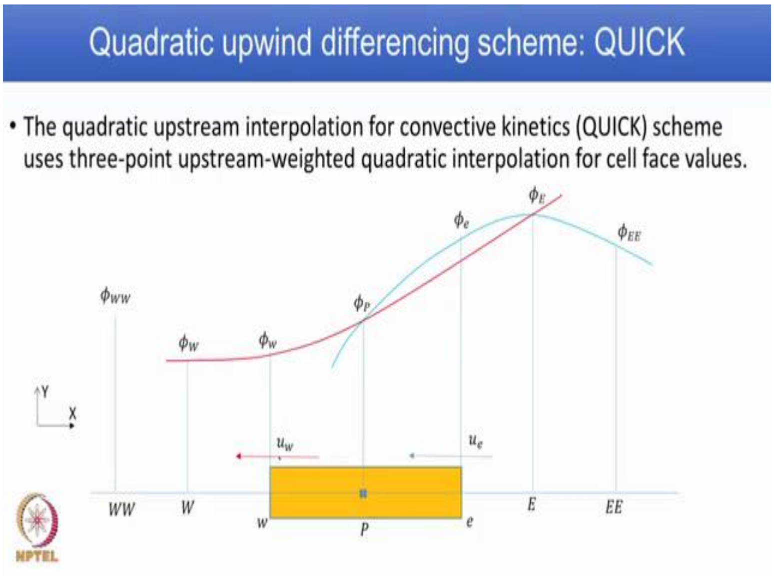

Just to recapitulate here is a illustration for QUICK scheme and we already learned in QUICK scheme, we have quadratic fit through three points, two points are on either side of the node of interest and one additional point in the direction of the flow.

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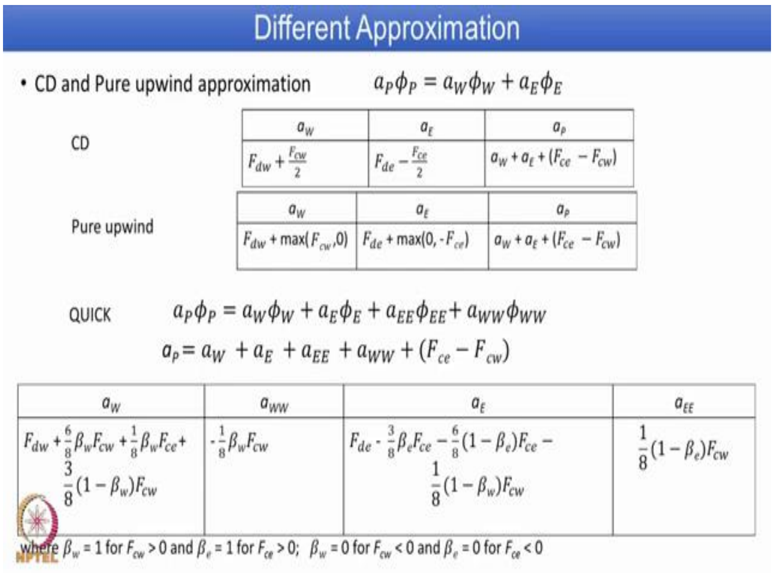

We had seen so far mainly three approximations, central differencing type, pure upwinding and QUICK scheme. In this particular slide, we will see all of them together get idea of how this coefficients are different for each scheme. For both CD type as well as pure upwinding scheme, finite volume discretized equation is written as shown a P phi P equal to a W phi W plus a E phi E. As we said many times before, P is the node of interest, and it runs from for the case of one-D, left to right; and a W and a E are neighboring nodes coefficients, so node of interest coefficients a P is related to neighboring node coefficient. In the case of CD, expression for coefficient, for a E, a W, a P are shown, similarly for pure upwinding coefficients for a W, a E as well as a P are shown here.

You can understand now very well, the difference in the term if it is central type and if it is pure upwinding. In the case of pure upwinding, we introduce a term called max of F c w and zero, so it takes between the two whichever the maximum, which is actually introduced for the purpose of direction of the flow. So, if the direction of the flow is from left to right, then it is positive F c w will be used in this expression. Similarly, if the direction is negative, then F c w will be negative, zero is the maximum then accordingly zero that contribution that node will not be consider. This way of writing is helpful when you are writing the code, it automatically switch between one side to other side based on this control.

Next we will see QUICK; in the case of QUICK then we get expression a P phi P equal to a W phi W plus a E phi E plus a EE phi EE plus a WW phi WW. We can immediately observe, the QUICK scheme has two additional terms. Once again this is the generic expression, generic in the sense, it accounts for any type of flow direction. So, when you compare CD or pure upwinding, QUICK scheme has two additional information from QUICK scheme has information from one additional node and that one additional node depends on the direction and that is put in the generic expression as shown here. And a P as we did in other two scheme once again relating to neighboring nodes, so a E, a W and a EE as well as a WW and additional term.

Correspondingly, you can write down coefficient, so a W, a WW, a E and a EE. We have introduced new coefficient beta, again that is decided based on the direction of the flow. If beta is equal to one, then the flow is from left to right, it is positive. If the beta is less than zero or equal to zero, the flow is from right to left then accordingly corresponding weightage taken to determine the coefficient values. So, for example, if beta is equal one the term in a W, this particular term is zero, there is no contribution from west node flux contribution from the west face then that is used here. Similarly, you can try to understand based on different values of beta coefficient expression, and it is the generic expression, so this helps in writing one code with different switching control option.

Now, we can also get a curious question where you can have code written employing for example, all the three schemes that is CD, pure winding and QUICK the answer is yes, because coefficients terms have same structure. As you can observe here, for example, a W has a similar structure only additional terms are introduced. So, you can calculate for CD and switch to pure upwind or switch to QUICK depending on the scheme that you are deciding that automatically is decided based on the direction of the flow.

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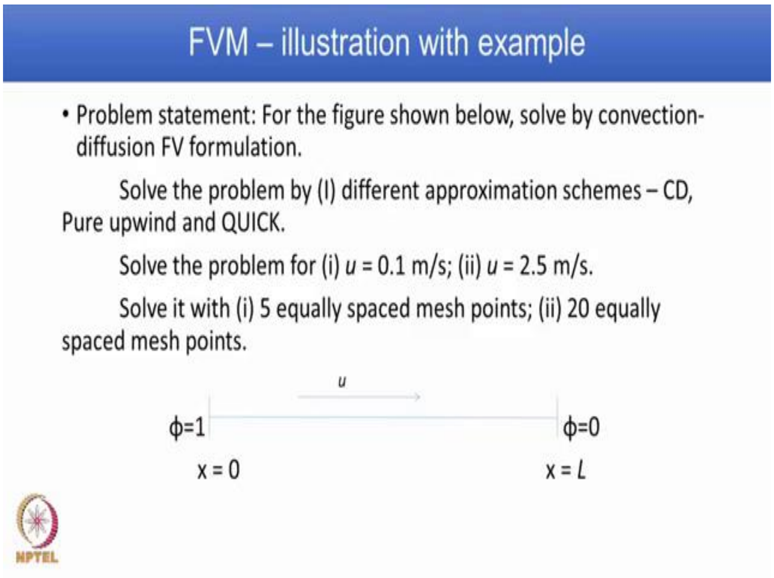

Now, we will take an example problem and try to get idea about performance of these schemes. So, problem statement is given here. The figure shown is one-dimensional, the flow is form left to right; x is from 0 to L, is the length of the problem, variable value at the left side is given as boundary condition phi at one, similarly the variable value at the right side, phi equal to 0 is given on the right side. The question is solve the problem by different approximation schemes – CD, pure upwinding and QUICK. Now, to get idea of how these schemes behave for different flow condition we take two different flow condition, u is equal to 0.1 meter per second another condition u is equal to 2.5 meter per second. Then to get the idea of for the same speed, if you discretized with the five mesh points, if you discretized with twenty mesh points, what is the different between five mesh points and twenty mesh points, the delta x is decided, the delta x term goes into the expression of coefficient calculation.

So, to understand how this mesh spacing also influences the accuracy, we will have another case solve the problem with two different mesh size, one for five equally spaced mesh points, another twenty equally spaced mesh points. Now, in the further illustration, we will not do all the combination, we will only look at few combinations and get the idea of performance.

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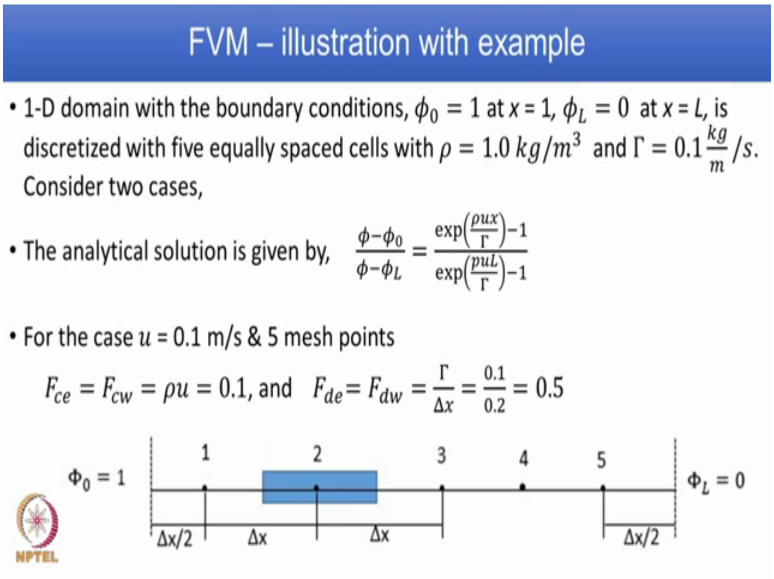

In the last slide, we explained the problem. Now, let us get some more explanation. 1-D domain with the boundary condition phi at zero equal to 1, phi at L equal to 0 when x equal to L is now discretized with five equally spaced. And for the problem define rho is taken to be 1.0 kg per meter cube and gamma taken to be 0.1 kg meter per second. Let us take two cases, and this is the domain. We discretized the domain with five equal nodal points, 1, 2, 3, 4, 5 and they are equally spaced. So, you see delta x as the distance between node one and two, same way the distance between node two and three and so on, because it is equally spaced. Node one which is the node near to the left boundary, similarly node five which is the node near to the right boundary, both are at the distance of delta x by 2 with respect to the respective boundary. And boundary conditions are already shown here, both left side as well as on the right side.

And for this problem analytical solution is already given and that is what it is displayed here. Now, we take one of the case that u equal to 0.1 meter per second and five mesh points. We calculate convective flux as well as diffusive flux; convective flux equal to rho into u; rho is already defined as 1 kg per meter cube and u is 0.1 meter per second, so rho u, you will get 0.1. Similarly, for diffusive flux, it is defined as gamma by delta x; gamma is 0.1, delta x is 0.2 and that is equal to 0.5. We want to evaluate convective flux as well as diffusive flux to get the idea of role of convective flux and diffusive flux. Now, this relation or weightage between convective flux and diffusive flux will inform or tell us about performance by different discretization scheme.

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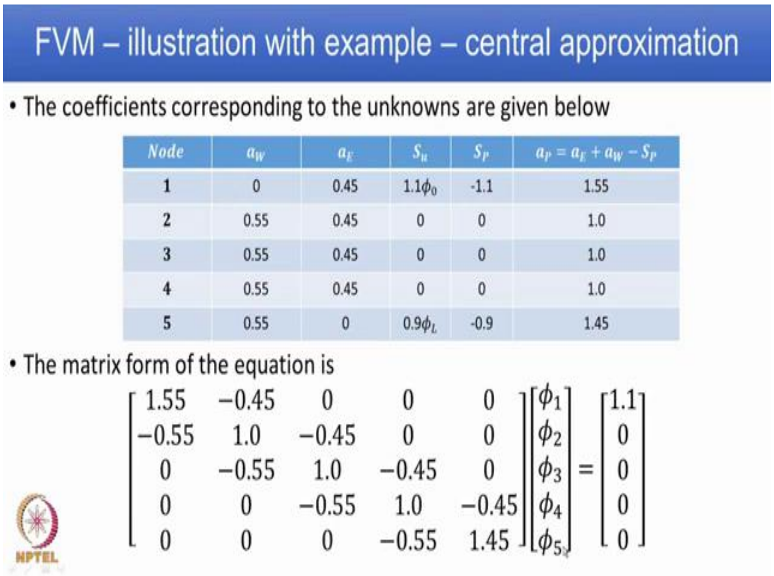

FVM – illustration with example – central approximation

- The coefficients corresponding to the unknowns are given below

Node	a_W	a_E	S_u	S_P	$a_P = a_E + a_W - S_P$
1	0	0.45	$1.1\phi_0$	-1.1	1.55
2	0.55	0.45	0	0	1.0
3	0.55	0.45	0	0	1.0
4	0.55	0.45	0	0	1.0
5	0.55	0	$0.9\phi_L$	-0.9	1.45

- The matrix form of the equation is

$$\begin{bmatrix} 1.55 & -0.45 & 0 & 0 & 0 \\ -0.55 & 1.0 & -0.45 & 0 & 0 \\ 0 & -0.55 & 1.0 & -0.45 & 0 \\ 0 & 0 & -0.55 & 1.0 & -0.45 \\ 0 & 0 & 0 & -0.55 & 1.45 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{bmatrix} = \begin{bmatrix} 1.1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We first do central difference approximation. We already seen expression for coefficients. Now, we substitute these values that is F c, F d values in the expression for each nodal location, and we get values as shown in this table. We can observe immediately node one, left side is the boundary condition, hence a W is 0; similarly for node five, right side is the boundary condition and a E is 0. These are cross checking. And the boundary condition on the left side as well as on the right side appears as a source term and that is also shown here. All for all internal nodes for this particular problem, there is no source, hence they go to 0. And we also mentioned point of interest node P runs from left to right, in this case, it runs from one to five. We also mentioned point of interest node coefficient a P is related to neighboring node coefficient and that is what is written here a P is related to a E plus a W minus S p.

And you can cross check here, take any one example, for example, node two, 0.55 for a W, and 0.45 for a E, and there are no source term, hence a P is just 1 and it is shown here. Now, you can write this in the form of matrix and that is shown here. So, if you write a P, phi p is equal to a E, phi E plus a W, phi W and that is what you will get the matrix. This coefficients are already calculated. Now, if you look at this, this is the coefficient matrix multiplying unknown variable column vector that is on the left side; on the right side, known value at the column vector. So, this P, runs from left to right that is from one to five and that is what is given here as unknown column vector.

So, this matrix if you look at, there is one main diagonal, and elements along the main diagonal, 1.55 1.0 1.0 1.0 1.45, and there is a immediate super diagonal and immediate sub diagonal. All other elements in this matrix are zero both above as well as below. Such a matrix which has only three one main diagonal just above and just below is what is known as tridiagonal matrix. And there is a separate procedure available what is known as tridiagonal matrix algorithm. We will be looking at that detail in subsequent class. So, once you invert this matrix by that procedure TDMA – tridiagonal matrix algorithm procedure then you will get all the values at nodes one to five.

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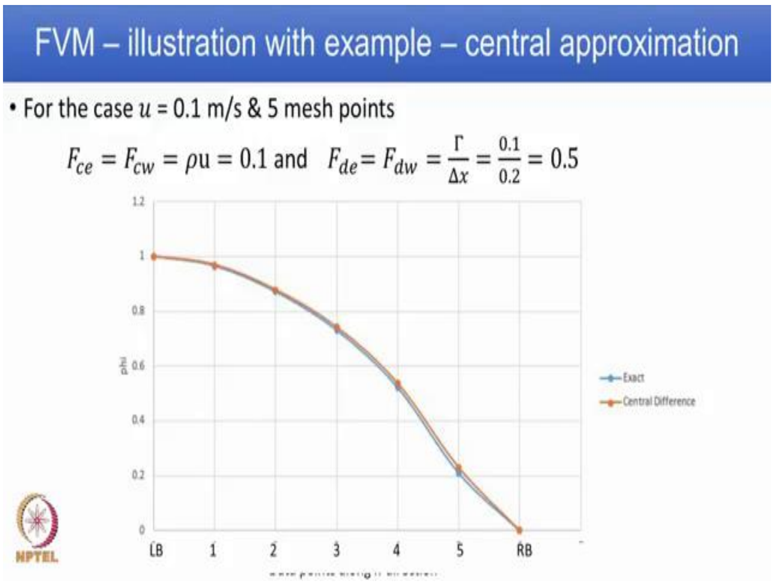

We also have a expression analytical solution, so we compare result obtain by analytical solution against results obtained by central type approximation. For this particular case, and that is what is shown in this graph, because the problem is very simple, the velocity condition is only 0.1 meter per second and five nodes only we are consider. The solution happens to be almost same as exact solution. So, in this case, there is no serious remark in terms of performance, it is only that for this condition CD exactly matches or very closely matches with the exact solution.

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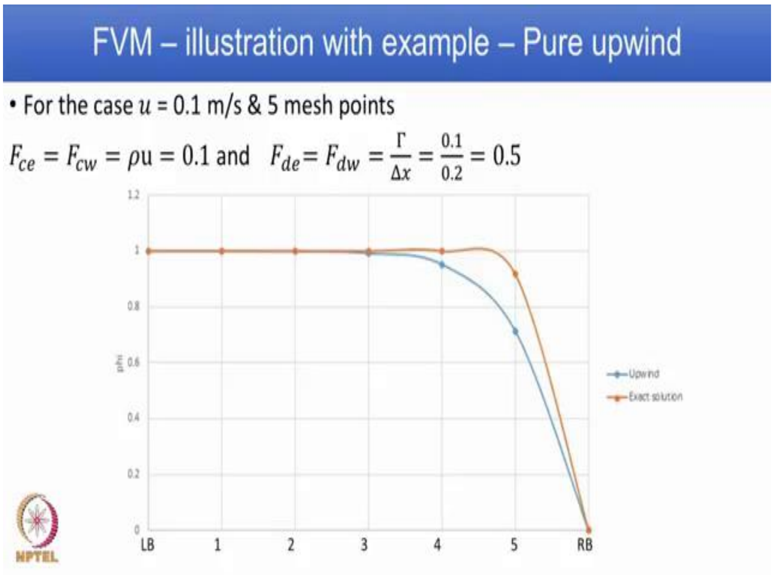

Let us take performance, let us look at the performance by upwind scheme. We are not going to the details of how each coefficients are calculated, because for this problem with this value of 0.1 meter per second and five nodes and density value and diffusion coefficient, it is very easy to calculate the coefficients value, you can get again from the generic expression, get the matrix and solve then you get solution by this upwind scheme – pure upwind scheme. You observe here pure upwinding scheme it matches with the exact solution for some points and then there is a deviation. So, this is pure upwinding, results obtained by pure upwinding and this is solution obtained by exact solution, there is a discrepancy here. Whereas in the previous case, for the same arrangement, the difference only in the scheme that is central difference scheme and here it is pure upwinding the solution almost matches.

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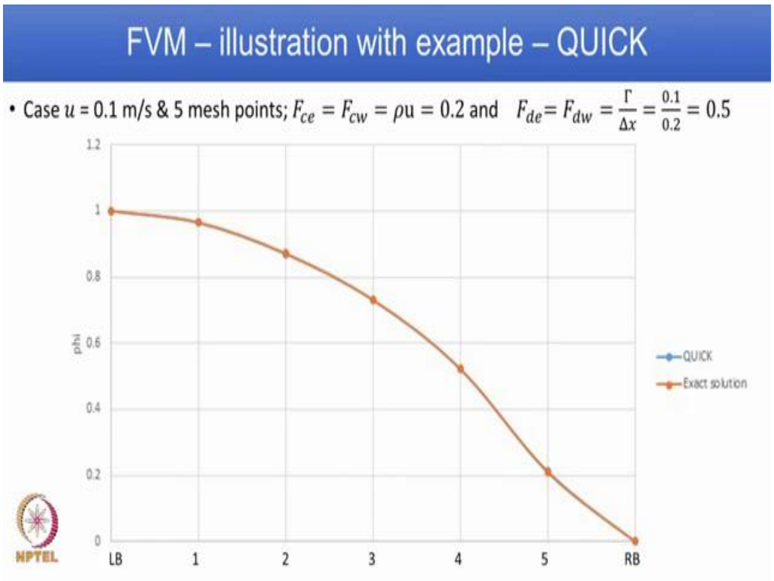

Let us look at the performance by third scheme that is QUICK scheme has three nodes; one node on either sides and one extra node direction of the flow. In this particular example problem, the direction of the flow is only one side that is from left to right always. And so the performance almost matches with the exact solution. We will look into this QUICK formulation in detail in next class. This idea here is to show performance by QUICK scheme also in addition to performance by pure upwind as well as CD.

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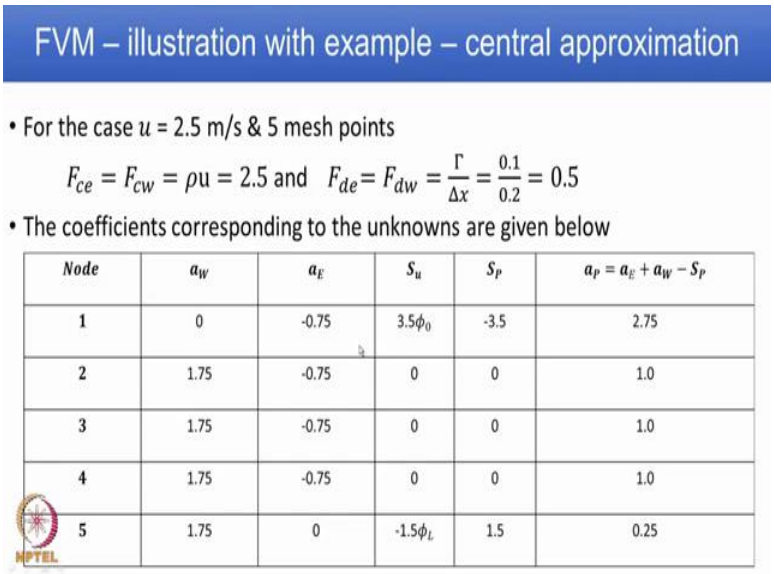

FVM – illustration with example – central approximation

• For the case u = 2.5 m/s & 5 mesh points

$$F_{ce} = F_{cw} = \rho u = 2.5 \text{ and } F_{de} = F_{dw} = \frac{\Gamma}{\Delta x} = \frac{0.1}{0.2} = 0.5$$

• The coefficients corresponding to the unknowns are given below

Node	a_W	a_E	S_u	S_P	$a_P = a_E + a_W - S_P$
1	0	-0.75	$3.5\phi_0$	-3.5	2.75
2	1.75	-0.75	0	0	1.0
3	1.75	-0.75	0	0	1.0
4	1.75	-0.75	0	0	1.0
5	1.75	0	$-1.5\phi_L$	1.5	0.25

Now, the second case is the higher velocity u is equal to 2.5 meter per second for the same problem that is same domain, number of mesh points are same as five, u is increased from previous case 0.1 meter per second now it is 2.5 meter per second, which means it is convection dominated, hence you get flux value calculated to be 2.5 and diffusion value calculated to be 0.5. Now, you can get idea of ratio of convection flux to the diffusion flux it is the convection dominated problem. And we know the transportive property is ensured better impure upwinding when compared to central differencing. So, we again get details of all coefficients by one scheme, this case it is central approximation with for this condition u is equal to 2.5 meter per second, maintain number of mesh points to be 5 and we get coefficient as shown here. These are individually calculated what is shown here only the final value.

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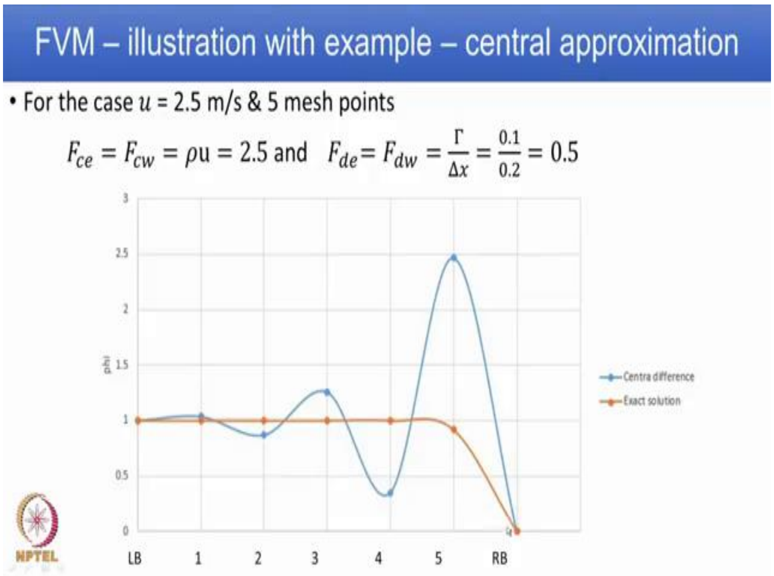

Now, if you look at the performance as I have been telling before velocity is increased from 0.1 meter per second to 2.5 meter per second, it is a convective dominated problem now. Hence the CD scheme will not be able to capture the direction and the performance is not up to the mark. Once again what is plotted here is the exact solution and performance by central scheme as you can see, there is a vigil, it is up and down and it is limited by a boundary condition on one side as well as on the other side.

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FVM – illustration with example – central approximation

- For the case u = 2.5 m/s & 20 mesh points

$F_{ce} = F_{cw} = \rho u = 2.5$ and $F_{de} = F_{dw} = \frac{\Gamma}{\Delta x} = \frac{0.1}{0.05} = 2.0$

- The coefficients corresponding to the unknowns are given below

Node	a_W	a_E	S_u	S_P	$a_P = a_E + a_W - S_P$
1	0	0.75	$6.5\phi_0$	-6.5	7.25
2 - 19	3.25	0.75	0	0	4.0
5	3.25	0	$5\phi_L$	-1.5	4.75

For the same problem that is velocity condition is 2.5 meter per second, but the number of mesh point is now increased from 5 to 20; in that case the diffusion flux, where delta x appear in the denominator is increased to 2.0. So, if you compare convection flux and diffusion flux, the order is almost the same that is two point and two point zero. This is convection flux, and diffusion flux meant for that particular control volume. So, for the same problem, velocity condition is the same 2.5 meter per second. All that we have done increase the number of mesh points from 5 to 20. Same scheme that is central approximation once again you calculate all the coefficients. So, you can see node one, which is near the left boundary and node five which near the right boundary and all others are internal nodes between 2 to 19, and we get coefficient calculated. As we have observed before, internal nodes there is no source generation, hence they are all zero, the boundary condition appears to source term for near nodes that is what you see here node and node five they have value. Once again, you have observe a P is related to neighboring coefficient, you can again cross check this values.

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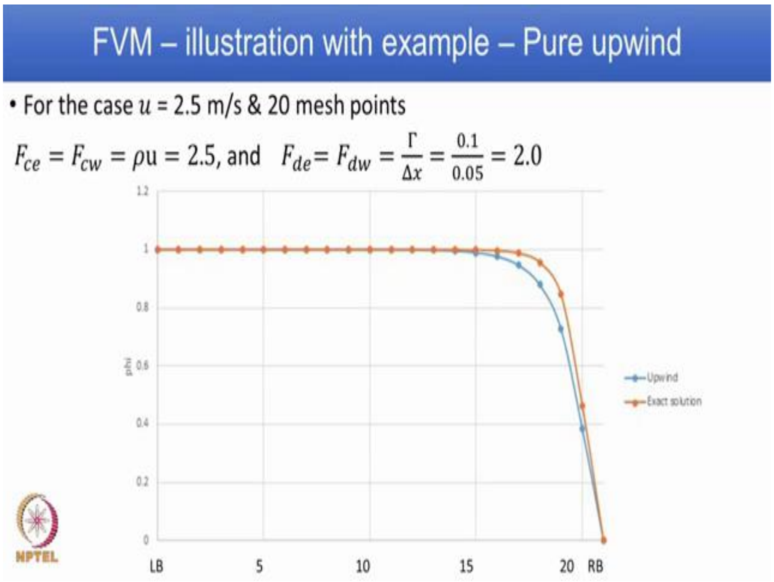

Now, you see the performance the same scheme central differences scheme for the same flow condition 2.5 meter per second when it was with five mesh points it exhibited vigil in the solution, the vigils are gone by increasing the number of mesh points from 5 to 20, and you get solution very close to the exact solution without any oscillation ups and downs. So, this is the way you can improve the solution accuracy if the order of accuracy is constrained. So, in this case, you would decide to have a central difference scheme, because it has a second order accurate, but it exhibited vigil or oscillating type of a solution. Now, you can increase the mesh size and get away with the oscillation and get a improved solution. For the same 2.5 meter per second with 20 mesh points, we would like to see performance by pure upwinding. We already know the pure upwinding has a property in the direction of the flow, hence in this case the performance is not surprising, the performance by the pure upwinding is almost same as the exact solution.

Now, with this, we looked at performance by two different schemes mainly central differences scheme and pure upwinding for one particular problem and two different flow condition and two different mesh condition. So, if you change the mesh from five to twenty for the same flow condition QUICK gives better performance or improved solution without any vigils as well as oscillation. Pure upwinding it performs better if you increase the mesh size, the diffusion part is reduced. So, in the next class, we will take the same example problem, we will try to understand performance by QUICK scheme, and then we will put performance by all the scheme together in one plot and get the idea of performance by all the schemes.

Thank you.