Foundation of Computational Fluid Dynamics
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Lecture – 14

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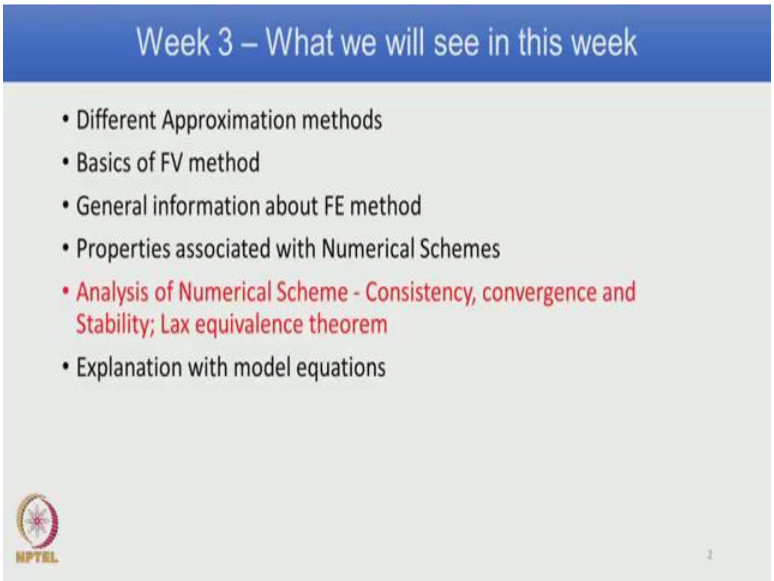

My greetings to you all, we are now onto the module four this week. In this week, we said we will look into different approximation method, which we have already done. Then some information about finite volume, finite difference and finite element method that also we have done. Then properties associated with numerical schemes in terms of conservativeness, boundedness and transportiveness; then we took important how to analyze a numerical scheme. We listed consistency, convergence and stability. So last class, we explained with the help of a model equation, get a clear explanation about consistency and convergence. And today's class, we will particularly talk about stability and we try to get a detailed explanation how to use or how to check stability and extent for a particular problem. And there is a theorem, Lax equivalent theorem, which connect all these three together. Then we will follow it with in the next class explanation with different model equations.

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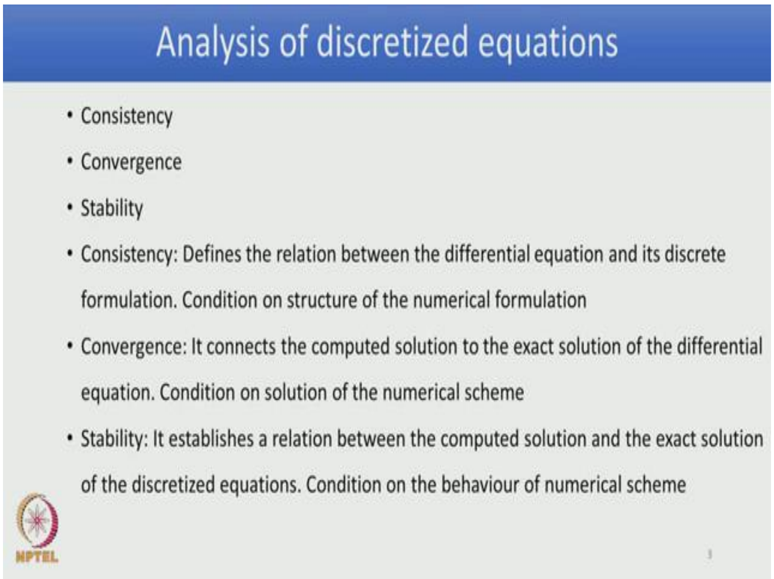

We said consistency, which defines relationship between differential equation and discrete formulation; it otherwise talks about condition on structure of the numerical formulation. Convergence it connects the computed solution to the exact solution of the differential equation; in other words, it puts the condition on solution of the numerical scheme. Third aspect is the stability, which establishes the relation between computed solution and exact solution of the discretized equations.

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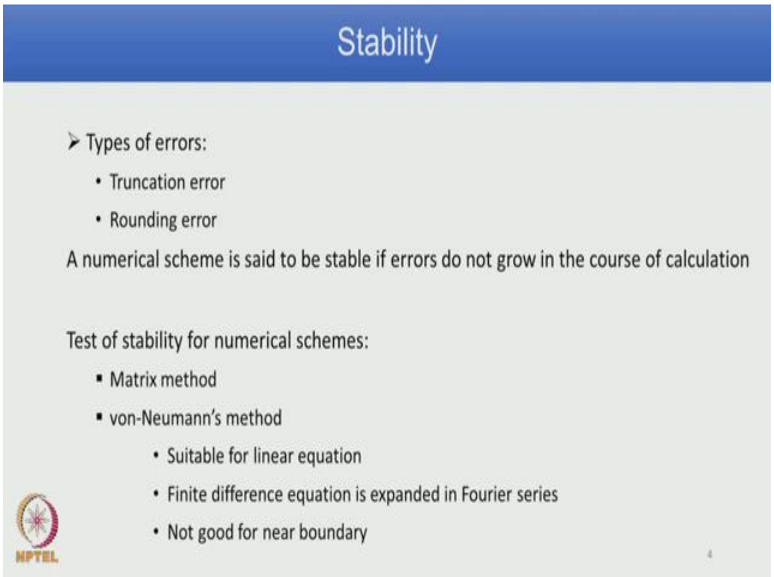

We take the third aspect – stability. And we mentioned last class, there are types of errors, truncation error and rounding off error. A numerical scheme is said to be stable if the errors do not grow in the course of simulation or calculation that is there are within some limits specified, if you get a solution then such a scheme said to be stable. how to check stability for a particular numerical scheme, there are two popular methods, one is the matrix method, second one is the von-Neumann's method. In matrix method, one find out the Eigen value and based on the Eigen value, one can explain where the scheme is stable or not. There is another method von-Neumann's method, where Fourier series explanation is used.

And today's class, we particularly talk about von-Neumann's method and how to apply for a discretized equation. As I mentioned last class, this von-Neumann method is particularly suitable for linear equation, but most of the equation used in for practical simulations are non-linear in nature; locally you can linearized and apply von-Neumann method, so there is a limitation to apply von-Neumann method for a practical non-linear problem. However, it gives idea whether the scheme is stable or not. And here as I mentioned, finite difference equation is expanded in Fourier series form. And just like linear equation is one limitation, this also not applicable near boundary.

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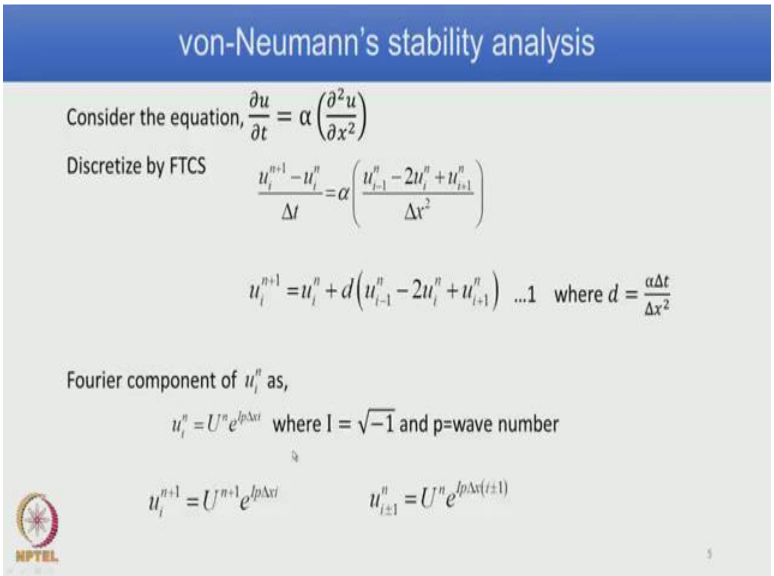

So to get explanation on von-Neumann's stability analysis, we take a model equation, so it is again one-dimensional equation, dou u by dou t equal to alpha dou square u by dou x square. We have already seen this equation, when we talked about consistency. Again we mentioned about what is FTCS, which is forward in time central in space. So, if you write down for this forward in time, u of i n plus 1 minus u of i n by delta t that is forward in time equal to alpha u of i minus 1 n u of i and so on, so this term is actually central in space, because it takes point of interest i and one node on either side i plus 1 and i minus 1, and this is evaluated at n time level. It is central in space and this is forward in time.

Just rearrangement, we take this delta t to the other side u of i n also known value, because it is evaluated at nth level also to the other side; only quantity to be determined is u of i n plus 1 so that is rewritten and you get this term u of i n plus 1 equal to u of i n plus d into this. So, what is d, d is alpha times delta t by del x square, so this del x square that is there in the denominator for the central space; and delta t, which is coming from the left hand side, which is because of the forward in time, and all are combined together we get coefficient d and it is written as alpha delta t by del x square. We try to express all these in term of Fourier component, so for example, u of i n as capital U n exponential I p del x i. So in this, where I is the imaginary number square root of minus one, and p is the wave number and this decides the component. And you get expression for each of these terms, u of i n plus one is written here; u of i plus or minus 1 n is also given here. We substitute this into this expression and let us see how to do.

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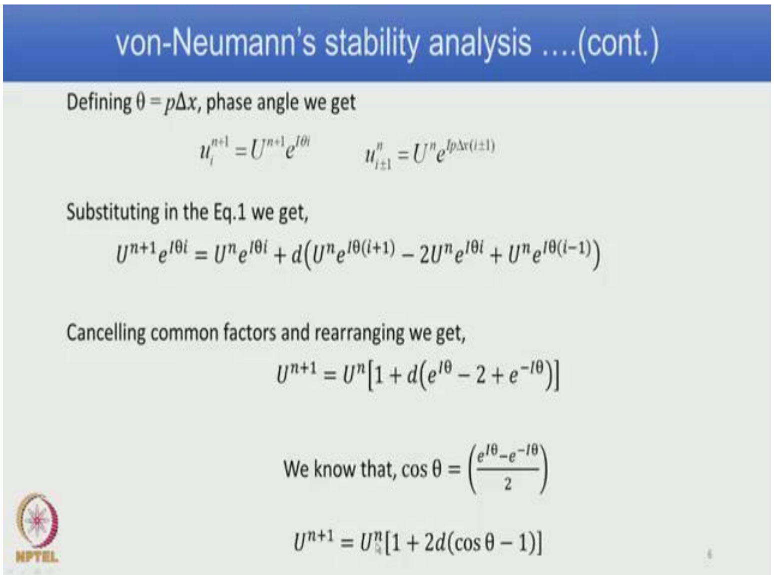

So, you also define theta as p into delta x, which is referred as phase angle. So, you can rewrite that expression as u of i n plus one equal to U of n plus 1 e power I theta i, similarly for other term. We substitute these into the original finite difference equation, which is written for one-dimensional diffusion equation that is what is shown here. we observed that e power I theta i appear both on left hand side as well as on terms on the right hand side. So, you can cancel some of the common terms and then rearrange, it is purely arithmetic. Once we do that then we get final expression as U of n plus one equal to U of n times 1 plus d e power I theta minus 2 plus e power minus I theta. we know e power I theta expressed in terms of cos theta or cos theta can be expressed in terms of exponential form as e power I theta minus e power minus I theta by 2. So, if you substitute explanation for cos theta and e in these two and rearranging then you get U of n plus 1 equal to U n times 1 plus 2 d cos theta minus 1. What you observe is U at n plus 1 is related to U of n times some quantity, and this is actually the error quantity and which you know will be used for doing the von-Neumann's stability analysis.

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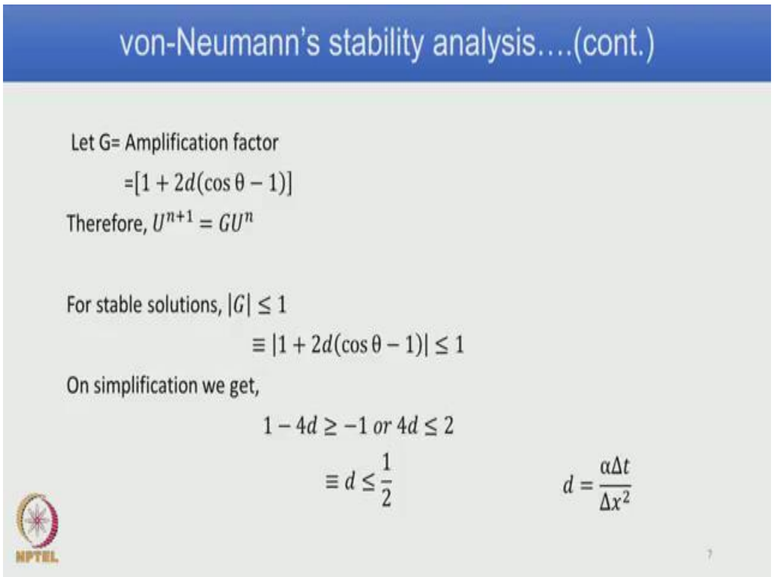

So, if you write the bracket terms, which is one plus two d cos theta minus one, and this is what is known as a amplification factor. And the usual symbol is G, so U of n plus 1 is related to G time U of n, so the if the error needs to be contained then for stable solution modulus of G needs to be less than or equal to 1; in other words, 1 plus 2 d cos theta minus 1 and modulus of that terms needs to be less than or equal to 1. So, you can understand how to obtain on this limit value for theta or value for d. We know maximum value of cos theta and we know d definition, so if we follow that you get a condition d should be less than or equal to half and we just put the definition of d here again, where d is related to alpha times delta t by del x square.

This alpha is a coefficient, diffusion coefficient, which is coming from the equation itself, so we do not have a much choice on alpha. Whereas delta t and del x are coming from discretization and that we have a control. So, you can adjust delta t and del x in such a way the d is less than or equal to one by two or half, in order for this scheme that you have chosen that is forward in time and central in space for that model equation, and solution to be stable. One has to satisfy the condition d is less than or equal to half.

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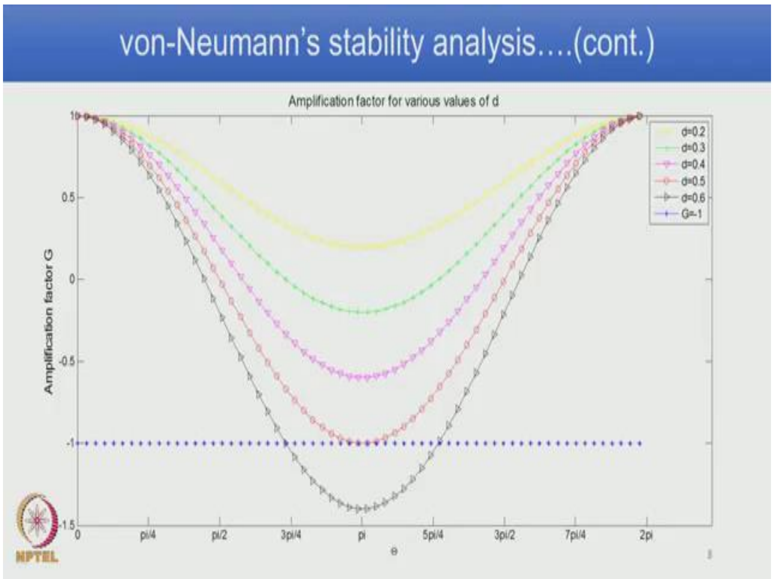

We will try to get explanation of the G graphically, so we know the expression and we work out different values of G for different values of d. And G is equal to one is in the blue color; and here we have not marked modulus of G, so you get minus sign here. And as you can see this black line is for the value d equal to 0.6; and we put the condition d less than or equal to half, so this is above that condition. And you can see here, for this range of theta, it exceeds that condition that G should be within one. Whereas if you take d is equal to 0.5 that is the limit that we set, d is equal to less than or equal to half that is the limit is 0.5. We can observe here, that is marked by this red color line, and it just touches G equal to one limiting line and then other points are well within that limiting line. Similarly, for other values of d, they are well within the limiting line, hence the scheme that we are chosen is stable for the particular that we have considered.

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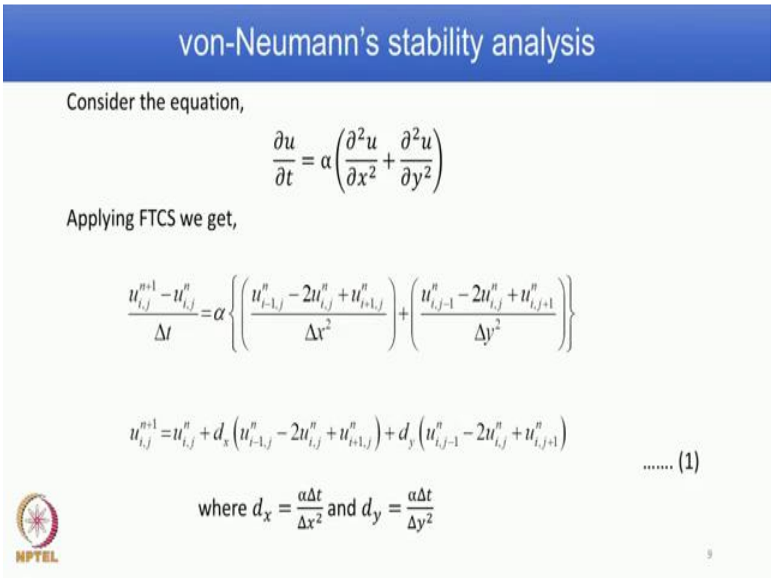

Let us explain this von-Neumann's stability analysis. For another equation – two-dimensional equation; we write down the equation again, but including two dimension that is the second dimension. So it is dou u by dou t equal to alpha dou square u by dou x square plus dou square u by dou y square. We apply again forward in time central in space scheme to this two-dimensional equation, and we get final expression as shown here. So, forward in time is on the left hand side, so n plus 1 and n appears; and on the right hand side, we have central in space, now applied for x-direction as well as in y-direction. So, the first term in the bracket this term is for second derivative in the x-direction, you can observe u evaluated at i minus one comma j, u at i comma j, u i plus one comma j, but all at nth level. Then second term in the parenthesis, this is for second derivative in the y-direction, but again central in space, so you can observe, u is taken from nodal location i comma j minus 1, u at i comma j, u at i comma j plus 1, again it is evaluated nth level.

We rewrite this equation, we are interested only in the term u i comma j n plus one, all other terms are known from nth time level. Hence we rewrite this equation by keeping the unknown on the left hand side, and taking all other terms to the other side. So for example, this delta t is taken from denominator on left hand side to the other side; then u i comma j n is also taken, it becomes positive. So, each term, for example, the first term, first set for second derivative in the x-direction, alpha delta t by delta x square is renamed as d x; similarly, alpha delta t by delta y square is named as d y. So, you get finally, expression as shown here. let us apply when von-Neumann's stability analysis to this final discretized equation, following the same procedure as we did for one-dimensional equation.

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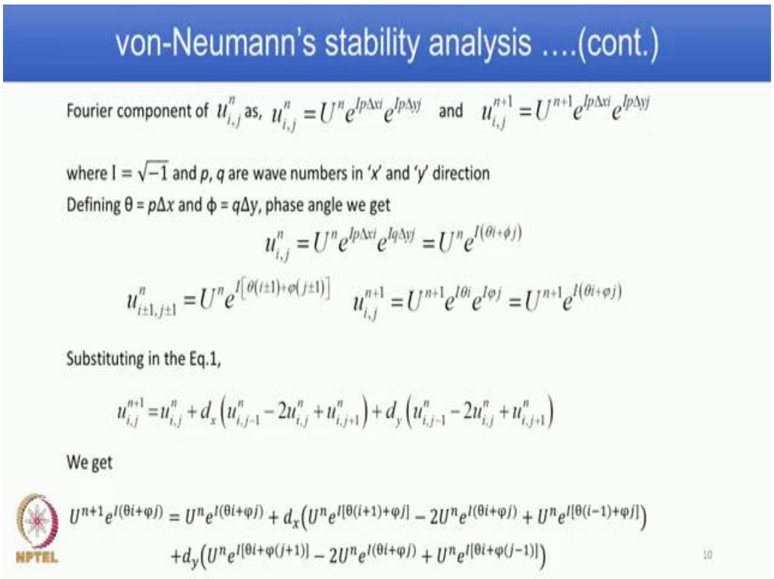

So consider the Fourier component of u i comma j n as u i comma j n equal to U at n, this capital U is for wave and e I p delta x i and e to the power of I p delta y into j. we have additional term, when you compare the 1Dsituation, so one term is for x-direction, the second term is for y-direction. We extend this for all other terms in that equation. For example, u i comma j n plus one is written as shown here. Similarly, for u i plus or minus 1, j plus or minus 1 at nth level is also written as shown here. As we did in one-dimensional case, we define I is actually square root of minus 1; p and q are wave numbers respectively in x and y direction. We define theta equal to p into delta x, phi equal to q into delta y. So, with this definitions, and expression for individual term, now we substitute in the final discretized equation and we get expression as shown here.

You can identify or recognize the left hand side, for example, U n plus 1 e to the power of I theta i plus phi j is for u i comma j n plus 1. Similarly, all other terms are independently written in this final expression as shown here. We will apply von-

Neumann's stability analysis to this equation; in other words, we rearrange and then find out amplification factor, limit for amplification factor and get idea on the stability condition.

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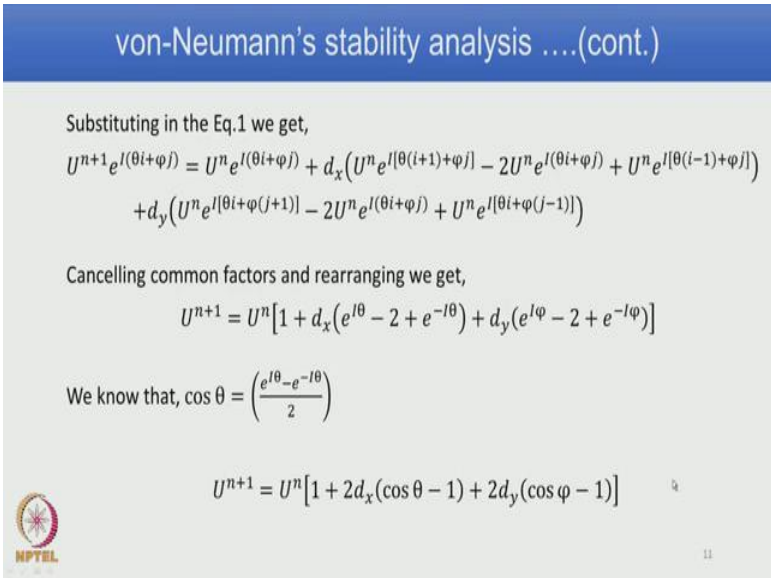

We rewrite again that expression, cancelling all the common terms from left hand side as well as right hand side, finally, you get expression as shown here as capital U n plus 1 which is related at n plus 1 level it is related to same variable at nth level in this way. Once again we can express exponential in terms of cos theta, so cos theta is used and for phi also again cos theta cos phi is used, so after this substitution and rearrangement, you get expression as shown here, which is in terms of cos theta and cos phi. just like we did in the one-dimensional situation, here also we have a condition for the terms which are written within this bracket.

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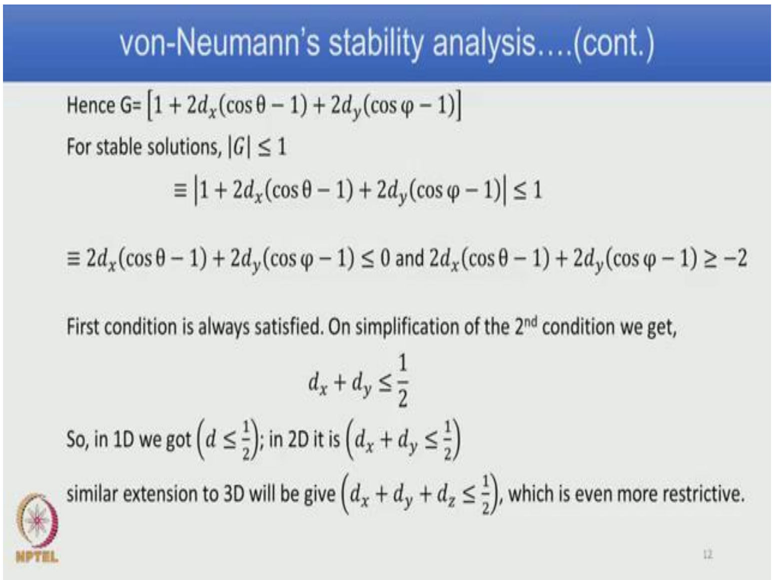

That is what is known as amplification factor, which is given as G equal to 1 plus 2 times d x cos theta minus 1 plus 2 times dy cos phi minus 1. So, for stable solution, condition is modulus of G should be less than or equal to 1. We can manipulate arithmetically and then first condition is always satisfied. The second condition is satisfied based on this condition that dx plus dy should be less than or equal to 1 by 2. So if you recall in one-dimensional, we got d to be less than or equal to 1 by 2; now in two-dimensional, dx plus dy to be 1 by 2. So, you can imagine, if you extend this procedure for three-dimensional, then it will become dx plus dy plus dz less than or equal to 1 by 2, so whenever you have such a summed up condition that is d x plus dy, which is summed up limiting condition then there is a restriction.

So you recall d is defined as alpha delta t by del x square, so similarly when it comes to two-dimensional d x is alpha delta t by del x square, d y is alpha delta t by del y square, and you have a summed condition d x plus d y less than or equal to one by two. So, whenever you have such summed condition, it is becoming the restrictive, now it becomes more restrictive in the case of three-dimensional where you have a summed up condition for all the three directions together, so d x plus d y plus d z is less than or equal to half. And you known delta t is one parameter you can control and del x and del y, del z are spatial distribution in respective direction that is another three parameter that you have to control, and together you have to satisfy this condition. For that particular scheme, that you have chosen for that particular equation that you are explaining and error to be or calculation to be stable.

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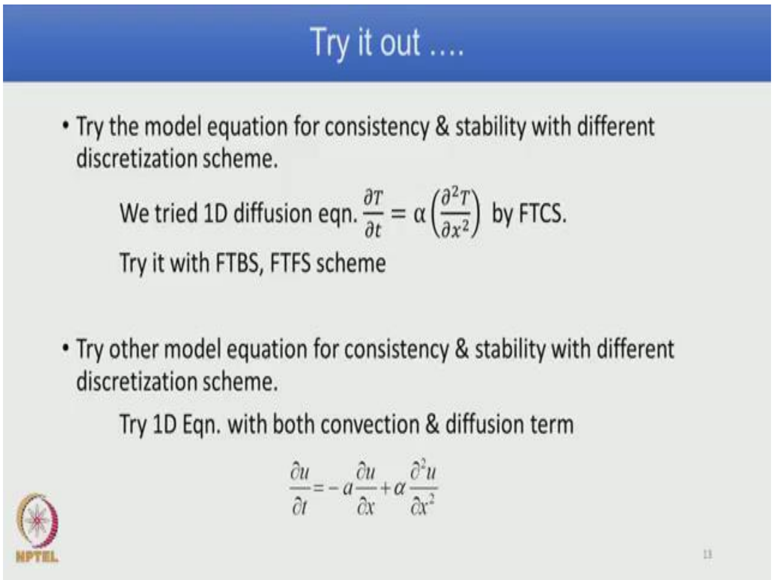

You can try it out, try taking the same model equation, we explained with forward in time and central in space consistency as well as stability. You can try the same model equation either in one d form or 2D form, and try it different other scheme that is backward in time sorry backward in time is difficult, you can forward in time forward in space or forward in time backward in space and so on. And try other model equation again for consistency and stability with different discretization scheme.

For example, try 1D equation, we have consider only the diffusion, you can actually include convection also as given here dou u by dou t equal to minus a dou u by dou x plus alpha times dou square u by dou x square. This particular equation specifically is called Berger's equation, this is Navier-Stokes equation without pressure term such a equation called Berger's equation. And there is a convection term, but this is linearized, so in the usual in the original convection term instead of a, there will be u the function variable itself u times dou u by dou x will be there. For the sake of understanding consistency and stability it is linearized, so you get a, a is some other coefficient, which is not related to the function variable u itself. So, such a equation is called Berger's equation. You can take other model equation and try with one of these schemes, understand consistency and stability.

So, in today's class, we have particularly talked about stability, explained stability. See you again with another important topic next class.