**Foundation of Computational Fluid Dynamics**
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**Lecture – 17**

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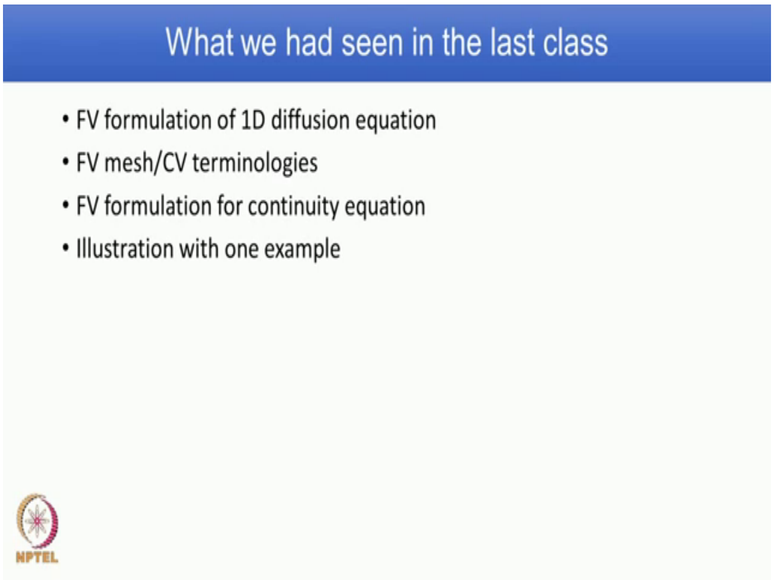


It is my pleasure to welcome you to this course again. Today is the module two for this week. Last class, we had seen finite volume formulation for 1D diffusion equation in detail. We understood the terminologies used in finite volume mesh, control volume arrangement, and how to identify distance between node and node distance between node and control surface. Then we explained in detail diffusion equation then we understand with a help of example problem the same formulation. We also try to do for near boundary node treatment. So, we took example problem and we did up to left boundary node for that problem. We have also written equation finite volume formulation for continuity equation.

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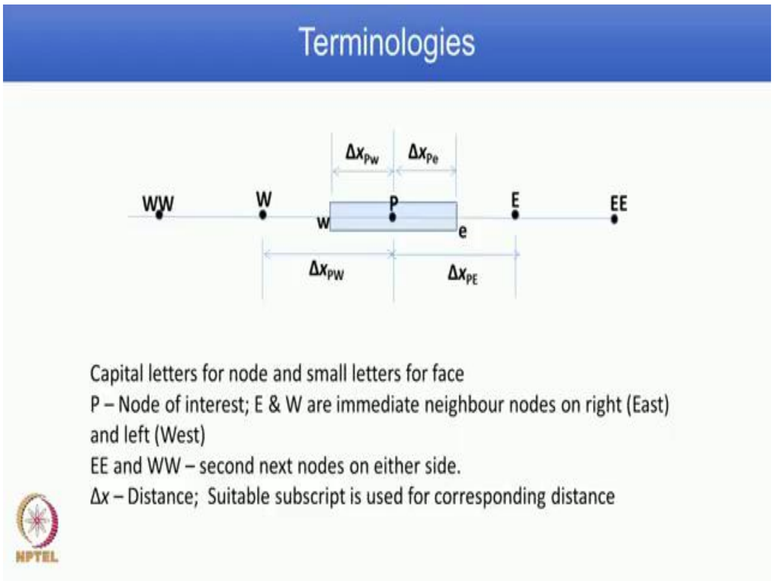


To recapitulate what we did, this important to show again understand again the terminologies used. Here is mesh arrangement, P is the point of interest, and the full circle is a node that is marked here as a P which is the point of interest. Then for a uniform spacing, the arrangement is shown, east side capital E and west side node is capital W. Similarly to the west of west it is given here as W W; to the east of east is given here as E E. Then a control volume is constructed around the node of interest. So, in this case, the full coloured shaded portion is a control volume, and the left side surface is shown here as a w - small case w. Similarly, the east side surface is shown here as small e; distance between node P and the face because it is one dimensional x-direction we say it is delta X P w; similarly, the other side delta X subscript P and small case letter e. Similarly distance between node and node, so here it is between node P and node W is again delta X, but it with a subscript capital P and capital W. We have seen all this.

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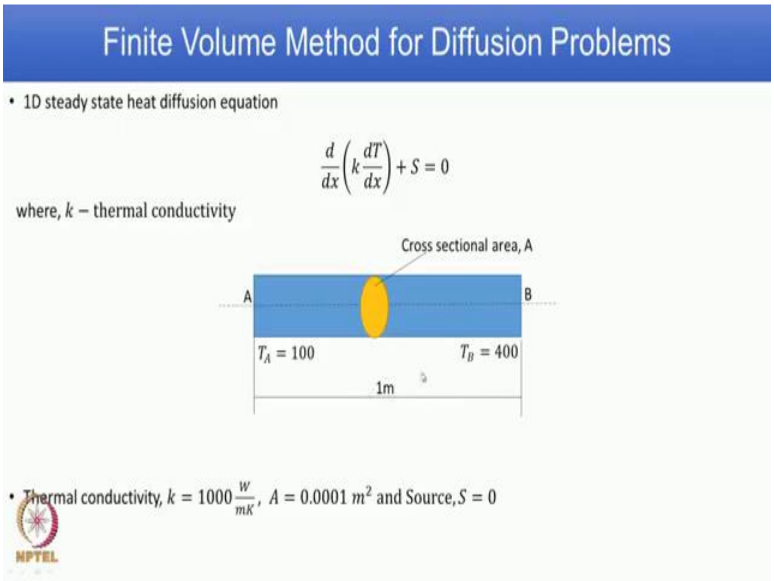


We took this example problem that is one dimensional heat diffusion equation and for this particular problem, we say source term is zero, it is very simple. And this is a illustration required for that problem and the cross section is circular, and area is given as 0.0001 metre square. The left boundary which is referred here as A, and temperature at that boundary is given as 100 degree Kelvin. Similarly, the right boundary which is referred here as B, and temperature at that boundary is 400 Kelvin; the distance between left boundary and right boundary is given as 1 metre.

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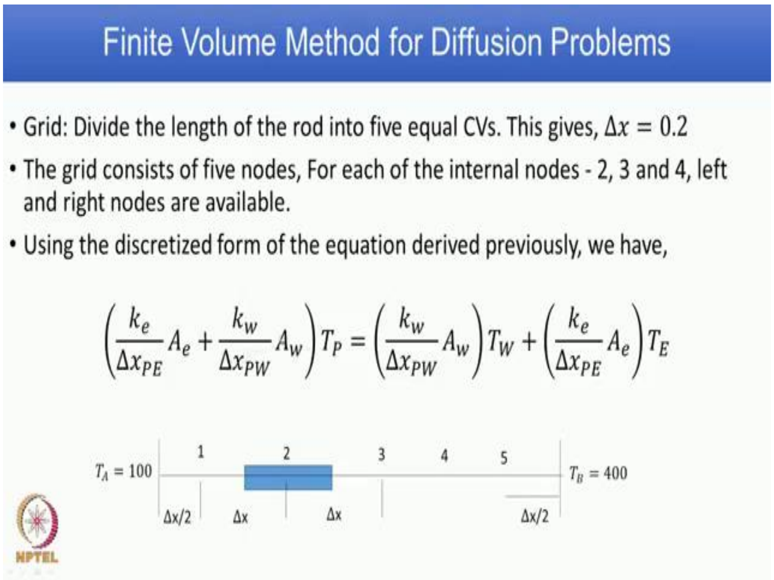

We said we will divide region into five equal control volumes. So, we have 1, 2, 3, 4, 5, and we understood all the distance between node 2 to node 1, node 2 to node 3, because it is a equal spacing. It is referred as delta x for all the nodes in between nodes and for the node one and node five which are near the boundary because it is equal spacing we have delta x by 2 as a distance between node and the boundary. So, in this case node between distance between node one and the left boundary is delta x by 2. So, we identify 2, 3, 4 nodes are internal nodes; 1 and 5 are boundary nodes. We have a discretized form of the governing equation as previously derived and this case is source term in that particular term is ordered and again in this case k that is a coefficient is constant and delta x is also same value we also identify terms in the coefficient for TP as same term as coefficient for TW and TE. So, this first term corresponds to coefficient for T at east node similarly the second term here corresponds to coefficient for temperature at west node. So, the TP coefficient of TP is summation of this two coefficient and that.

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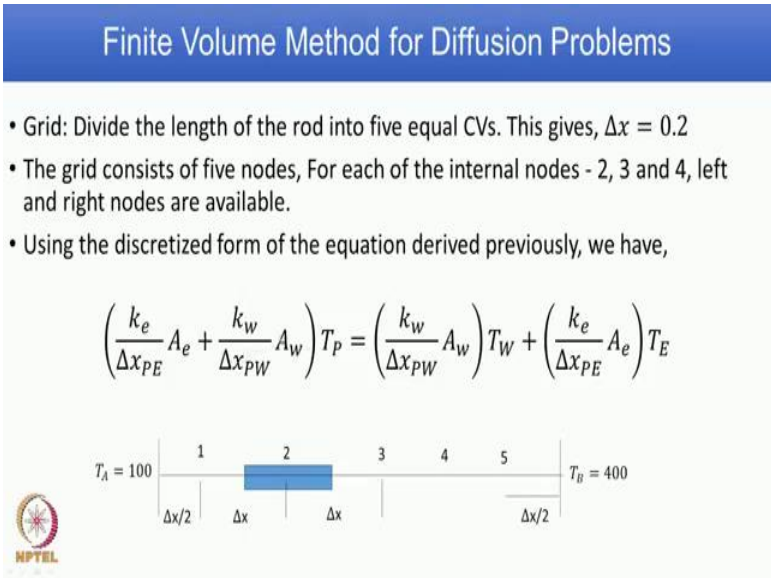


So, we write as a p T P equal to a E T E plus a W TW with coefficient written as shown here and we say p is a node of interest transform left to right in this case we initially consider only internal nodes. So, p will actually run from for node two three and four and we are able to evaluate corresponding values k by delta x and j for node one the left side is boundary referred here as a and temperature is given as a boundary condition for node one right side that is east is available hence for node one this is written separately k A T

E minus TP by delta x were as on the other side it is boundary condition TP minus TA which is a boundary condition the distance is delta x by 2.

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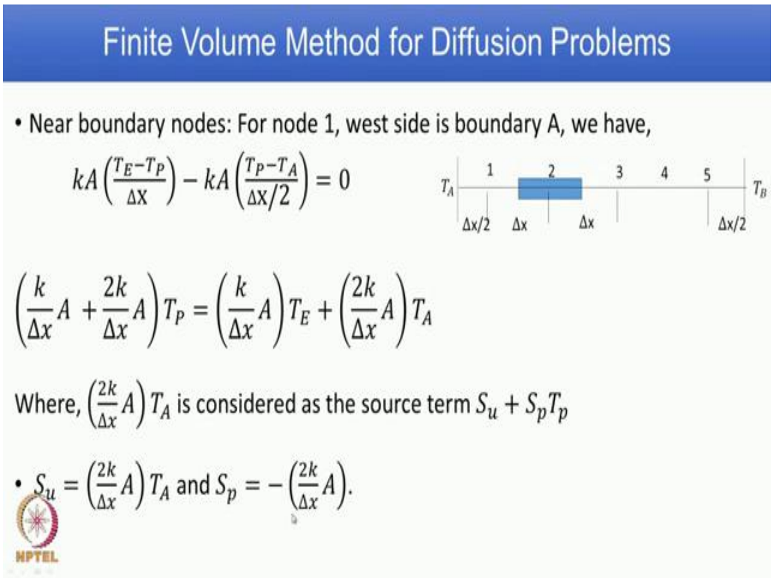


Now, we will also try to do similarly for other node; before that we will identify that T P is a node of is a node of interest is appearing in term here as well as in the term here. So, we collect this coefficient. So, k by delta x a two k by delta x a T P equal to terms on right hand side two k by delta x a and T A is considered as a source term S u plus S p T P S u is two k by delta x a into T A and S p is minus 2 k x delta x into A.

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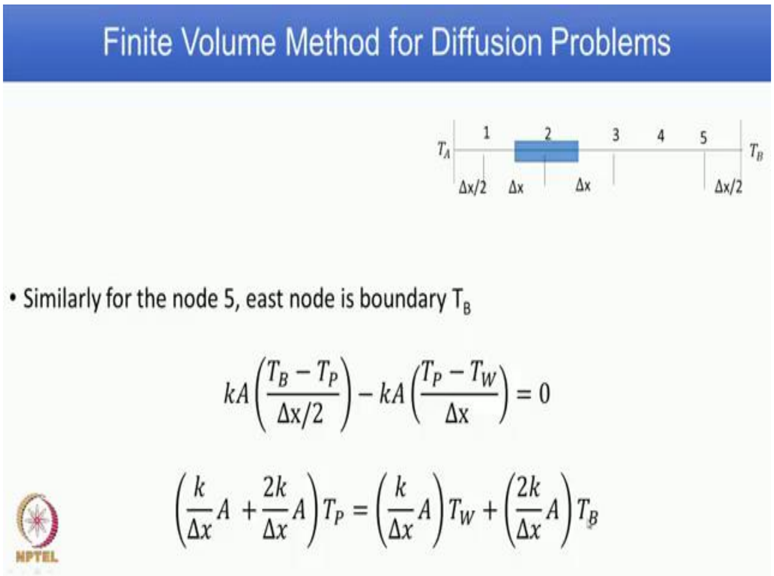

So, if you extend this procedure for the other node in this case 5 node 5 is a node next to the right boundary and k A T B is coming from right boundary minus T P by delta x by 2 again for node 5 west side that is four is available. So, we write full term T P minus T w by delta x equal to 0. Again the identify T P that is the point of interest appearing in both this term collect corresponding coefficient. So, you are able to rewrite this equation in this form k by delta x into a plus 2 k by delta x 2 is coming because of this delta x by 2 into T P equal to k by delta x a t w plus 2 k x by a 2 k by delta x into A multiplying temperature T B.

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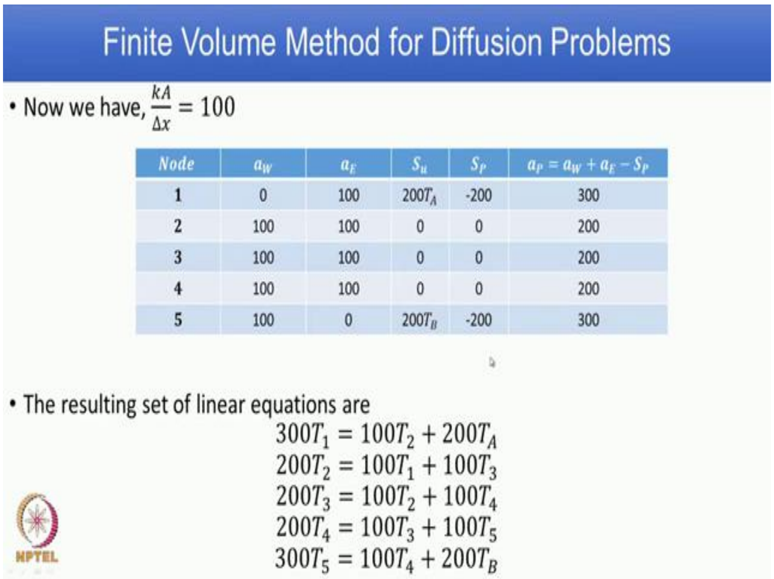

Finite Volume Method for Diffusion Problems

- Now we have, $\frac{kA}{\Delta x} = 100$

| Node | $a_W$ | $a_E$ | $S_u$ | $S_P$ | $a_P = a_W + a_E - S_P$ |
|---|---|---|---|---|---|
| 1 | 0 | 100 | $200T_A$ | -200 | 300 |
| 2 | 100 | 100 | 0 | 0 | 200 |
| 3 | 100 | 100 | 0 | 0 | 200 |
| 4 | 100 | 100 | 0 | 0 | 200 |
| 5 | 100 | 0 | $200T_B$ | -200 | 300 |

- The resulting set of linear equations are

$$300T_1 = 100T_2 + 200T_A$$
$$200T_2 = 100T_1 + 100T_3$$
$$200T_3 = 100T_2 + 100T_4$$
$$200T_4 = 100T_3 + 100T_5$$
$$300T_5 = 100T_4 + 200T_B$$

So, we have full discretized form for each node now we are done all this without actually substituting this number in particular example problem that we taken k is defined a is also known and delta x depending on the mesh that you are considered and k is considered we have five nodal points one metre is a distance. So, delta x becomes point two we can substitute those values and evaluate the conscience. So, k A is delta x is one such value happens to be 100, if you substitute this 100 which is k A by delta x in this discretized equation for all the nodes node 1 to 5 then corresponding coefficients are evaluated and they are tabulated here.

Let us try to understand this table the first column is for node 2, 3 4 are internal nodes full discretized equation is available, hence 2, 3, 4 can be evaluated, and you get this coefficient values evaluated as shown here. And we also learn in the problem statement there is no source term, hence the external source term are all zero here. Now, for the node one, the left boundary is there, but there is no west node hence the coefficient a w is actually zero, whereas east node is available and you are able to evaluate east coefficient for the node one. And similarly for node five there is no east node the east surface is the boundary condition location itself, hence a E is zero and a W you are able to evaluate the boundary conditions which are enforced appears as a source term and they appear for the corresponding node one as shown here and for node five as shown here while writing the discretized equation you also observed that point of interest coefficient that is a p is the summation of neighbouring nodes coefficient.

So, in this case a p is equal to a w plus a e and then source term of course, in this case source term is not there it is zero and we get here this a e plus a w is evaluated as shown here right. So, for example, 100 plus 100 equal to 200 you are also able to pros check right now this minus S p is 200 here and that is added to 100 in 300 here.

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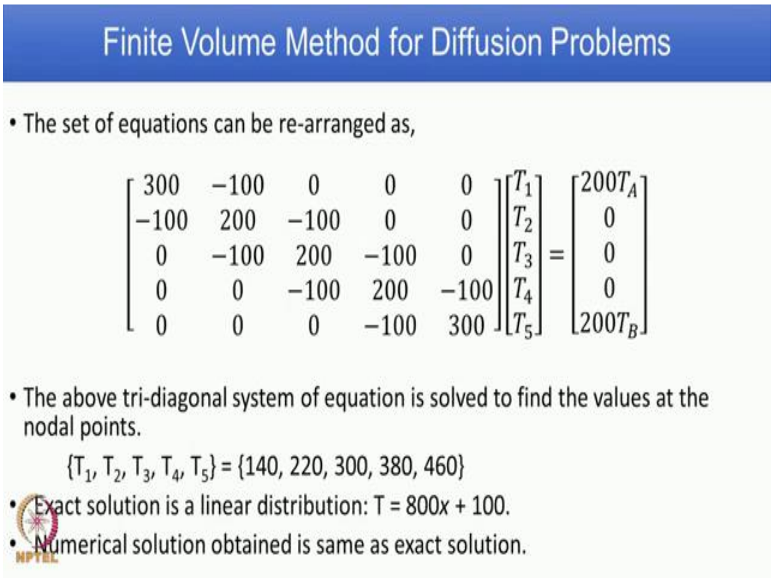


So, if you write this each if you write this set of linear equation in the form of matrix then you will get a matrix and you can get a structure like this. So, this a T 1, T 2, T 3, T 4, T 5 are the values to be determined and this is what is known as a column vector of unknown value to be determined. And to the left of it is a coefficient matrix, this corresponds to each row corresponds to one particular variable in this column vector. And the right hand side there is another column matrix and that is a known value. So, in this case for example, temperature at A is coming from boundary condition, temperature at B is coming from boundary condition, hence it is a known value.

If you look at this matrix structure, you observe along the diagonal, so the values are here for example, 300, 200, 200, 200, 300. This is main diagonal then you have one diagonal following the diagonal just above that is called super diagonal;, in this case it is minus 200 minus 100 minus 100 minus 100. Similarly, there is one diagonal immediately following the main diagonal just below minus 100 and so on. So, you get a structure what is known as a tridiagonal matrix, so one main diagonal, immediate super diagonal and immediate sub diagonal. We have already seen different matrix structure in last week class penta diagonal penta diagonal of two different forms tridiagonal and only diagonal matrix. There is a separate procedure available how to invert this matrix, we will reserve the discussion for next week class. So, in this case, if you happen to invert this matrix then you determine these values of T 1 to T 5.

Above tri diagonal system of equation is solved to find the values at nodal points and it so happen values are determine, and it is given here T 1 to T 5 in this form. And for this simple problem, analytical method of getting the solution is possible, and that is what is referred as exact solution, because it is a simple problem without any source term, it happens to be linear distribution as temperature is equal to 800 x plus 100, and this runs from left side to the right side. You can substitute value of x depending on your computational node and get exact solution at the computational node and compare with your computed CFD computed results, it so happen in this case they are almost the same. So, numerical solution obtained is same as exact solution. It is a very simple problem, hence it is not a surprise that the both match. Later we will see there are situation were exact solution is not matched.

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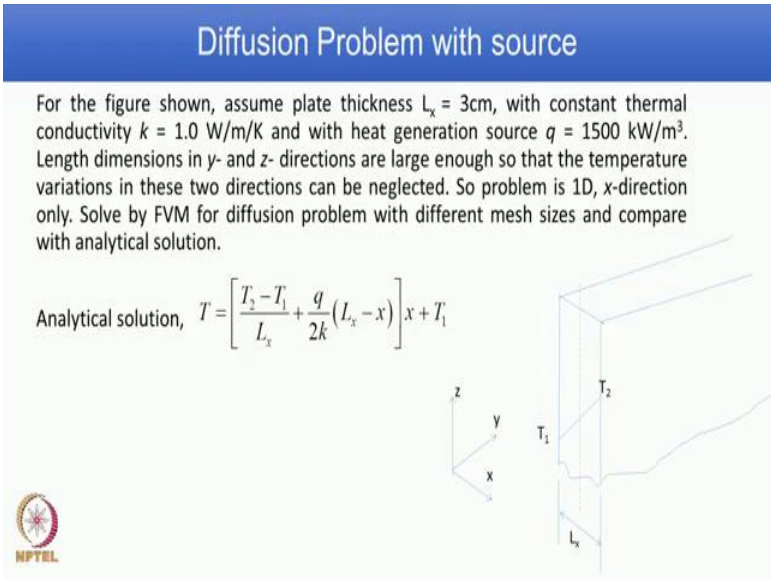


So, what we have done and the example problem is without source term. So, you can try this problem with a source term in this case the figure is shown L x is length, and because it is 1D, it is one it is in x direction. And there is a temperature distribution from one side to the other side; k value is given, length is given. In addition there is a heat generation source term value is given as 1500 k that is kilowatt per metre cube. Then though it will shown as a three d that is dimension in y direction as well as in z direction. There are so large that temperature variation along y direction and z direction can be neglected, hence the problem is 1D. Solve this problem by diffusion method that procedure explain for a diffusion equation with the different mesh size. The last problem we have explained only with five nodal points. So, in this case you can try starting with five and again try with eight and twenty what is happening. And there is again the analytical solution available for this problem and that is also given here and compare this scene.

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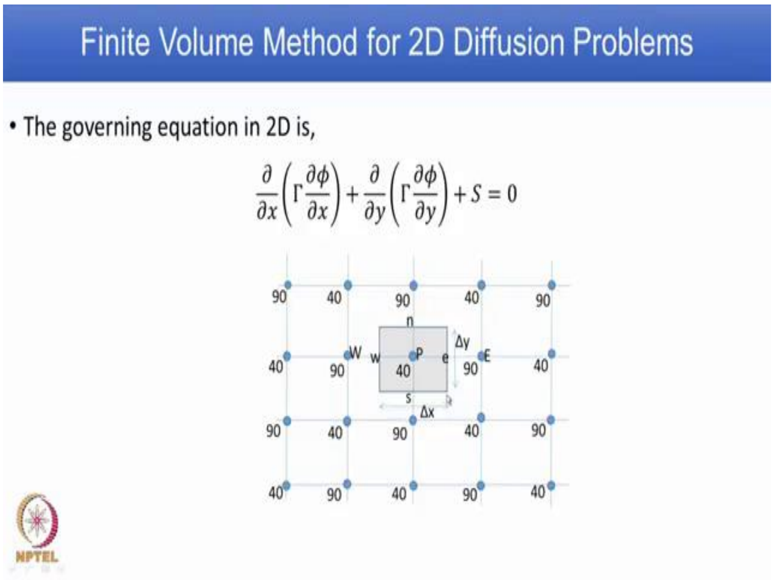


We have seen so far one dimensional diffusion equation and we got explanation with the help of an example in now extreme for two dimensional now in the case of two dimensional governing equation becomes dou by dou dou by dou x of gamma dou phi by dou x plus dou by dou y of gamma dou phi by dou y plus S equal to zero and the mesh are the finite volume is shown here in case of one d it was only east and west now we have north as well as south.

So, let us look at this figure in detail. So, p is the point of interest or node of interest and you concept the contra volume around the pod of interest now in two dimensional you have a delta y as well as delta x define in the size as control volume to the interest p on the right side, we have the east node; and the left side we have a west node in addition you have one node on the north and one node on the south as we did in one d you also have to define to control forces. So, this face you already learned is marked as w with a letter small letter case and east side is marked with a letter case e with small case in two d you have two more face one on the north and there is shown in small case letter n similarly on the south you have one more face which is north the letter s small case letter s.

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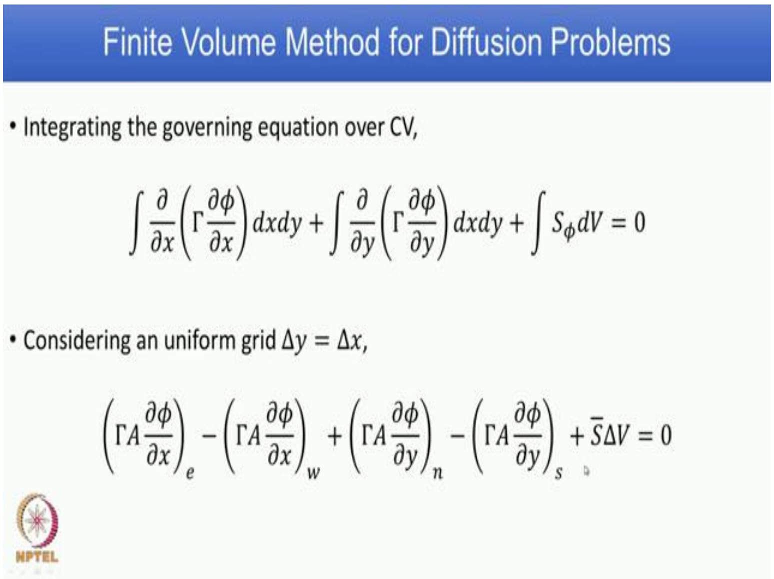


Now, that we have described control volume and terminologies for the two dimensional situation let us now go to the procedure of getting the discretized equation for two dimensional equation when you integrate the governing equation for the control volume and that is what is shown here. Let us now do the first term that is derivative in the x direction for the sake of simplicity you consider uniform grid delta y equal to delta x. So, the first term is gamma a dou phi by dou x evaluated at the east face minus gamma a dou phi by dou x evaluated at the west face plus gamma a dou phi by dou y evaluated at the north face minus gamma a dou phi by dou y evaluated at the south face. Now when you compare what we did for the one dimensional case we have now two additional term for the second direction plus source term S bar delta v equal to 0.

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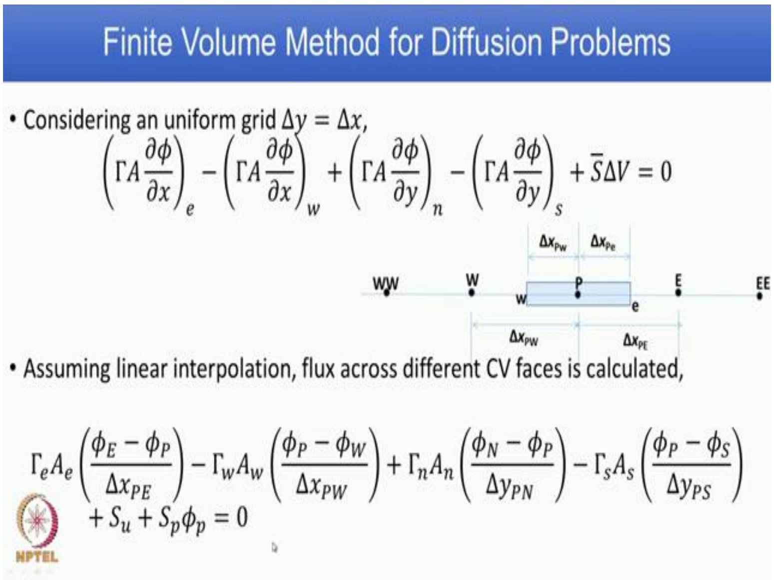


We rewrite that equation again here. Now you need to evaluate dou phi by dou x at east face west face; similarly dou phi by dou y at north face and south face. For that sake we reproduce finite volume terminology as shown here. Now we have already done evaluation of fluxes at east face and west face in one dimensional and that is what is shown here. So, gamma e a e phi at east node minus phi at p node divide by delta x p e minus gamma w a w phi at p minus phi at w by delta x distance between node of interest p and west node. We have now two more terms corresponding to y direction they are shown here that is gamma n a n phi at north node minus phi at p node divide by delta y that is a distance between p node and north node. Similarly, the other term that is gamma S u S phi at P node minus phi at S divide by delta y p S, and source term is evaluated as we did in one-dimensional case that is S u plus S p phi p equal to 0.

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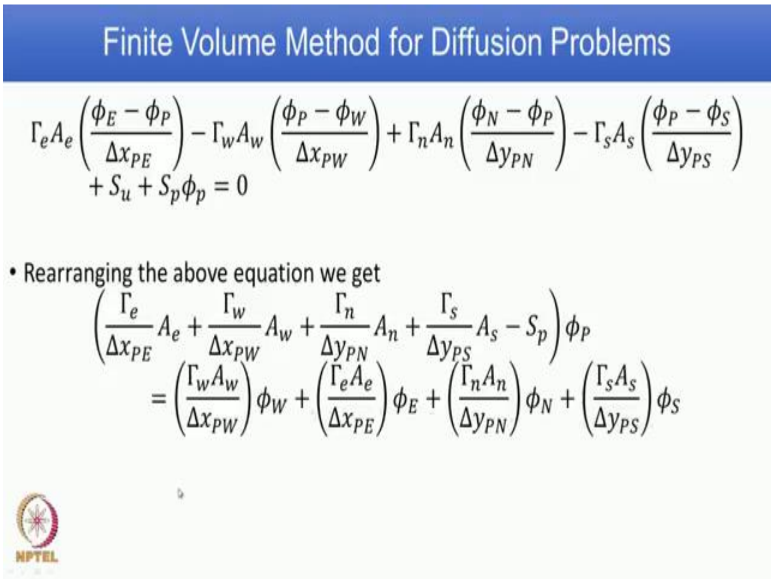


The same discretized equation is shown once again here. Now we need to rearrange as we did in one dimensional case the point of interest term that is phi at p appears in all the terms. So, we rearrange we collect coefficients multiply phi at p. So, when we rearrange the above equation we get final equation as shown here. So, this equation on the left hand side has coefficients multiplying phi at p equal to all other terms on the right side. If you look at this equation left hand side is unknown right hand side is contribution from neighbouring nodes; in the case of two-dimensional case neighbouring nodes are west node, east node, north node and south node.

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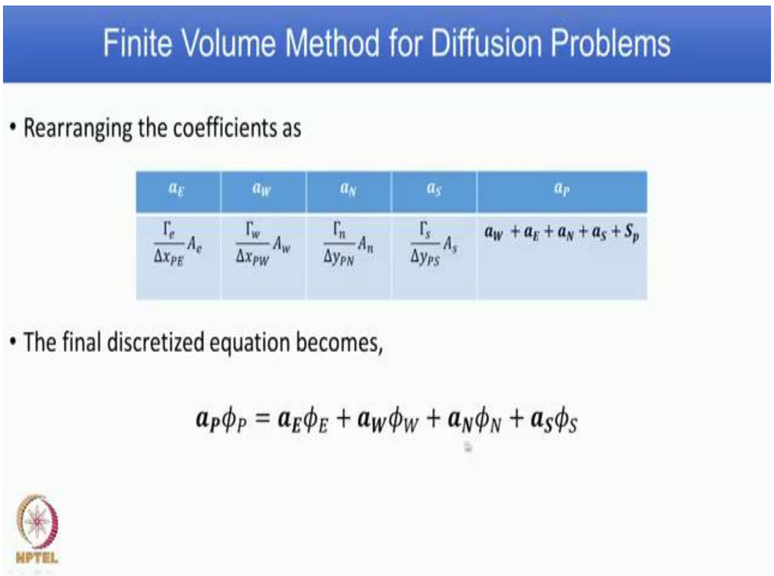

Let us rearrange the coefficients as we did in one dimensional case in the form of the standard expression that is a e a w now we have a n and a S also included. So, you can immediately recognize a E is gamma E by delta x p E a E then a w is gamma w delta x p W a W gamma N by delta y p N a N is for a n gamma S by delta y p S a S is for a S. The node of interest coefficient a p has contribution from neighbouring nodes that is why it is written here as a W plus a E plus a N plus a S plus source term. So, the final discretized equation becomes a P phi P equal to a E phi E plus a W phi W plus a N phi N plus a S phi S.

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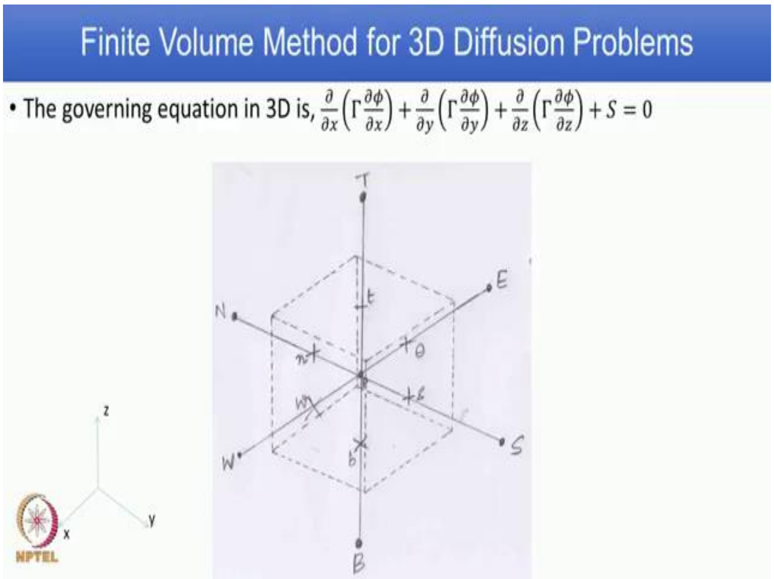


Now in 3D, it will have one more term added in the governing equation correspondent to third direction that is z direction. So, the term new term is dou by dou z of gamma dou phi by dou z and source term equal to zero. So, in 1D, it just element; in 2D becomes a area; in 3D,it becomes a volume and that is a sketch, and this is the control volume have to shown. For a simple case a cube and always mesh lines corresponds to that cube representation and the coordinative is also shown here x y z.

So, in x direction, you have along the x direction east node capital e east node and west node and this is the point of interest p now along the y direction we have the north and south node similarly along z direction we have a top t and bottom d is surface and node capital t and bottom node capital b. So, for all the surface the centre of the surface is actually marked here and that actually correspond to particular surface for example, small case letter t is centre point for this particular top face k similarly the b is marked small case letter d marked as a centre point for this particular bottom face and so on. Other surface once again this is for simple situation, a cube is taken, hence it is very simple all sides are equal and centre is able to fix imagine how will you do for un structure term mesh.

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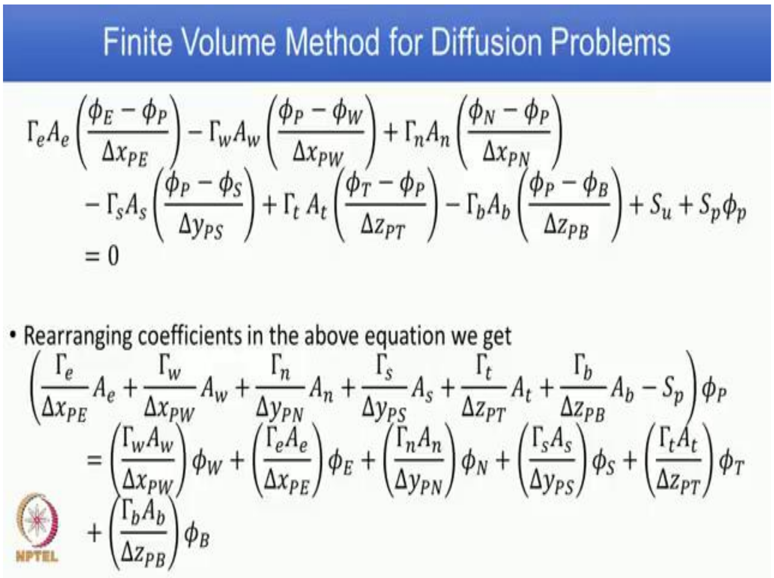


So, procedure is the same as we did for 1D and 2D as we did in the case of 1D as well as 2D we identify control surface and all the derivatives require at control surface are evaluated based on functional values at the respective nodes. So, you observe here two new terms one corresponds to top, another one corresponds to bottom. And we observe here the derivative phi t minus phi p by delta z p t the distance between nodes p and nodes t and this will give dou phi by dou z; similarly for bottom and source term is there. Again you identify that node p which is the point of interest appearing in all the terms in all the terms. So, we collect all terms there phi p z put them together, and you get the coefficient for phi p this is the node of interest or in other words it is unknown.

So, we retain on the left side, all other terms are brought to the right side. So, we get phi w phi E as we had seen in 1D, additional term north and south as we seen in 2D. Now two additional term phi T and phi P for three d. Similar observation terms inside coefficient for node phi P are nothing but term for neighbour node. So, for example, first term gamma e by delta x A e is coming from gamma e by delta x A e for node at E and you can observe the similar for other neighbouring nodes. So, this becomes the summation of individual neighbouring node contribution plus the source term.

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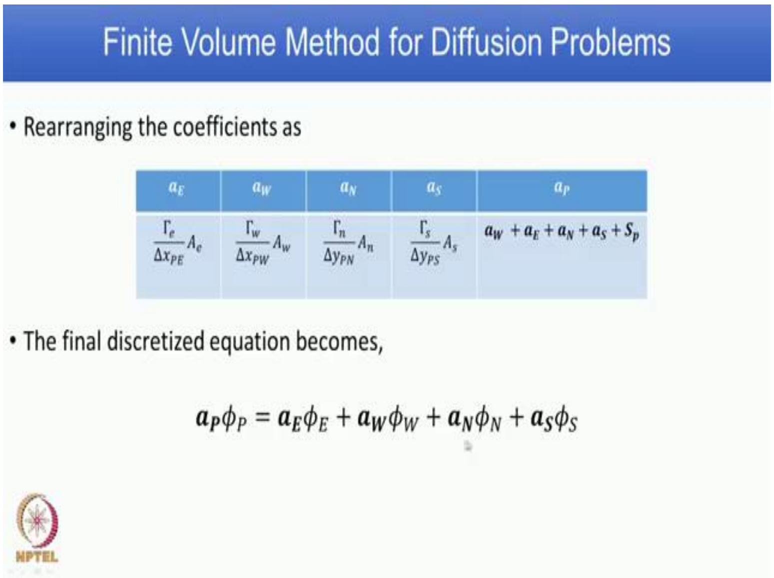


So, similar pattern, we will follow a E a W a N a S, now a T and a P, these are coefficients rewritten and a P is coefficient for P node. And as observed before it is summation of all neighbouring nodes coefficient. So, final discretized equation becomes a P phi P equal to a E phi E plus a W phi W plus a N phi N plus a S phi S plus a T phi T plus a B phi B, source term is not there source term is already included in phi B. Now you can imagine how it will be by writing code or solving the 3D problem you have so many coefficients to be evaluated, and we will see how to handle with is in future class. With this, I will come to the end of this class.

Thank you.