Foundation of Computational Fluid Dynamics

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Lecture – 11

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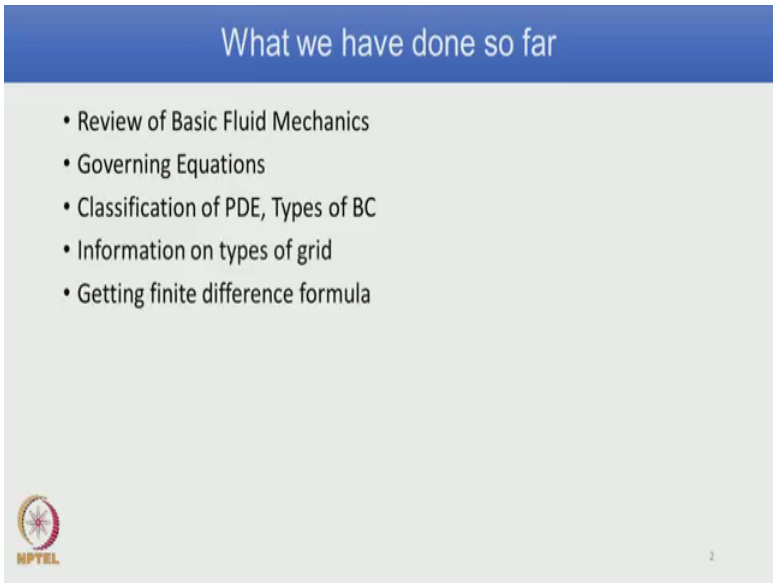

It is my pleasure to welcome you all again to this course on CFD. Today, we are on week three, we will be teaching some more important aspects of this course. Before going to this week content, let us see what we have done so far in this course. We have reviewed in some important aspects of basic fluid mechanics, governing equation , non-dimensionalization , some information about grid, then Taylor series expansion, how to different finite difference scheme, then boundary condition, and nature of equation and its solution.

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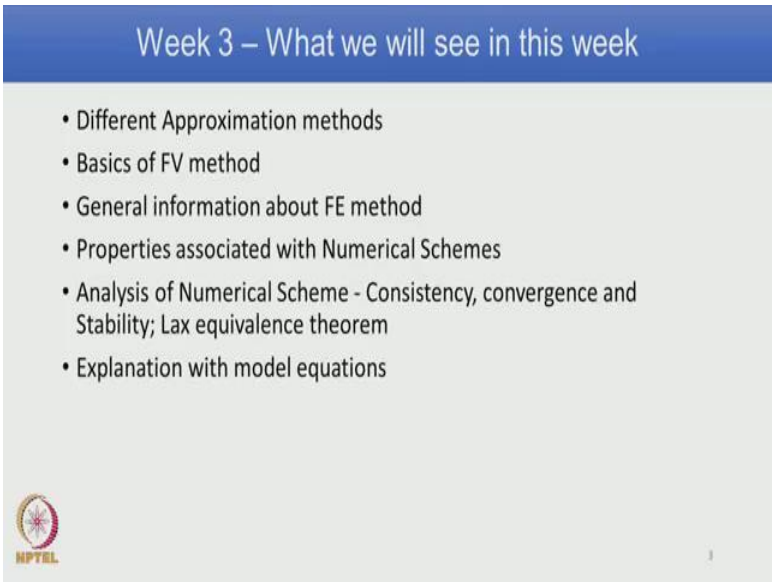

This week we will particularly talk about different approximation methods available; then some basic information about three important approximation methods finite volume, finite difference and finite element methods. Then each of these schemes, they have some properties associated, we are going to have a definition of these properties and explanation about these properties with some examples considered. Then whether numerical scheme that you are chosen is good enough and what it should have and this is what is known as analysis of numerical scheme. And there are three important aspects of numerical scheme, we are going to talk about them definition of them and with explanation, we understand these aspects. Finally, I will also try to give explanation with some standard model equations available to understand some of the aspect that you will learn in today's class.

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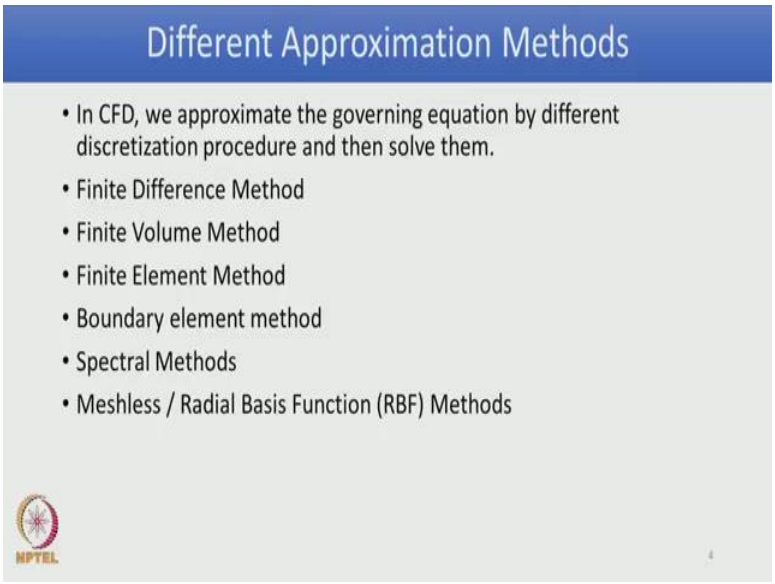

In general, CFD, we approximate the governing equation by different discretization procedure and solve them. We have many methods available, but we will take or look at only few methods which are very popular, finite difference method, finite volume method and finite element method, these are three approximation methods commonly used. But there are also special methods available for special situation, boundary element method, spectral method, and there is another class of approach what is known as a meshless method, one of them for example, is radial basis function methods, this is also something called smooth particle hydrodynamics - SPH method. These are all advanced topics in this course, we will limit ourselves to finite difference, finite volume and finite element methods. And in this course, particularly I will focus on finite volume method in detail, but next few lectures we will get a idea of what is finite difference method and finite element method, and explanation with some example situation.

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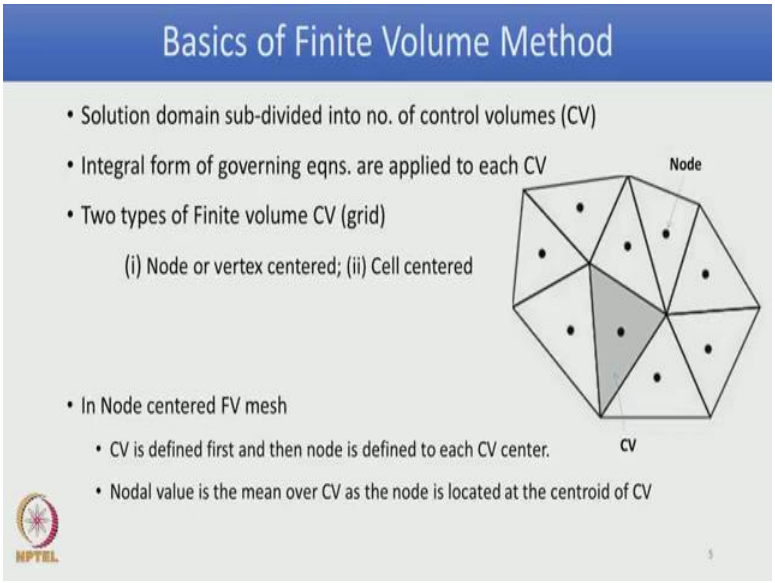

So, in finite volume method, we divide the domain into number of control volumes, then we learned governing equations in two form, one is the differential form, another one is the integral form. So, in finite volume method, we will mostly use integral form of the governing equation, and this is applied to each control volume. In general, there are two types of control volume grid, one is called node or vertex centered method, other one is called cell centered method. So, in node centered finite volume mesh, CV is CV means control volume it is defined first and then node is defined to each CV center. So, here is the illustration, so if you look at this figure, black filled dots that what I am showing is a node and you construct a finite volume around that node such that this node is at the center of the control volume that you have consider. So, we define the CV, then we define the node. How do we define the node, they are all mean of each faces, this is what is known as a node centered finite volume.

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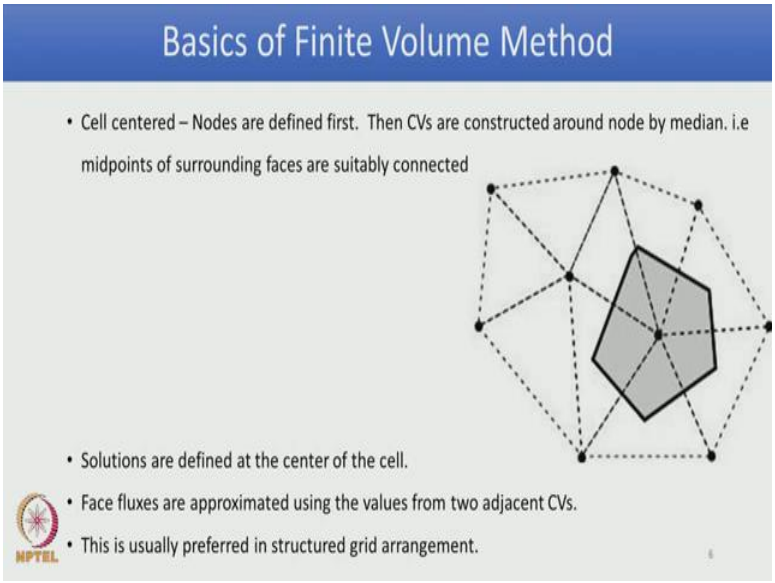

There is another method called cell centered finite volume mesh. In this, nodes are defined first, then control volumes are constructed around the node by median; in other words, midpoints of the surrounding faces are suitably connected. So, if you look at that mesh arrangement, so this each one of them are vertex then you construct a control volume around that vertex by taking midpoint of that two vertex responsible for that control volume. So, if you look at this figure, there is one vertex here, immediate adjust vertex is here and you take a midpoint, and you have a face, control surface of the control volume passing to that midpoint. So, CVs are defined or constructed around the node by median or midpoints of surrounding faces, which are suitably connected. So, for each of this control volume, we construct a face, then if you see they are all cell centered control volume.

In this procedure, solutions are defined at the centered of the cell, and face fluxes that is flux crossing each control surface are approximated using the values from two adjacent control volumes. So, if you look at this surface then whatever flux crossing is defined based on two adjacent control volume. And these are usually preferred in structure grid arrangement, so we can have a finite volume mesh both in structured as well as unstructured mesh arrangement.

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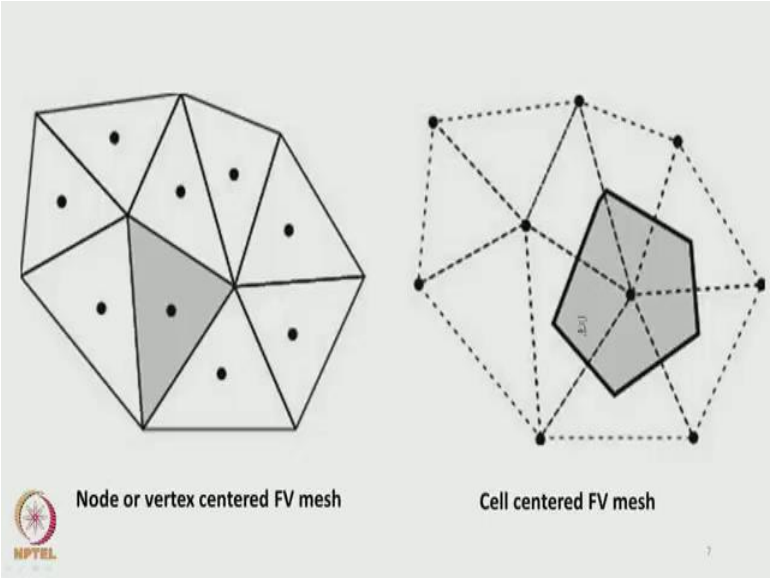

So, we put both these mesh arrangement together, so this is the node or vertex centered finite volume mesh, and this is cell centered finite volume mesh. By now we should able to appreciate the difference between these two type of mesh; so in this, the vertex is centered for the volume and in this cell is centered for the volume.

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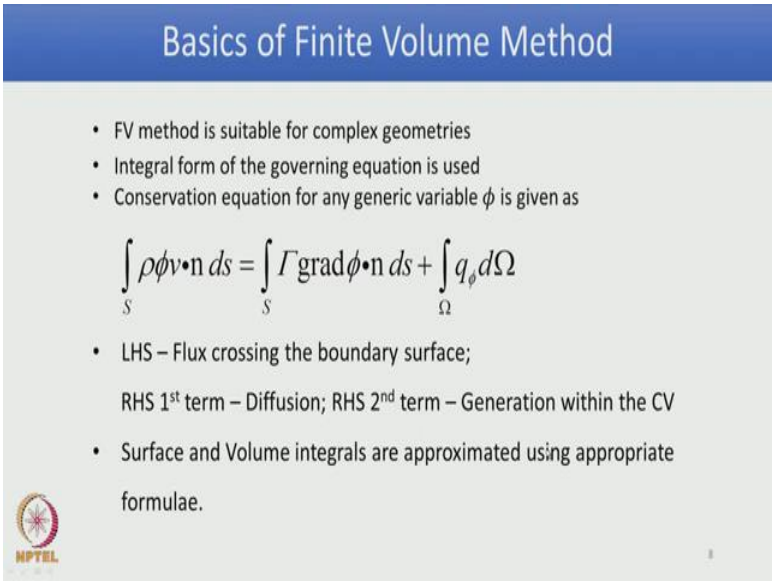

But we do not look at these in detail, they are usually suitable for complex geometries. And then as I mentioned before, integral form of the governing equation is used. Conservation equation for any generic variable phi, say it is given in integral form as

shown here. So, integral over the surface rho phi v dot n into ds, so this will give you the flux crossing control surface s; so it may be entering, it may be going out, so in general it is called flux crossing. It is equated to any diffusion again integrated over the surface, so if you have a control volume with phi surface then this is consider over phi surface. And then any generation in that within that volume, so this d big omega represent volume, so any generation within the volume is also accounted. So, this is nothing, but balance, so budget accounting for flux crossing, diffusion as well as any generation. So, first principle behind deriving governing equation, L.H.S – flux crossing the boundary surface; R.H.S first is the diffusion, R.H.S second term is generation within control volume.

Now, we need to evaluate these integrals, surface integral as well as volume integral over finite volume mesh that you have considered. And they are approximated using appropriate formula. So, when you say appropriate, we have volume, we have surface integral, so we need to take those things into account. Second in the case of flux crossing, if we are considering convection term, it is a non-linear term as we have mentioned before and treatment of non-linear term requires separate procedure. So, we have integration value or integration estimated differently for different quantities in this equation that is meaning of the word appropriate formula.

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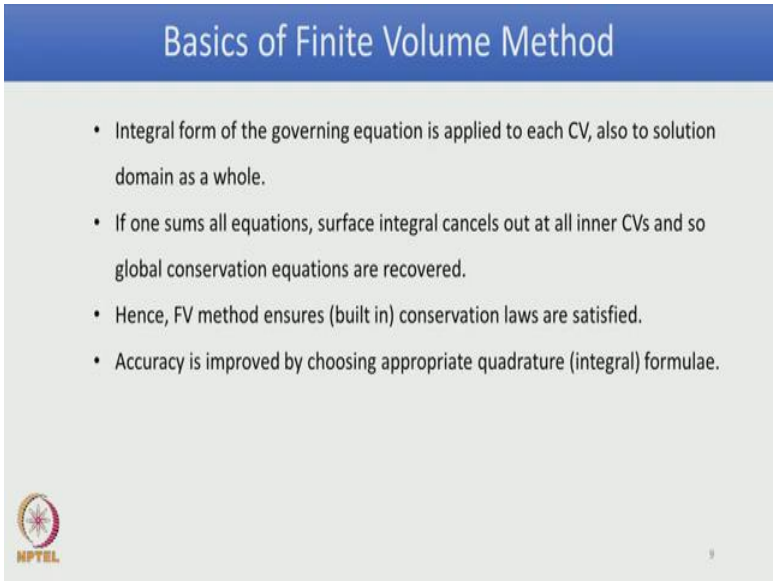

Now, the integral form of governing equation is applied to each control volume, so separately you apply for each control volume and we mentioned in the beginning the entire domain is divided into number of control volumes so you apply this governing equation to each control volume to all the nodes or all the control volume, so in total for the entire domain as a whole. Then if you sum all the equations, so you have equation for each control volume, and if you sum all of them together, then what happens all the surface integral which are evaluated at the control surface for flux going in and for the flux going out. If they two adjacent control volumes are there, they represent the same quantity that is for volume one, it may be flux going out, and for the immediate adjacent; volume two – control volume two, it may be flux going in. And if you extend this explanation for all the control volumes together in a domain then you can understand the surface integrals cancel out. Hence the conservation equation on a global basis is also satisfied.

So, the finite volume method ensures conservation of laws definitely. It is not so in the case of finite difference method, because we do not have such kind of integral form and all the derivatives are expressed as the function of Taylor series expansion. Hence there is a difference between finite volume and finite difference method. And in the case of finite volume one more point that we have to note as I mentioned before, though it is integral over the surface, we have the different quantities, one is the convection flux, other one is the diffusion flux. And they need to be treated separately for some problem or they may be treated same in some other problem or in a same domain, they may be treated same or differently depending on the nature of the solution within that problem description. So, accuracy is improved by choosing appropriate integral formula or quadrature formula, which approximates expression in the integral equation.

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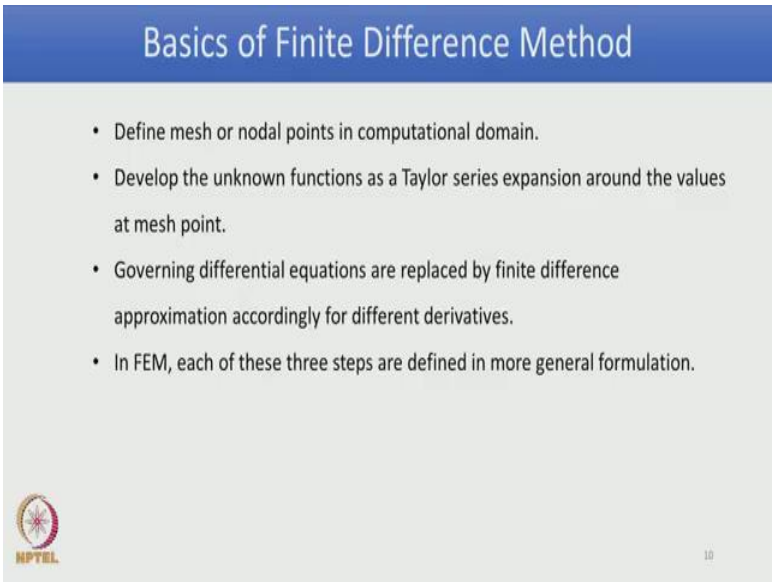

So, next we will see some important aspects of finite difference method. So, as I mentioned before, there are three methods finite volume method, finite difference method and finite element method. Now, in this course, I am going to just touch upon finite difference and finite element, but we will focus mostly on finite volume method, a detailed procedure, how to construct discretized equation based on finite volume method will be dealt in future classes. Now, let us see some important aspects of finite difference method. Again as we did in finite volume, here we talk about individual nodes, so define mesh or nodal points in the computational domain. Then develop all the function as the Taylor expansion around values at each mesh points. So, entire governing differential equation is now replaced by a finite difference approximation accordingly for different derivatives.

So, you have for example, if you take Navier-Stokes equation, x-momentum Navier-Stokes equation, convection terms are first order derivative, diffusion term is the second order derivative. So, we need to have a expression separately for these derivatives. Also you decide based on the order of accuracy, so we can have a convection term with second order, third order accurate, diffusion term can be second order accurate, unsteady term can be first order accurate; whichever order of accuracy that you are deciding accordingly, you can have a different difference formula for each of the term. Then you put them together then entire governing equation is now replaced by finite difference

equation. In FEM, that we are going to see in next, all these three steps are defined in more general formulation, we are going to see with the example in the next slide.

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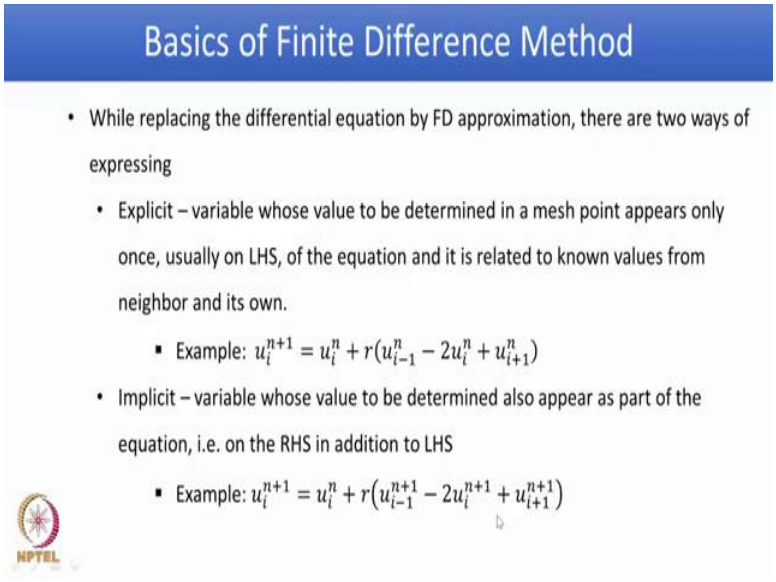

So, while replacing that is differential equation by finite difference approximation, there are two ways of expressing. First one is explicit way of expressing. So, in this case, variable of interest, for example, if you are solving u-momentum equation, u is the variable interest, and if you are solving say energy equation, temperature may be a variable interest. So, whichever variable you are interested and whatever the nodal location, if it is a one-dimensional, you represent by i form as x of i or u of i; if it is a two dimension, you have i comma j, similarly for third direction, same way for temperature. Similarly, if you are solving unsteady equation, and you are computing in iterative procedure way then you have one more index to represent the time integration level.

For example, unsteady equation, unsteady momentum equation u at any nodal location for two-dimensional i comma j and then one more index superscript n to represent that you are doing time integration and you are trying to find out or you know the value of u at n location of the time. So, variable whose value to be determined at each mesh point appears only once in the equation, depending on what scheme you are following. So, usually it is on the left hand side, in the first step then it is moved to the right hand side in the subsequent simplification steps. And this value variable whose value to be determined is always on the left hand side, so all other quantities are moved to the right

hand side, so you have new value to be determined is equal to function of all old values. Those old values are either from its own location a point of interest, but determined from the previous time level in the case of unsteady formulation or it is related to for the same time level, it is related to all the neighboring points. Such formulation is what is known as explicit formulation.

The second one is what is known as a implicit formulation, where the variable whose value to be determined also appears as part of the equation, which means you will have if you write the finite difference equation, you will observe to the variable for which you are trying to solve the equation appears both on left hand side as well as on the right hand side. So, you may not be able to bring out a very clear expression for variable to be determined, they also become part of right hand side equation, hence some rearrangement needs to be done. This is what is known as a implicit formulation.

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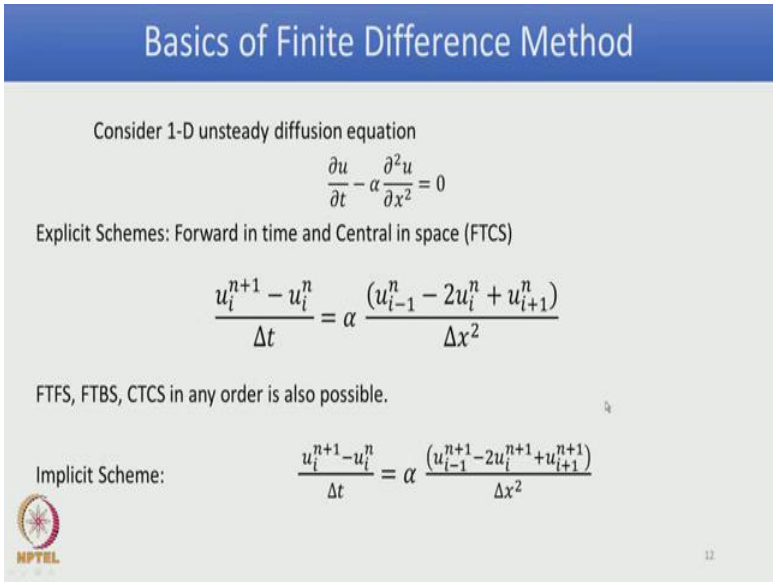

Now, let us get explanation of these with examples. For example, you take unsteady diffusion equation one dimension and that is shown in this dou u by dou t minus alpha dou square u by dou x square, this is very generic equation. In this case, we have taken variable as u for velocity, it can be temperature also. Now, let see how to write as explicit formulation; so we have put forward in time and central in space. Now, please recall, last week class, where we have explained how to get first derivative, how to get second derivative in forward difference, backward difference, central differencing.

Similarly, how to get lower order or how to get higher order accuracy scheme, again by any of these procedures for any derivatives, first derivative, second derivative, third derivative, so please recall all those.

Now, that is only a generic procedure, it is used for any equation, for example, in this case, first we have a time derivative component dou u by dou t and in the second term is the second derivative, but for the spatial expression. So, we can apply whatever Taylor series expansion whatever procedure we got from there for forward difference, backward difference can be applied to this equation also, whether it is time derivative or spatial derivative it does not matter. In the sense, if you say forward in time and central in space and you write it as FTCS. So, let us see how to write when we get equation like this, so u of i n plus 1 minus u of i n by delta t equal to alpha times u of i minus 1 n minus 2 u i n plus u of i plus 1 n by del x square. Now, let us understand this equation; in this, for every variable for example, u you have a subscript i and you have a superscript n plus 1, so subscript i defines the nodal location, in this case, it is only one dimension, so it represent x i particular interest of point of interest, and superscript n or n plus 1 stands for time location. So, in this case, n plus 1 means you are trying to find out value at new time location and n, so u of i n represent you know the value at the same location i, but from a previous time iterative value, so n corresponds to previous, and n plus 1 corresponds to current.

So, what we said forward in time, so you can understand it is forward in time, so n plus 1 minus n, that is how you should try to understand. So, u of i n plus 1, u of i n same variable at same x location, but at different time level and forward in time level that is why it is forward in time, n plus 1 and n. Similarly, central in space; that means, it is central only in space, so all the time quantities are already known that is why on the right hand side, you have u of i minus 1 n as a superscript, similarly two u i of n and u of i plus 1 n, so you can understand all these variable values u are determined from nth level in time. So, nth level in time is something that you have already determined before, hence the time level is explicit and spatially if you look at spatially i is the point of interest, i minus 1 and i plus 1, you take values from either side of the point of interest that is why it is i minus 1 which is on the left hand side and i plus 1 which is on the right hand side, and i is the point of interest. But all the three quantities are known from the previous

time level that is why you have superscript n appearing, so it is central in space, forward in time.

Such a equation as I explained just now, only n plus 1 needs to be determined and that alone will be there on the left hand side, you can rearrange the equation by considering u of i n move it to the right hand side and club it along with this, and then you get explicit expression to determine the value of u of i n plus 1. Now, I will explain all these by following one finite difference equation form that is forward in time and central in space. You can also write down expression based on forward in time, forward in space; forward in time, backward in space; central in time and central in space. Now, we will see the same formulation, how to write implicit form, so you can understand this is forward in time still on the left hand side, because n plus 1 and n. Whereas on the right hand side, if you look at the right hand side, it is central in space, i minus 1, i and i plus 1 as I explained before, i is the point of interest. So, it is central in space, but you look at the superscript, all the superscripts are all at n plus 1 level. So, this n plus 1 level is a point to be determined for at i node, i plus 1 node, and also for i minus 1 node. Sometimes i minus 1 node, which is on the left hand side, you would already determined, because it is on the left hand side, whereas, i plus 1 node, you do not know.

So, if you look at the overall formulation, n plus 1 u of i n plus 1 appears on the left hand side, and u of i n plus 1 appear on the right hand side, so this becomes in implicit formulation. We do not go into advantage of explicit scheme, implicit scheme now. We will reserve that explanation in the subsequent class. We just get idea of writing different finite difference approximation. So, in this class, I have started telling something about three different basic discretization procedure. We finished finite volume, we came to know different mesh arrangement then we got some idea about finite difference procedure. We will continue the next class the third type finite element method and then remaining interesting topics.

Thank you.