Microfluidics
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Lecture - 09
Micro-scale Fluid Mechanics

Okay, we have been looking at Poiseuille flow you know we are talking about Newtonian incompressible fluid flow through microchannels of you know long microchannels of constant cross-sectional area which is called Poiseuille flow, right. In partially flow, the flow rate is directly proportional to the pressure difference. So knowing that, we can establish what is called hydraulic resistance.

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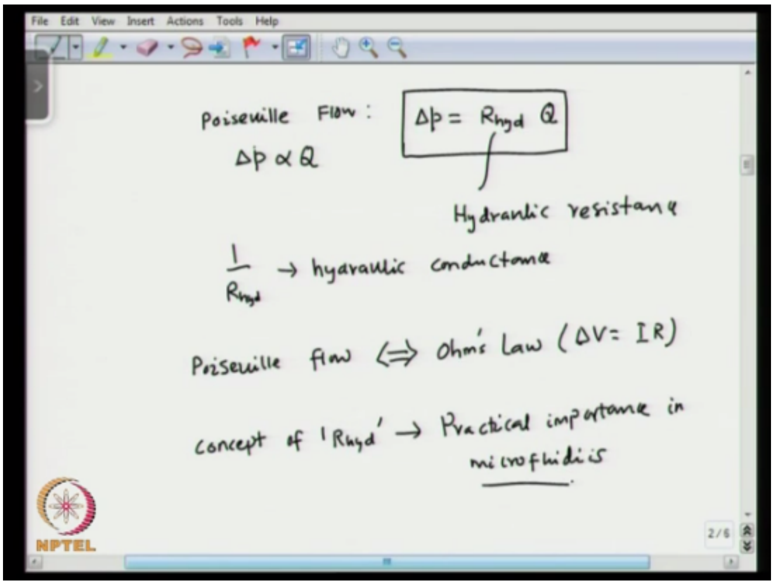

Let us look at what is called hydraulic resistance? So in Poiseuille flow, we can write delta p will be = R hydraulic * Q, okay. So, here delta P and Q are proportional to each other. So, you can write delta P is R hydraulic * Q. So this is known as hydraulic resistance and the inverse of hydraulic resistance is known as hydraulic conductance. So what do we see here is, the Hagen-Poiseuille law which is delta p = R hydraulic * Q is similar to Ohm's law.

In case of Ohm's law, the voltage difference is a product of current and resistance, okay. So you know, Poiseuille flow is equivalent to Ohm's law which is delta V = I*R, okay. Now, the concept of hydraulic resistance is very important in microfluidics because it helps in characterizing microchannels, it helps in designing microchannel ne2rk, okay. So, the concept of hydraulic resistance as practical importance in microfluidics, okay.

We have another terminology called hydraulic complaints, okay equivalent to you know capacitance in an electrical circuit. In hydraulics, the equivalent term is called hydraulic compliance, okay. So, hydraulic compliance is equivalent to the electrical capacitance and the hydraulic capacitance is defined as change in volume over change in pressure and there is a negative sign, okay because the volume is usually reduced when the pressure is increased, okay.

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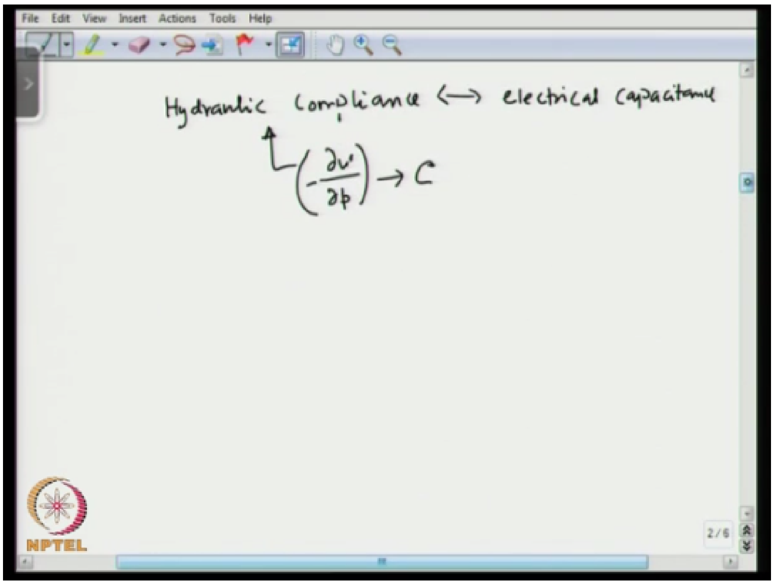

So, this term is called hydraulic capacitance, okay denoted by C. Now, let us look as viscous dissipation equivalent to Joule heating in electrical circuit because of finite resistance and finite current. In hydraulic circuits, we have viscous dissipation when you have you know a flow Q is occurring because of the pressure difference delta P; we would have a viscous dissipation.

Let us imagine, we have a Q and we apply a time t < 0, we have a pressure difference delta P between the 2 hands unless a time t = 0, we take out a delta P, okay. So what will happen is, the initial energy will get used up to give us to the viscous dissipation, okay. So, if you look at you know this case here, so let us consider we are talking about viscous dissipation.

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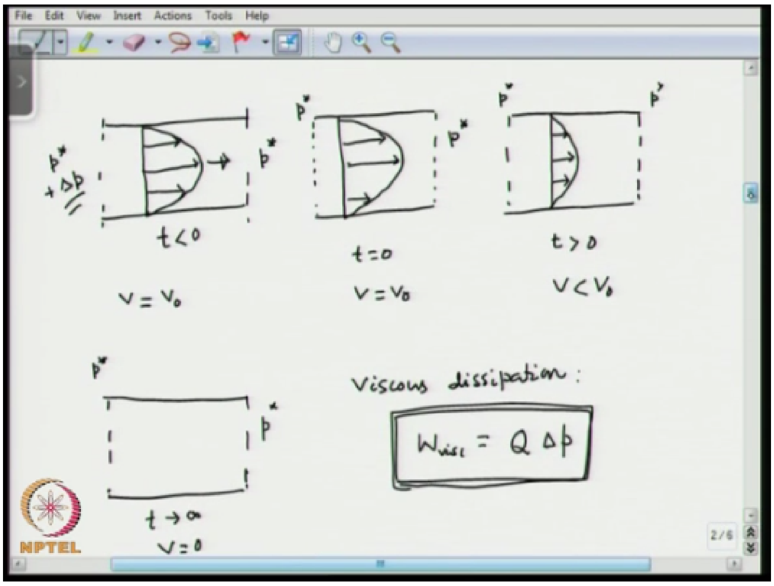

Let us talk about you know flow between parallel plates, so this is the inlet, this is the outlet. We have pressure here P* + delta P and pressure at the outlet P* and this is at time t < 0. So you get velocity v = some v0, right. Now a time t = 0, lets you take out the pressure drop delta P, now this pressure will be = this pressure, right. So initially, the velocity profile as you know it will be parabolic.

At time, t = 0, when you just take out the delta P, the profile would still be parabolic, okay. But as time progresses, let us say time t > 0, here v, velocity = v0. When time progresses, this is P*, this is P*, the velocity profile will become smaller, okay will become flatter, right. So, this v is going to be < v0. Now as time t tends to infinity, you will have this end P* here and P* here and the velocity will become 0, right.

So, the energy of the fluid is utilized to over the viscous loss, okay. So, the viscous dissipation can be expressed as W viscous is expressed as Q * delta P, so delta P is the pressure difference here and Q is the flow rate that is occurring. So, you should take out delta P, you know that energy will be utilized to overcome the viscous laws, okay and that actually helps us to understand the hydraulic resistance.

Because if there is a finite resistance offered by the walls, the viscous dissipation occurs and the energy is dissipated to overcome the viscous dissipation, okay. So, let us consider hydraulic resistance in case of a circular channel cross-section. So, we first consider hydraulic resistance analysis and first to consider circular cross-section, okay. So you know we have let us say, the hydraulic circuit anything like this.

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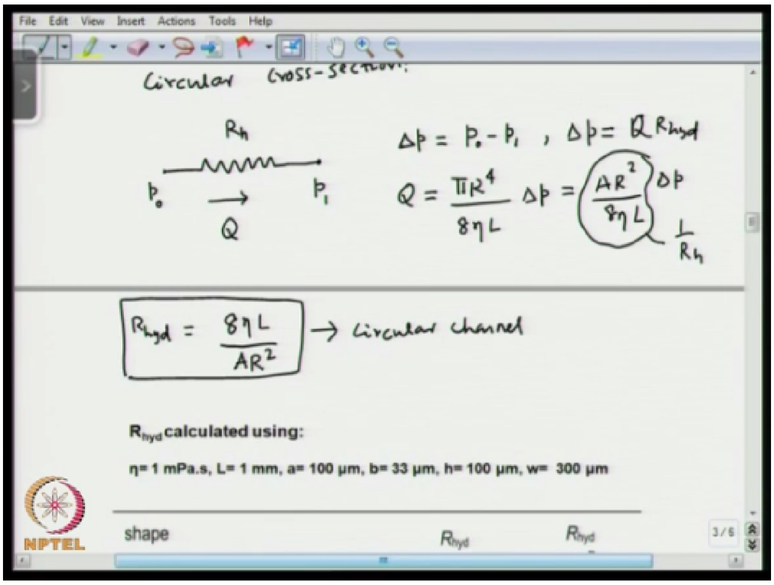

Here, pressure is P0, here pressure is P1, this is hydraulic resistance, they have flow Q occurring, okay. So, the pressure difference delta p is going to be P0-P1 and from Hagen-Poiseuille law, what we have seen here, okay, this is Hagen-Poiseuille law, you can write delta P = Q * R hydraulic and for a circular channel, we know that Q is pi R 4 over 8 eta L * delta P and pi R square is the area, so you can write it as AR square * delta P divided by 8 eta L, right.

So this is the hydraulic resistance, okay. So, R hydraulic is going to be 8 eta, so this is actually 1 over R hydraulic, 8 eta L divided by AR square, okay. So, this is the expression for hydraulic resistance for circular channel. Now, let us look at here hydraulic resistance of different channel cross-sections. First look at the circular channel cross-section and here the hydraulic resistance is what we just obtained 8 eta L over AR square, which pi A4, A being the radius of the circle.

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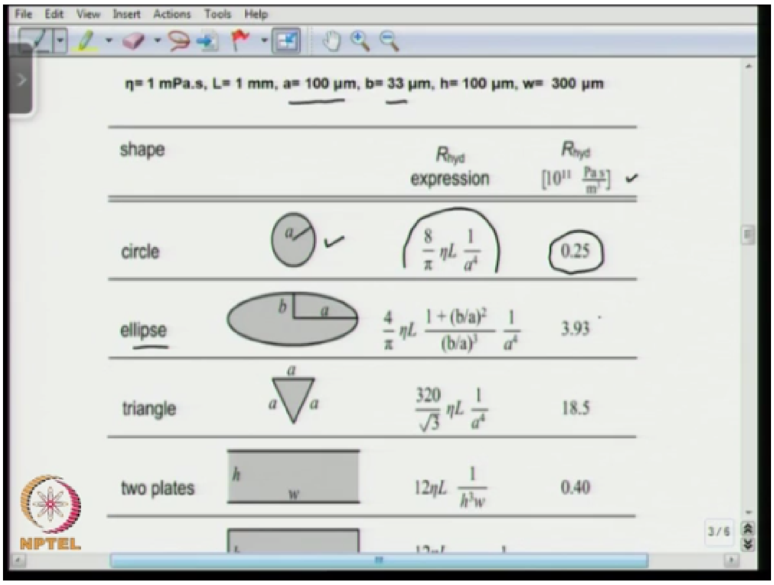

And if you consider a is under micron, we can see the hydraulic resistance is about 0.25*10 to the power 11 Pascal second per meter cube, okay. Now, if you consider an elliptical channel, where a is the half of the major axis and b is the half of the minor axis, a to be 100 and b to be 33 micron, then the hydraulic resistance is going to be 3.93, okay and if you look at the triangular cross-section, you know this is expression for the hydraulic resistance.

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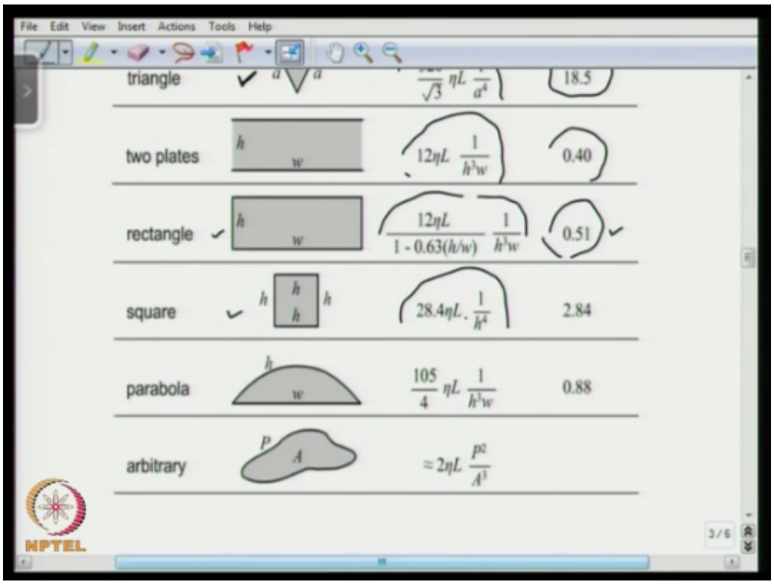

And if you compare with that circular cross-section, this circular cross-section has radius a and here, the triangular cross-section has each side A, you can see that the hydraulic resistance is about you know 70 times the circular cross-section. So, hydraulic resistance of a triangular channel is very high.

Now, if you look 2 parallel plates, the hydraulic resistance is given by this expression here and this is its value when we talk about a suppression distance under micron, okay and for all cases, we are considering viscosity to be 1 millipascal second, right. Next for the rectangular cross-section, this is the expression and it is not very different compared to the parallel plate case, okay. Resistance is not very different.

In case of a square channel cross-section, you can see this is the expression here and for a square cross-section of each side 100 micron, the resistance is 2.84. It is definitely higher than the circular cross-section, but lower than the triangular channel cross-section and here is the expression for the hydraulic resistance in case of parabolic channel and this is its value when h and w are given here. H is 100 micron and w is 300 micron.

Now, this is the general hydraulic resistance that we had derived, when we talked about Poiseuille flow in arbitrary cross-section. So with that, let us move on to talk about shape dependence of hydraulic resistance. You know if you know the shape of a channel, how we can determinates hydraulic resistance, okay. So, we talk about shape dependence of hydraulic resistance, okay.

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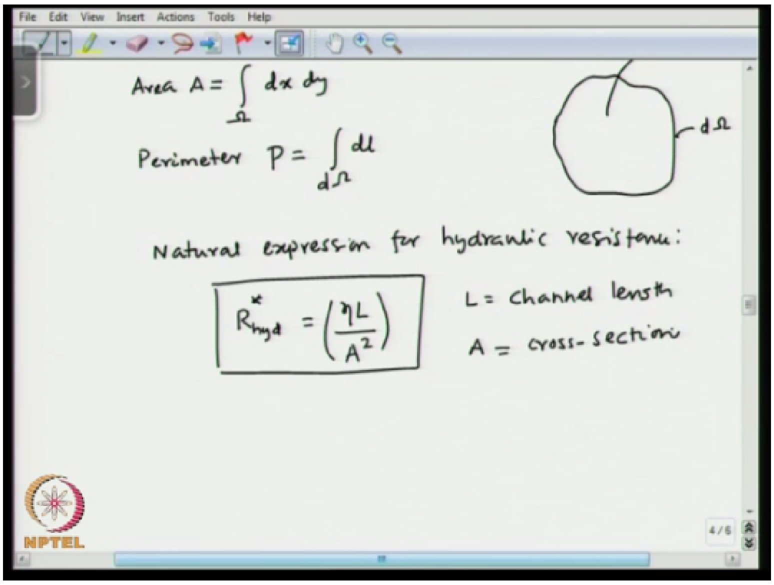

We can write let us consider an arbitrary channel say let this surface be d omega and the inside cross-sectional area is omega, okay. Now the area A can be written as integration dxdy, integration over omega is in Cartesian coordinate system. The perimeter P can be written as dl, integration over d omega, okay. Now you can come up with a natural expression for hydraulic resistance.

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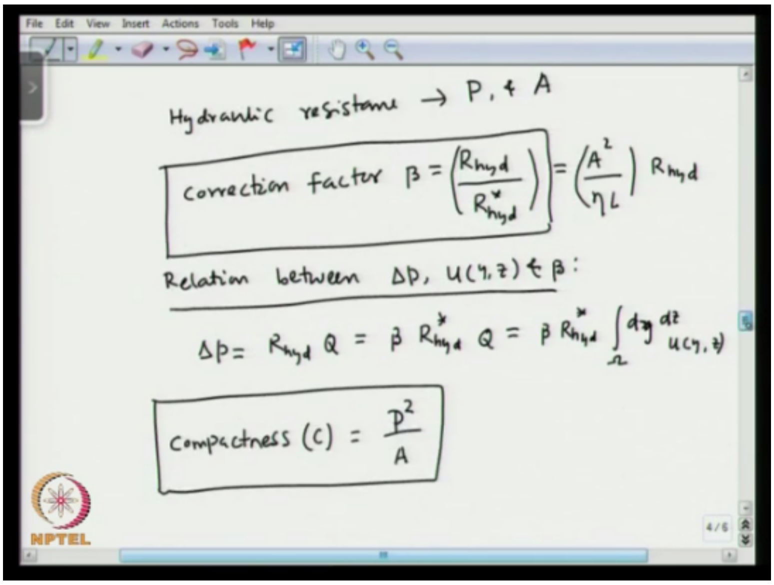

If you look at in other circular channel case, the hydraulic resistance is varying as eta L over AR square, so which is eta L over A square, right because the area is pi R square. So, this is how it is varying, right. So, we can write a natural expression for a hydraulic resistance to be let us call it R hydraulic* to be eta L over A square, okay. So, where L is the channel length and A is the cross-section.

Since we are talking about flow through channels, where on the channel perimeter mostly boundary condition will be valid, then we can intuitively think that the hydraulic resistance will be related to the channel perimeter as well as the cross-sectional area. So, we need to correct the natural expression for the hydraulic resistance. So, we include what is called a correction factor.

So, we include a correction factor beta which is R hydraulic divided by R hydraulic*, okay. So that will be A square over eta L, so that is how we define the natural hydraulic resistance * R hydraulic, okay. Now, we can write the relation between delta P, the velocity filled U let us say y, z and beta. So, delta P could be R hydraulic * Q, which is beta * R hydraulic* * Q, which will be beta * R hydraulic* integration dydz U y, z, okay.

So that is how you can relate delta P U and beta. Now in microfluidics, the Poiseuille flow is you know coupled with many additional phenomena. Most of them are surface phenomenon. For example, electrophoresis, electro-osmosis these are surface phenomenon. So in addition to Poiseuille flow, these surface phenomenon are going to be important. So this is because of the high area to volume ratio at micro-scale.

So in that prospective, we defined a parameter called compactness, okay. So we defined a parameter called compactness, which is c = P square over A, your ratio between square of perimeter divided by area cross-section and this correction factor beta is the ratio between the hydraulic resistance divided by the natural hydraulic resistance, okay. So, now here our goal will be to relate the correction factor beta with the compactness c.

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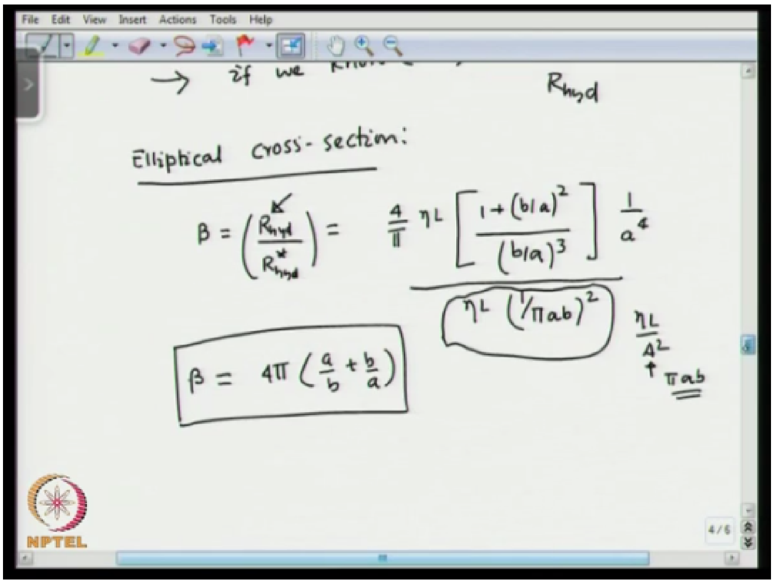

So that for any arbitrary cross-section if we know what is going to be you know the shape of cross-section, we know what is going to be the value of P square over A, so we know the compactness, knowing compactness we should be able to find out beta and hence the hydraulic resistance. So, here our objective is to find relationship between beta and c. So, this will allow us to if we know c, then we can calculate R hydraulic.

Let us first take an example of elliptical cross-section. In an elliptical cross-section, we can beta, the correction factor to be R hydraulic divided by the natural hydraulic resistance R hydraulic* which can be written as, so we have an expression for hydraulic resistance from the Poiseuille flow, so we can write it as 4/pi * eta L * 1 + b/a square divided by b/a cube * 1/a4 divided by eta L * 1/pi ab square, okay.

So, this is the natural hydraulic resistance which is eta L over A square and area of an ellipse is pi ab, right. This is what doing now, so this is the expression for the correction factor. So, if you simplify you get 4 pi * a/b+b/a, so this is the correction factor for an elliptical cross-section. Now for a circle, if it is a circle, then beta will be 8 pi, a = b, so beta will be 8 pi. So we can write the compactness for elliptical cross-section.

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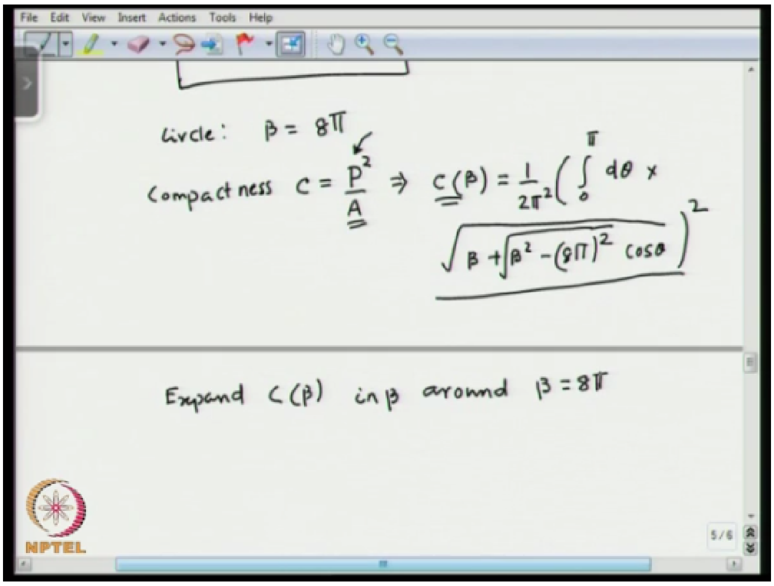

Compactness c is P square over A, if you can represent the expression for the perimeter of an ellipse in terms of an equation and divide with the area of cross-section, right and then we have an expression for beta and we try to represent c in terms of beta, so we would have compactness as a function of beta, right. This is the equation we ended up with, 1 over 2 pi square * 0 to pi d theta * beta + beta square - 8 pi square, square root, then * cosine theta whole square root.

So that is the expression for if we can write an equation for the perimeter and then divide by the area of the ellipse and then write since we have an expression for beta, write express compactness in terms of beta, okay. So this is the expression will end up with, right. Now, if you expand c beta, if you expand this expression for c beta in beta around beta = 8 pi, okay. This will say expansion that you have to do and then do inversion.

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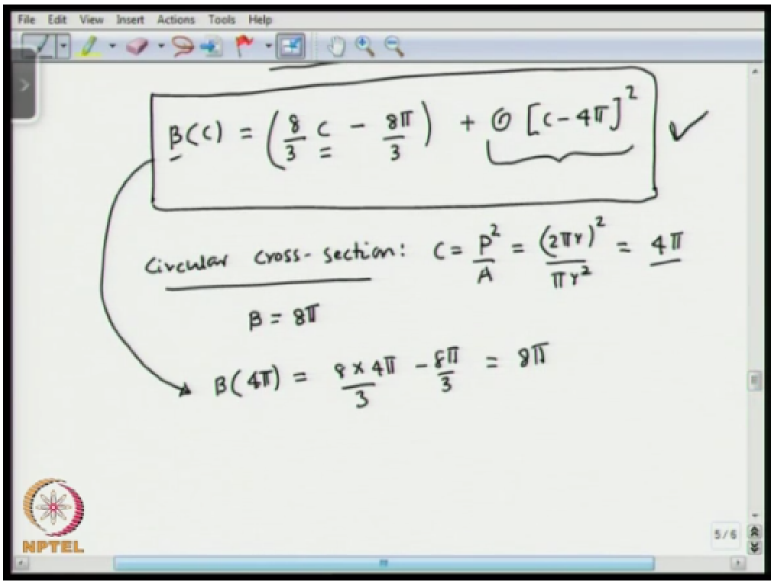

So now here, you are expression compactness in terms of beta. So I am asking to invert means beta in terms of compactness, okay. So if you do that, you would get beta in terms of as 8/3 c – 8 pi/3 + a term which is of the order of c - 4 pi square, okay. This is the remainder that you would get, okay. So, this is the expression for correction factor in terms of compactness for an elliptical cross-section.

Now, if you check this for a circular cross-section, our c is P square over A, right. So, you have 2 pi r is the perimeter square over pi r square, so we get 4 pi and beta for cross-section, we have calculated it here, it 8 pi, so beta is 8 pi. Now if you use this equation, okay so beta for c is 4 pi, then we get 8 * 4 pi divided by 3 – 8 pi/3, so you get this as 8 pi, okay. So this expression is holding through for circular cross-section as a special case of ellipse.

So this is the general expression that can be used for any elliptical shape, if you know what is the compactness we can find what is its hydraulic resistance. Now if you look at you know the plot here, in the x axis, we have compactness c and the y axis we have the correction factor beta. If you know the correction factor, you know hydraulic resistance. So what you see here is for elliptical cross-section, this is the line okay.

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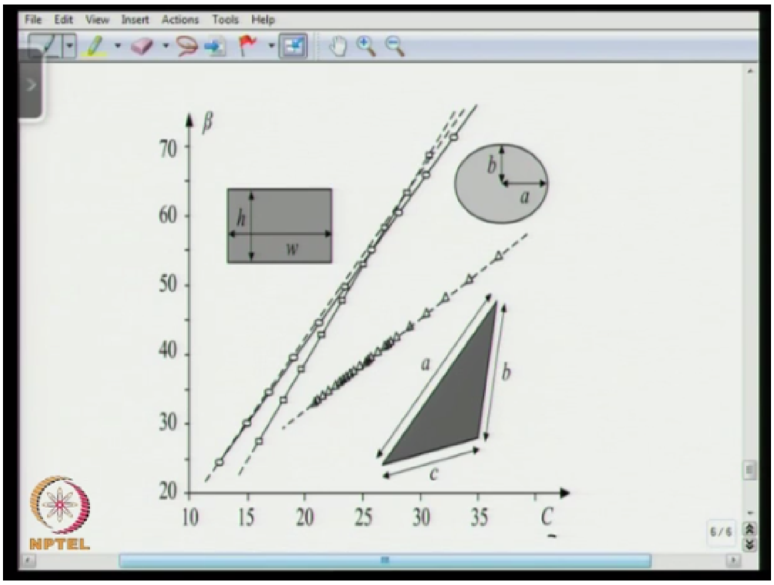

Sorry, this is the line here, a is for elliptical cross-section and you see that beta is linearly dependent on c, okay. So the correction factor is linearly dependent on the compactness, okay. So that is what we see, we see that beta is linearly dependent on c, okay. So, next we look at a rectangular cross-section, okay. Now for rectangular cross-section, we can write beta to be R hydraulic divided by the natural hydraulic resistance R hydraulic*.

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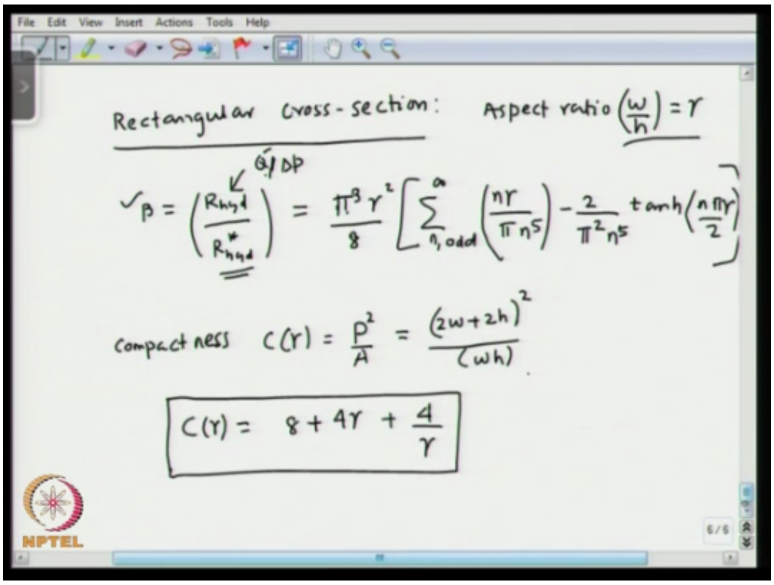

So, R hydraulic can be obtained by Q over delta P, so we have an expression for Q for rectangular cross-section which is a Taylor series, right. So, we can divide that with R hydraulic natural, so we get an expression which looks like this we get pi cube and if you say the aspect ratio of the rectangular channel, okay.

We say width divided by height = say gamma, will write pi cube * gamma square divided by 8 * Taylor series and r infinity * n gamma divided by pi and n5 - 2 divided by pi square and pi * tangent h, tangent hyperbolic of n pi gamma over 2, okay. So, this is the expression we get for the correction factor beta. Now, we can find out compactness c gamma as perimeter square over area.

So, this is 2w+2h that is the perimeter divided by area is wh, so we can write this as 8+4 gamma, gamma is a width to height ratio + 4 over gamma, okay. So this is the expression for c in terms of width to height ratio, okay. So now, we know that tangent of hx = 1 for x >> 1. Now, if you used that we would be able to simplify this expression here. So, if you this condition, then you can write that beta of gamma will be = 12 pi 5 gamma square divided by pi 4 gamma – 186 * gamma to the power 5, okay.

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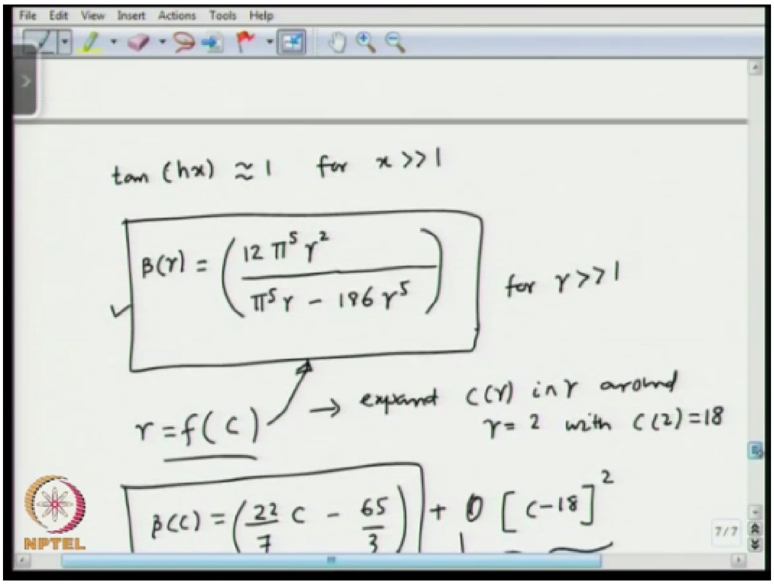

So, we write down the simplified expression for the correction factor and this is called gamma >> 1, okay. So, now if you substitute this c gamma, okay so you find out gamma in terms of c and substitute in this equation here for beta, what you get is so you find out gamma in terms of c, okay and the substitute in this equation you get beta c is going to be 22 divided by 7*c–65/3.

So, you substitute here and then expand c gamma n gamma around gamma = 2 with c up to as 18, which you can check here, okay. So, if you do an expansion of this after substitution, you would get 22/7c–65/3 + a term which is of the order of c - 18 square. This is the additional term that you get, okay. Now, if you consider a square cross-section, for a square cross-section, gamma is going to be 1, right.

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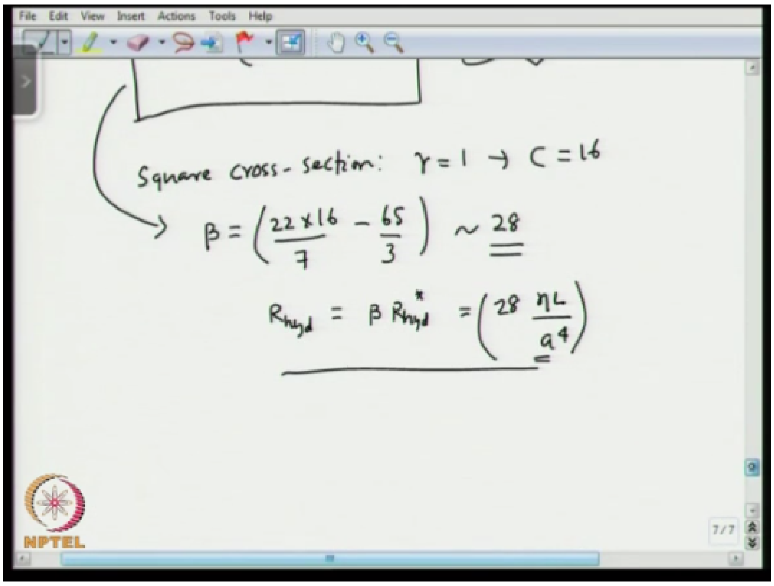

So, if gamma = 1, then you can evaluate what is going to be c here. You can evaluate what is c, c is going to be 16, right. So c is going to be 16, so the beta is going to be 22*16/7-65/3 that will come around 28, okay and this is what we had seen for a square cross-section, right. So for a square cross-section, R hydraulic will be equal to beta * R hydraulic*, so this will be = 28 * eta L over A square, for a square cross-section, it will be a4, thus a is each side.

This is the expression for square cross-section and this is what we saw here, right. This is what we had seen here. For a square cross-section, the hydraulic resistance is about 28.4 eta L over h4, right. So let us consider a triangular cross-section, before we move onto triangular cross-section, we can check in the plot here, we can see here that for a rectangular cross-section, which is this line here.

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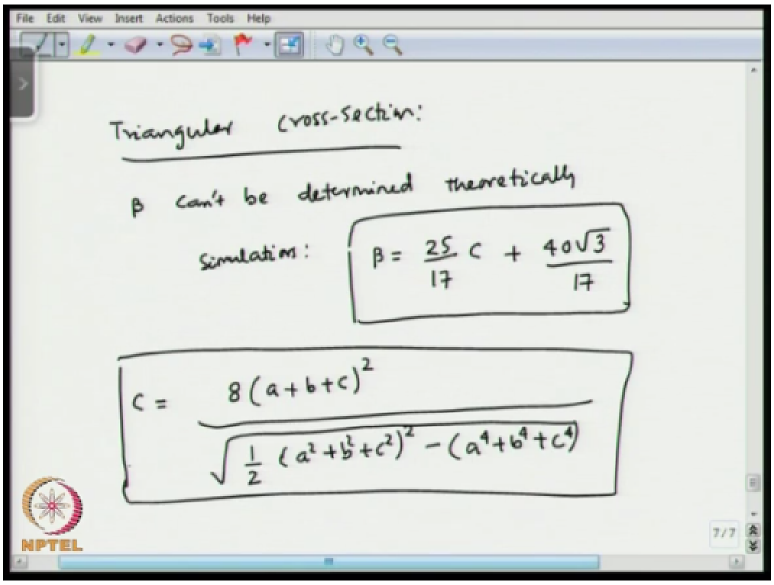

For respective of the value of w and h, beta is going to be linearly dependent on the compactness c, so that is what also we predicted on theory, okay. So for a triangular cross-section, we do not have an expression for beta. Beta can only be determined from simulation, okay. So, beta cannot be determined theoretically and it was in simulation, beta is found to be 25/17*c+40 square root of 3/17, okay.

So, this is for a triangular cross-section, half beta is related with compactness. So, here c is given as 8*a+b+c whole square, where a, b, c are each sides of the triangle divided by 1/2 * a square + b square + c square whole square - a4 + b4 + c4 square root, okay. So, this is the expression for c, which you can find for any triangular cross-section, okay. So once you know beta, you can calculate R hydraulic, right.

Knowing a triangle, you can calculate compactness, knowing a, b, and c for a triangle, you can calculate compactness and for that triangle, once you know the compactness, you can find out beta using this formula and then, you can calculate the hydraulic resistance. Now, you can look at this plot here.

Respective of that shape of the triangle, it may be right angle triangle, it may be you know isosceles triangle or acute obtuse angle triangle, all the different triangles fall on a same line, okay and their beta is going to vary with the compactness in the same way that is what we see here. So what we see here is, relation between beta and c for different triangles fall on the same line, okay on the plot.

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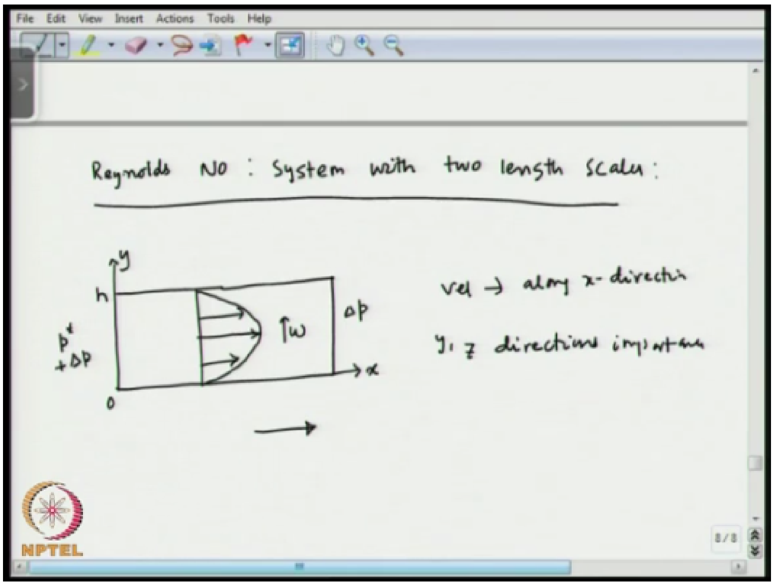

So with that let us move on and talk about Reynolds number in case of a flow when we talk about 2 different length scales, okay. Let us consider flow between in a parallel plates and you now when considering a flow in the x direction, okay then the Reynolds number can be determined based on the length of the plate, okay. But, if you have a nonzero velocity component in the vertical direction, then the Reynolds number can be based on the gap between the 2 plates.

So in this situation, where we have 2 different length scales, you know what is the correct way to determine Reynolds number, okay. So, we look at Reynolds number in case of system with 2 length scales, okay. So, we have seen this infinite parallel plate case, we have 2 parallel plates and this is h, this is P* + delta P, this is delta P here and this side is 0 and we get a velocity profile something like this, okay.

So here, this is x direction and this is y direction. So in this case, we know that there is no variation in the velocity along the x direction, okay and normally, y and z direction are important, okay. So we say that velocity is along x direction, so y and z directions important. Now, although the system is invariant along the x direction, if you consider that we have the nonzero velocity along the y direction, then in that case, we cannot neglect the x direction, okay.

So, when we have the velocity let us say w coming in along the y direction, then the x direction independence breaks down, okay. So in that case, the governing equations will get modified, right. So let us do some scalene, so we have 2 length scales here, one is L, the other one is h, so we can say x = L * x* by inverse scalene and you can say that z = h * z*, right.

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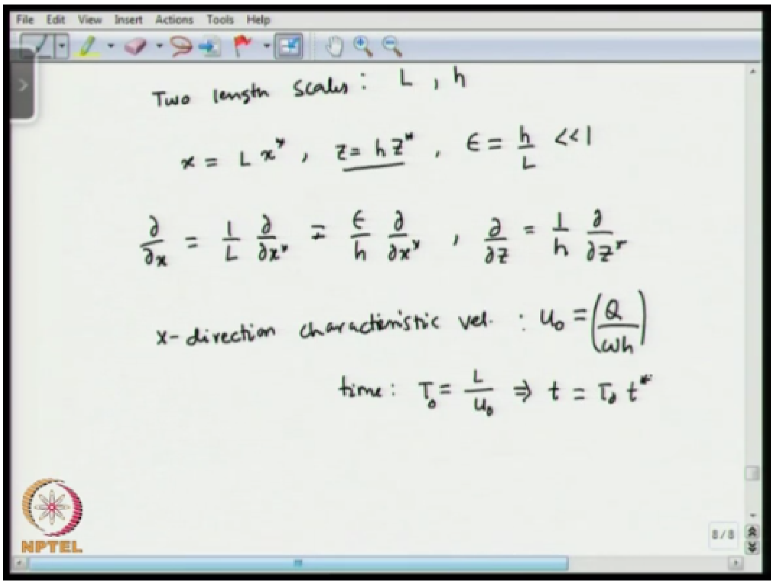

And where we can say that the aspect ratio is going to be h/L which is going to be << 1, okay, so this is what we can say, now this is here, we are taking this as z that is why z = h z*, right so and we assume that in the y direction, the length is infinite, okay. So in that case, we can write the derivatives delta over delta x to be 1 over L * delta over delta x*, right. So, this also you can write it as epsilon over h * delta over delta x*.

So and we have delta over delta z can be written as 1 over h delta over delta z*. Now, the characteristic velocity along x direction characteristic velocity u0 will be Q divided by w * h, w is the width of the plate and h is the gap between the plates and the characteristic time can be T will be =, let us call T0 will be L over u0, so and we can say t = T0 * t*, okay. Now, we can write the velocity components u as u0*u.

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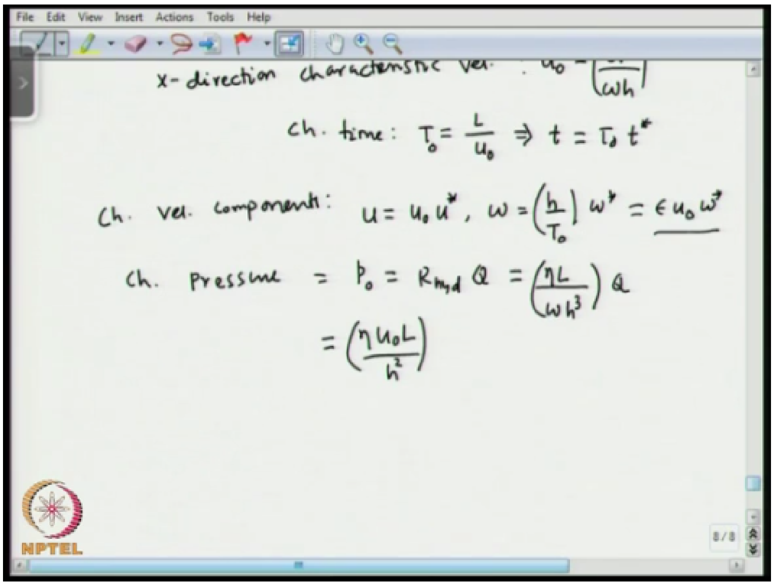

And we can write w will be = h over T0 * w*, right. So this is u*, u0 * u*, w = h over T0 * w*, which will be = epsilon u0 * w*. Epsilon is h over L and L over T0 is u0, so we get epsilon u0 * w*. The characteristic pressure, this is characteristic time, this is characteristic velocity components, characteristic pressure P0 can be written as R hydraulic * Q which will be = eta L over w h cube is the hydraulic resistance in parallel plate flow.

If there is a factor 12, but we are not considering here, because we are doing a scalene here * Q, so that will be = eta u0 L over h square, okay. Now, if we follow the convention that you know the Reynolds number can be defined based on the smallest length scale, then we can define Reynolds number Re as rho u0 h over eta, okay.

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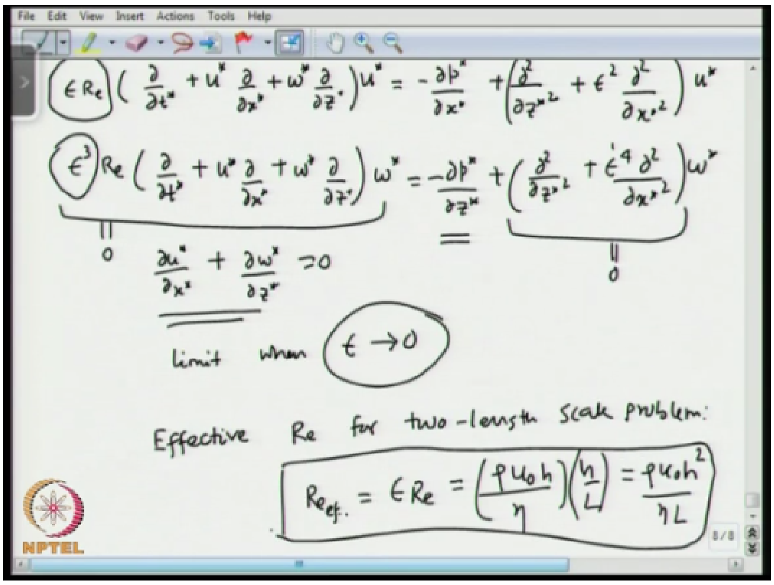

But here, we have 2 length scales, so what we will do is, if we write down the Navier–Stokes equations using all the non-dimensional terms here, okay you would have 2 different momentum equations because we have 2 different components of velocity now. So, we can write Navier–Stokes equation, one it will be epsilon * Re * delta over delta t* + u* delta over delta x* + w* delta over delta z* * u* will be = - delta P* over delta x* + delta square over delta z* square + epsilon square delta square over delta x* square * u*.

And the second equation will be epsilon Q, Reynolds number * delta over delta t* + u* delta over delta x* + w* delta over delta z* * w* will be = - delta P* over delta z* + delta square over delta z* square + epsilon 4 delta square over delta x* square of w*, okay and we have the continuity equation as delta u* over delta x* + delta w* over delta z* to be 0.

Now, here if you in the limit when you know epsilon tends to 0 which is the case when the gap between the plate is small compare to the width of the plate, then in that case, if you see here this term is cubic, okay, so all this term as well as this term are going to vanish, so we will have only pressure gradient term present here and you know, there will no change.

So what we see here since these terms will vanish because epsilon is very small, instead of Reynolds number epsilon appears as an important parameter. So, this is called effective Reynolds number for 2 length scale problem is Re effective is going to be epsilon* Re which is rho u0 h divided by eta * h over L, okay. So, this is rho u0 h square over eta L. So that is how you can express Reynolds number or 2 length scale problems.

And here the effective Reynolds number will be small compared to the actual Reynolds number because it is multiplied by epsilon, which is very small compare to 1, okay. So, we stop here.