Foundation of Computational Fluid Dynamics
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Lecture - 12

Greetings to you all, we are going to next module for this week, course on CFD. So, last class, we just started taking something about finite volume and finite difference method. As I mentioned in last class, there is another popular method available in CFD, which is finite element method. Basically the finite element method was developed for structural analysis then people started using it for fluid flow and heat transfer calculation as well. So, the one group of researchers, they use only finite element method for solving different problems. Just like commercial CFD software available based on finite volume method, you also have a commercial CFD software available based on finite element method. And there are also attempts to combine advantage of finite volume, finite difference and finite element methods; there are works based on this hybrid approach as well. In this course, as I mentioned before we will focus mainly on finite volume method and application of finite difference method. We will not talk in detail about finite element method; I intend to give only a small introduction, and there are separate courses available on finite element method and so you are encouraged to look those courses as well as there are many reference books available.

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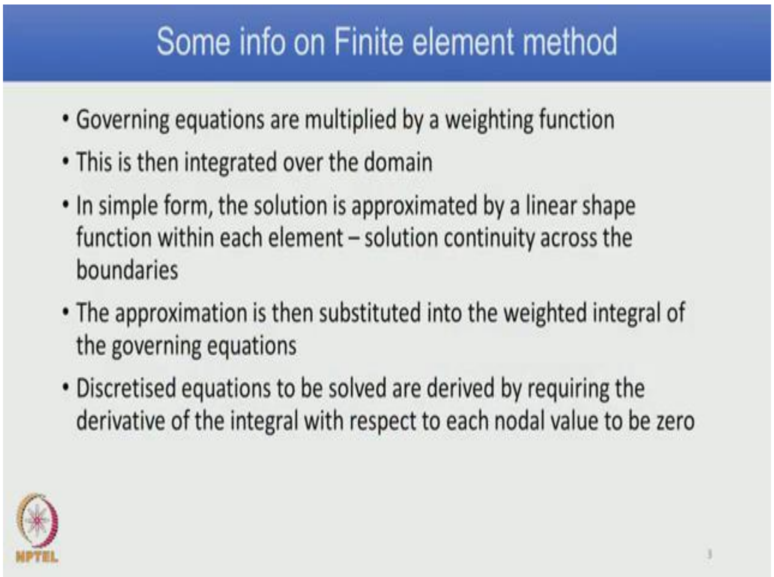

So, in finite element method, governing equations are multiplied by a weighting function. Then this is integrated over the entire domain. In simple form, so what is the approximation, it is a linear shape within each element, so we divide in finite element method entire domain into number of small elements, just like in finite volume, the solution continuity across the boundary of each element is also ensured. This approximation is then substituted into a form what is known as a weighted integral form of the governing equation, and this is where two terminologies are used what is known as strong form and weak form. The strong form of the governing equation is the original equation with the corresponding boundary condition; the weak form is the approximation to the governing equation that is why it is called weighted integral form of the governing equation and in other words, it is called weak form of the equation. Discretized equations to be solved are derived, so when we say weighted integral, we have made some approximation, it needs to satisfied some properties; the property that needs to satisfied is derivative of the integral with respect to each nodal value to be zero, this is one of the requirement.

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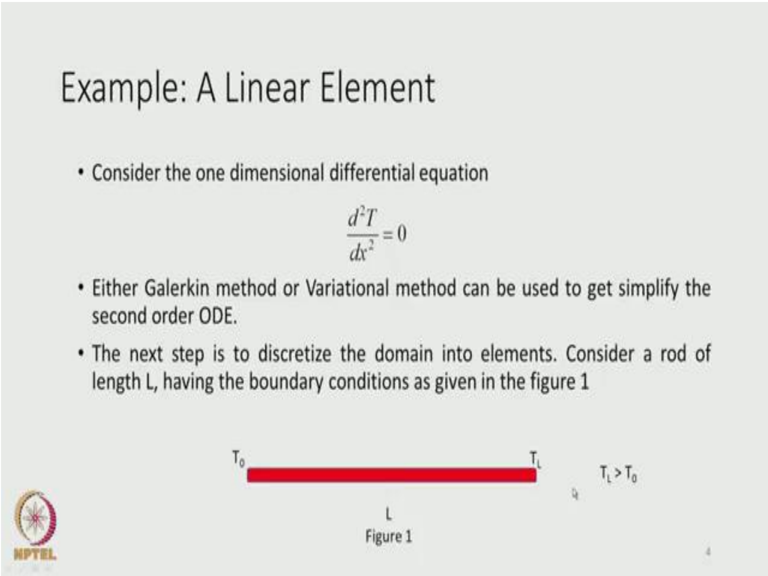

So, we will try to explain with some example problem. Consider one-dimensional differential equation, so second derivative because it is one-dimensional. We have written an ODE as such d square T by d x square. And there are again procedure Galerkin procedure and Ritz method. We will not go into the details of this, so one of these methods is used to simplify the second order ODE. Next, we discretize the domain into number of elements. So, consider a rod with the length zero on left hand side, and the length running to the other side, the length corresponds to L, and temperature on one side is T zero, and temperature on the other side is T L. We make a small restriction the T L is greater than T zero that is right hand side temperature is greater than T zero.

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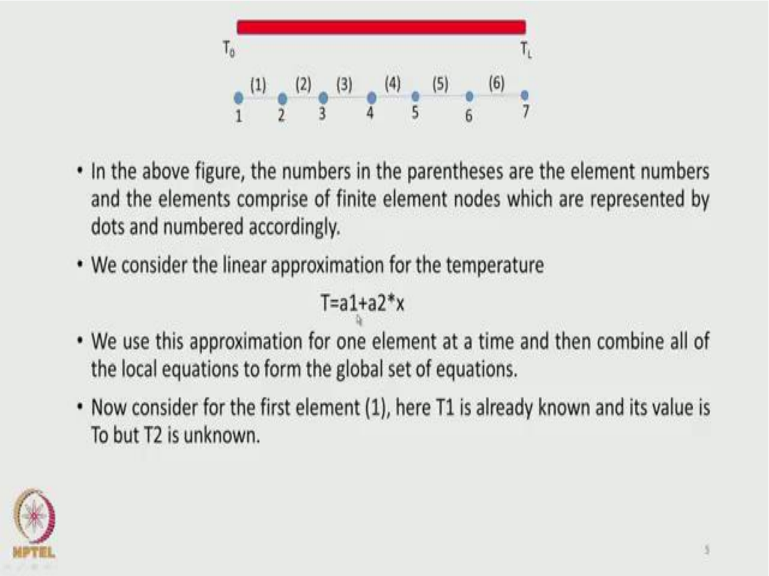

Now, this figure, previous figure is drawn again here, in addition to representing it with help of finite element nodes. So, in this figure, number 1, 2, 3, 4, whatever is given in the parenthesis, whatever is given in the bracket are element numbers and each elements comprise of nodes, for example, element one has node one and two; similarly element four has node 4 and 4. So, we need to establish a connectivity between node and the element, which are represent by dots and numbered accordingly. Element three for example, is connected by node 3 and node 3. So, in this, we consider linear approximation for the temperature T equal to a 1 plus a 2 into x, so we need to evaluate the coefficients a 1 and a 2.

We use this approximation for one element at a time and then combine all of the local equation to form what is known as the global set of equations. So, consider the first element, for example, element one T 1 which is already known, T 1 is left side temperature, which is given from the boundary condition as T 0, but T 2 is not known, that is temperature at nodal location two is not known. So, from this situation or from this condition, we can substitute in this linear approximation and evaluate coefficient a 1 and a 2 we have to do.

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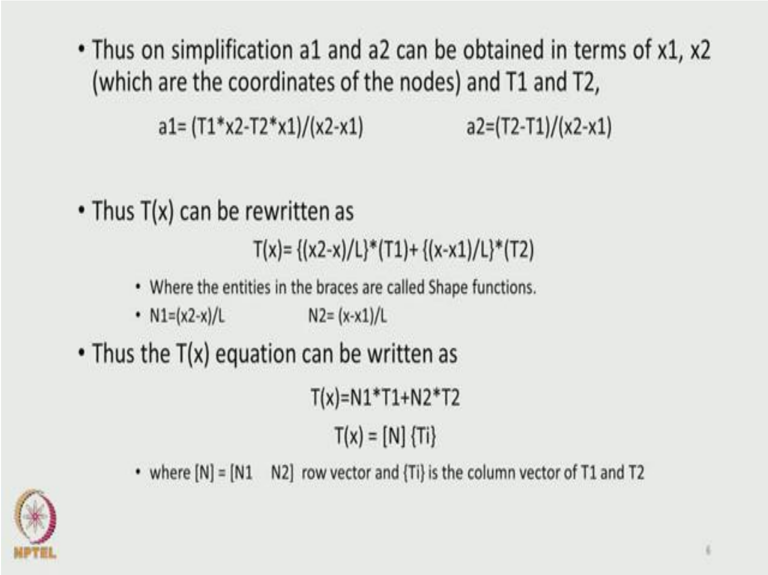

We will get a one as T 1 into x 2 minus T 2 into x 1 by x 2 minus x 1, and a two as T 2 minus T 1 by x 2 minus x 1, this is straight forward procedure. You substitute the linear equation, you get two equations, two unknowns, solve them and you get coefficients values. The original equation T of x can be rewritten as using this coefficient a two and a one, and given here x 2 minus x by L into T 1, similarly x minus x 1 by L into T 2. Where entities in the braces are called shape functions, usual symbol is letter N, so there is a shape function corresponds to nodal value T 1, there is another shape function corresponding nodal value T 2, and they are written here x 2 minus x by L, and x minus x 1 by L. So, you can rewrite the equation again, using the shape function definition, so this T of x is written here with N 1 into T 1 plus N 2 into T 2. We are now explain for taking one element and two nodes one and two; imagine or assume that you are going to extend this for the entire one-dimensional node, one-dimensional element that you have consider. So, we have so many elements, extend this procedure for each element or in generic form, you can write it as T of x N and T of i.

Now, in this case, N is the shape function and this is the matrix form. We are assuming linear variation between two nodes and we are assuming that linear assumption is same throughout the domain. So, we get same function for throughout the domain. So, it is very simplified procedure, but we can have a different variation. So, this T of x is written as N into T of i, and N is the row vector and T of i is the column vector.

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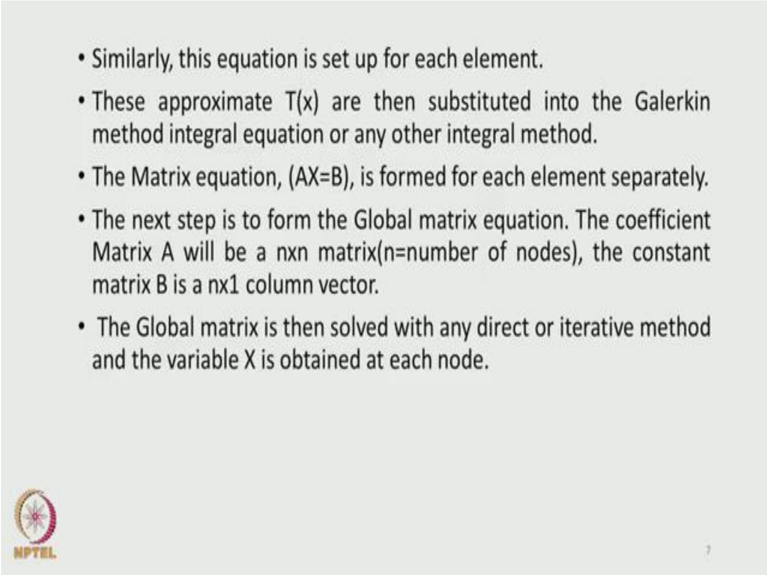

Similarly, this equation is set up for each element. This approximation of T x is then substituted in the Galerkin method integral equation or any other way of writing the weighted form of the equation. Then after assembling matrix for each element to get a coefficient matrix in the global form, then you can solve. So, matrix A will be a n by n matrix, which corresponds to number of nodes, and constant matrix B will be a one column vector which is on the right hand side. X is the unknown to be determined. So, once you set up this equation A X equal to B and solve then you get variable X to be determine for all the nodal location. So, global matrix is then solved either by direct or iterative method, right now do not worry about what is direct or iterative method, we are going to have a separate detailed explanation on matrix solution procedure, there will get to know what is direct procedure, what is iterative procedure. So, variable X is obtained or in this case, it is temperature T obtained in all the nodes and then you proceed like that. So, we have explained what is finite element method by taking a simple example for one dimensioning.

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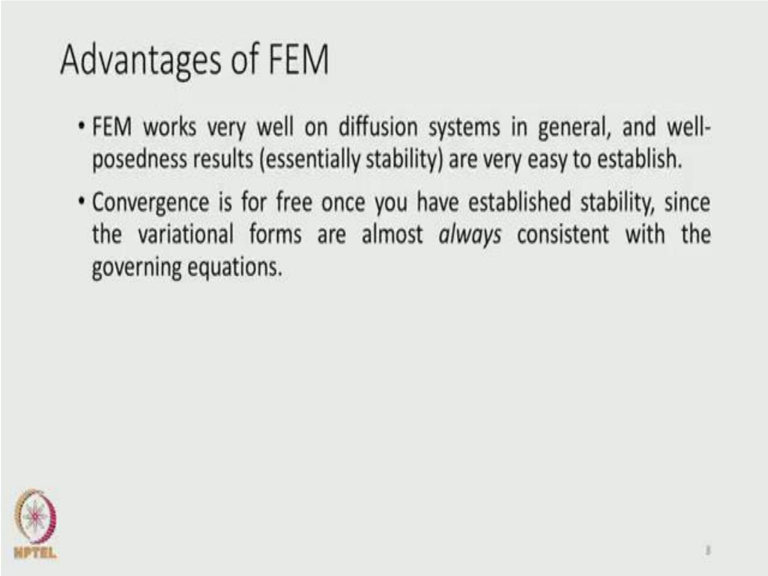

What are the advantages, as I mentioned before, we can use this finite element method in completely unstructured form, and it is very good for complex geometries. It works very well on diffusion systems in general, and for well-posed results and which means essentially the stability criteria is satisfied. Again we are going to have a next lecture, where we are going to talk about how to analyze a numerical scheme, there are three important aspects – consistency, convergence and stability, so there we will talk about what is stability and how to obtain or how to understand the stability. The FEM is supposed to ensure the stability criteria for a diffusion system, and which happens to be a well posed problem.

Then convergence is also for free once you have established stability, since the variational forms are almost consistent with the governing equation. So, once again, we need to wait to understand these definitions and the connection between convergence, consistent and stability, there is a theorem called Lac equivalence theorem. If a scheme is consistent and stability is ensured then convergence is automatically ensured that is the Lac equivalent theorem. We are going to talk about it again later. So, FEM ensures convergence, because the variational form, which is used as a weighting to get a weighted form of the governing equation or to get the weak form of the governing equation, it always consistent. And we already seen the FEM procedure mostly ensures stability, so once you ensure stability and consistent may convergence is automatically satisfied. This is the major advantage with FEM procedure. As I mentioned before, we are not going to talk in detail about FEM procedure that done totally in a different course.

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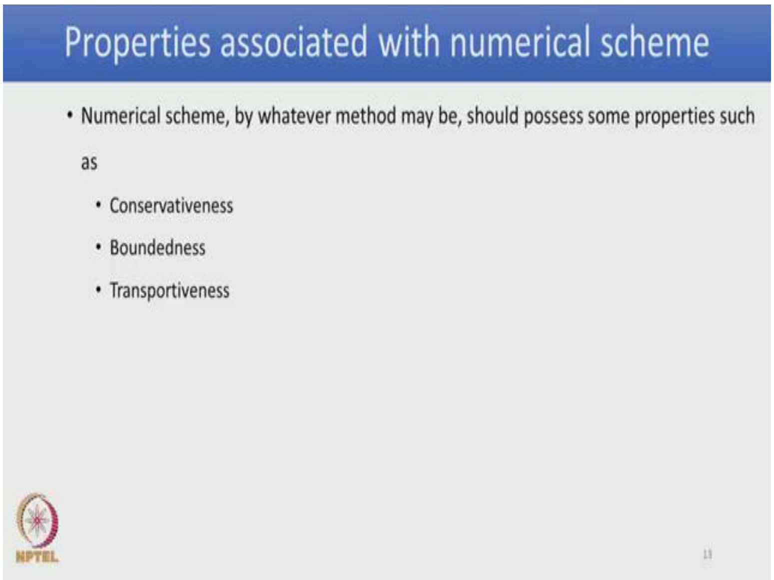

Another important aspect, what are the properties associated with the numerical scheme. Numerical scheme, whatever method may be, so either you follow finite element, finite volume, finite difference or combination of any of these, similarly any order of accuracy you use, we are expected to have some properties satisfying for this scheme first one conservativeness, boundedness and then transportiveness.

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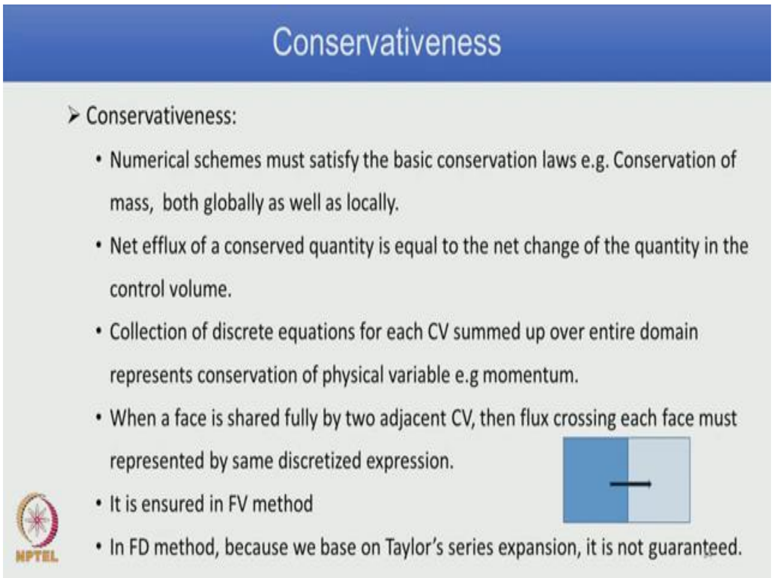

Let us get explanation on conservativeness, boundedness and transportiveness with some examples. So, conservativeness – numerical schemes must satisfied all the basic governing equations, whatever equation we have taken, all the governing equations are derived based on some conservation laws. So, for example, continuity equation is for conservation of mass, so for momentum equation for conservation of momentum and so on. So, any numerical schemes whatever you have following, because it is written for the governing equation, it should satisfy the basic conservation both locally as well as globally. So, if you consider finite volume method, I already mentioned, the flux crossing is accounted properly between two adjacent control volume and if you sum them together for the entire domain, then conservation is automatically satisfied both locally as well as globally.

So, net efflux of conserved quantity from one volume is equal to the net change of the quantity in the control volume. So, if you consider the entire domain, then net quantity is also conserved. So, collection of discrete equation for each control volume is summed up over the entire domain, and it represents conservation of the physical variable. So, if you are trying to get estimate of conservativeness or trying to get the definition of conservativeness explained using a variable say momentum then that should be satisfied for the entire domain. So, when a face or control surface is shared fully by two adjacent control volume, then flux crossing one face or flux leaving the one face must be exactly equal to flux entering the adjacent control volume or if you write in a form of discretized way in one it will be positive, in other it will be negative, balancing each other.

So, sketch wise, so this control volume for a simplified control volume, whatever is going out and this exactly the control surface is shared fully by two adjacent volumes and whatever is going out is equally represented whatever is coming in here that is the discretized form of this leaving must be same as discretized form what is entering for the next control volume. So, if you ensure like that for each control volume and sum them up for the entire computational domain then conservativeness can be ensured. Then as I mentioned before, in finite volume, it is automatically ensured, whereas, in the finite difference method, because we are simply following Taylor's series expansion procedure for each nodal location, the overall conservation satisfaction is not guaranteed.

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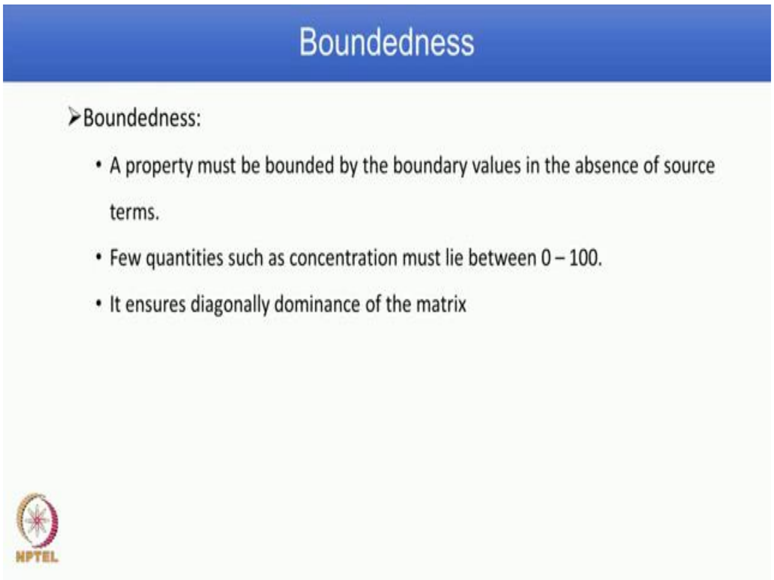

Next property we are going to discuss is boundedness. We have transport equation for every variable. In the transport equation for every variable, we have a balance term, sometimes there is also a source term then we solve the transport equation in the computational domain with boundary condition imposed on the domain. So, when you solve, there is a specific property called boundedness by which we say the values obtain should be within the values specified in the boundary condition, if there is no source term in the equation or in the domain. So, when you say source term, or source generation for example, momentum injection is one additional source term in such cases then you can have values more than or beyond what is specified in the boundary location. So, we define boundedness as a property must be bounded by the boundary values in the absence of source terms.

Let us take for example, a multi species situation then we have a species equation for each species. We try to find out concentration then in that case concentration of each species can be between zero percent and hundred percent. concentration of any species cannot go less than zero, in other words, negative values such situation is sometimes otherwise it is also called realizable condition. Now, this boundedness ensures what is known as diagonally dominance situation in a matrix. Once again when we discretized the governing equation, finally, we arrive at system of linear equations, and we put them in the form of a matrix A into X equal to B, where A is the coefficient matrix, X is the unknown column vector, and B on the right hand side is the known matrix. Now, we need to invert this matrix to get solution vector, this inversion procedure depends on the coefficient matrix nature. To have a stable solution, we should have diagonally dominant matrix. Now, this boundedness ensures the matrix is diagonally dominant.

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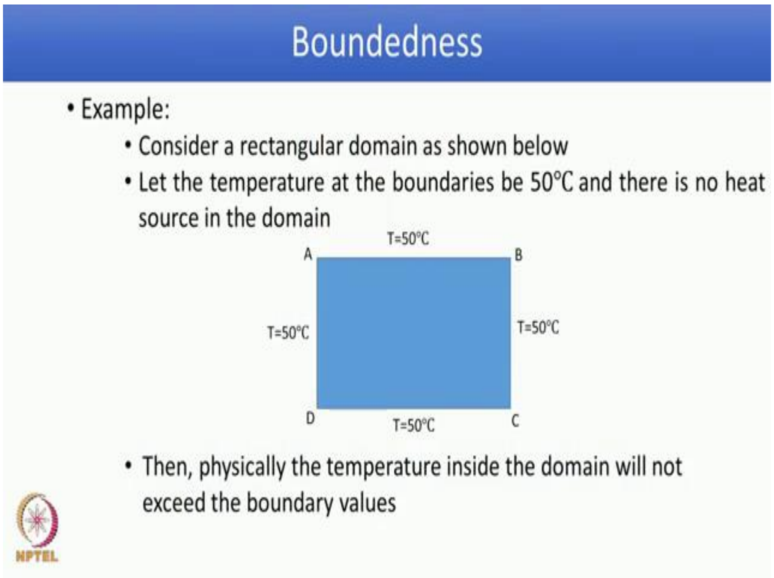

Let us try to explain this boundedness with the help of a simple problem. We consider a rectangular domain as shown here, a simple domain, big the boundary is marked as A B, B C, C D, and D A. Now, for simplicity sake, we consider this problem without any source generation. And boundary condition specified or temperature along all the surface and as shown here. So, along A B, it is fifty degree Celsius; along BC, it is fifty degree Celsius; along CD, it is fifty degree Celsius, again along DA, it is fifty degree Celsius. Now, to solve this problem, what do we do, we discretized the domain either by finite difference or finite volume method. It will have internally nodes described, you solve the transport equation for energy which contains the temperature term. We already mentioned there is no source term inside the domain, or there is no source term in the equation. In such situation, when you solve the transport equation for temperature, finally, the temperature values should be within the boundary condition value that is specified. Now, for this problem, it is specified as fifty degree Celsius, hence the values of temperature inside the node should be within the boundary values that is fifty degree Celsius. Physically the temperature inside the domain cannot exceed the boundary values.

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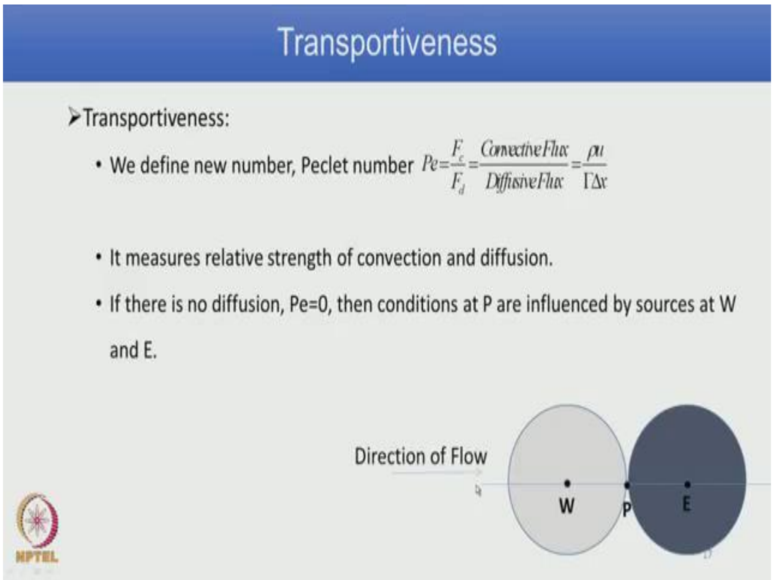

So, we go onto the third property what is known as the transportiveness. For this, we define new number what is known as Peclet number, which is usually given by letter Pe, and it is ratio of convective flux to the diffusive flux, and if you write convective flux as rho into u, because u is the velocity and that tell you which direction how much it is going, so that it becomes convective flux. The diffusive flux gamma is the diffusion coefficient, and delta x is the gradient distance that we decide, so diffusive flux. A ratio of convective flux to the diffusive flux is what is known as a Peclet number. It measures relative strength of the convection term and diffusion term. So, any equation, governing equation, for example, momentum equation, we have convective term on the left hand side, diffusion term on the right hand side. Later, we will also see in turbulent flow, we also a term equivalent to convection term, term equivalent to diffusion term. So, the Peclet number gives the idea of the role of or the weightage of convection with respect to the diffusion.

If there is no diffusion then Peclet number goes to zero, then condition at P which is the point of interest is influenced by sources at W and E. So, I am just giving one figure to explain this, so there are two control volume, for example, shown in here circle , P is the point of interest that is the nodal value where you want to find, and E and W are two adjacent nodes; E basically for east, and W representing west. If there is no diffusion, then variable value at P or the flux at P is completely influenced by W and E that is equal weightage. Now, this is the direction of the flow.

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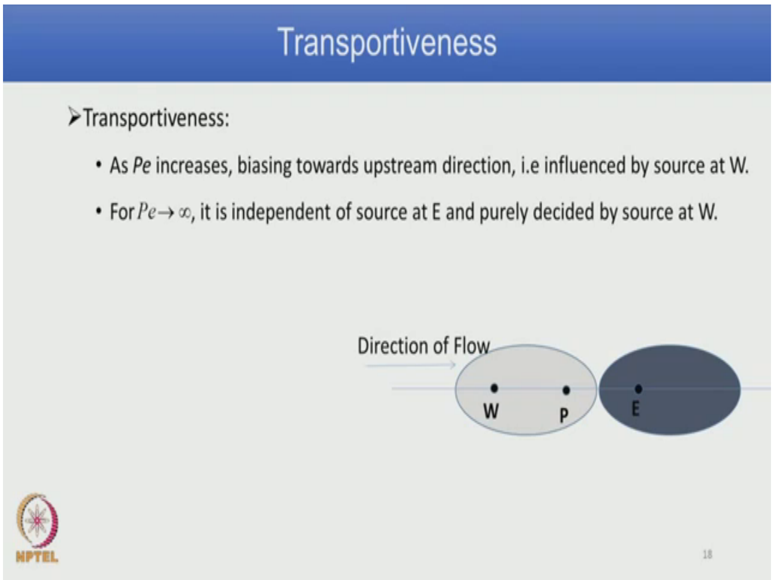

Now, the other case is as Pe increases that is convective flux to diffusive flux Pe – Peclet number increases, the biasing towards upstream direction, it is influenced by source at W. So, as you can see, the size of the domain of influence, which is given here, P comes into the domain of influence of W node. So, the extreme case, when Peclet number is almost infinity then the independent of source at E happens and it is purely decided by value at W. So, this transportiveness, boundedness and conservativeness are three important properties, which any numerical scheme should have. I also mentioned about realizability, we already talked bit about it that is concentration. If you are talking about multi component system, concentration value cannot go negative, it can have either lowest value of zero or highest value of hundred that will be in a situation..

Similarly, if you consider energy, kinetic energy for example, or stress term – square of the residual stress, they are can have only a positive value, it cannot have a negative value. Individually they can go without squaring the term can go negative, but when you square it, it has goes to positive only. So, kinetic energy for example, it can have only a positive value. We will talk about this when we talk about turbulent flow, so this property is called realizability condition. So, in this class, we learned little bit about finite element method, procedure in a finite element method with help of example problem one-dimensional diffusion equation. Then we talked about properties arising in a numerical scheme or what properties in a numerical scheme should satisfied. So, next class, we are going to have another important topic, how to analyze a numerical scheme with some key definitions.

Thank you.