Microfluidics

Dr. Ashis Kumar Sen
Department of Mechanical Engineering
Indian Institute of Technology – Madras

Lecture - 07
Micro-Scale Fluid Mechanics

Okay. So let us continue our discussion on Poiseuille flow. We have derived Navier-Stokes equation for Poiseuille flow assuming that cross-sections can be arbitrary so we have tried to find an expression for velocity as well as flow rate then we looked at the expression for the velocity and the flow rate in case of flow between parallel plates. Now we go on and talk about flow in a elliptical channel. Okay.

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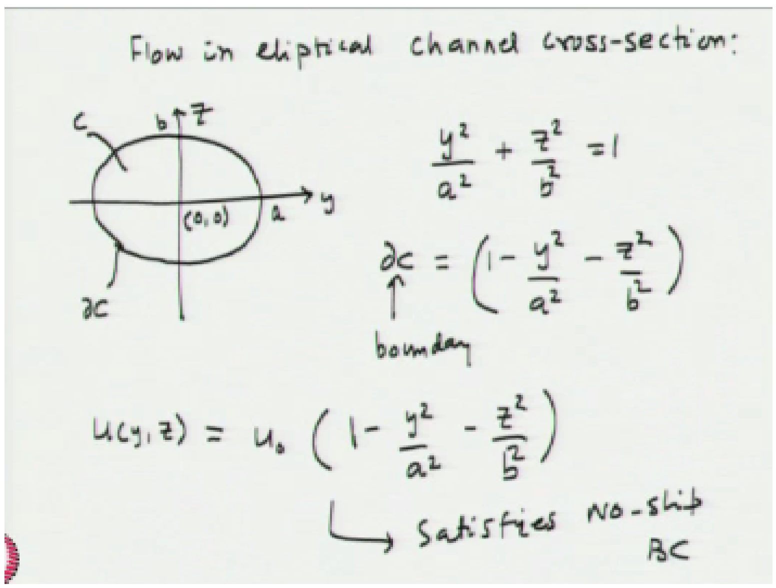

Let us talk about flow in elliptical channel cross-section. Okay. So elliptical channel cross-sections can be fabricated using silicon substrate if we use bulk micromachining Isotropic etching without agitation, okay then you would be able to fabricate elliptical channel cross-sections, so that is a practical importance, okay. So we can draw the cross-section here, so this is the elliptical channel, we can say this is y-axis and this is z-axis.

The origin is at (0,0) and this is a and this point is b and this is the surface which we denote as del c and the cross-sectional area is c. Okay. So we can write the equation of ellipse which is x square over a square + you know in this case we are using y and z, so y square over a square plus z square over b square = 1. So the one cross-section del c becomes 1-y square, this is-- sorry the boundary, the LC becomes 1- square-z square-b square.

So for this case we can assume a trial solution, okay we can assume a trial solution and make sure that the boundary condition is satisfied that means the velocity at the wall is 0, okay. So we can assume a trial solution u(y, z) which is u0*1-y square over a square okay 1-y square a square – z square over b square. So this equation satisfies No-Slip boundary condition, okay.

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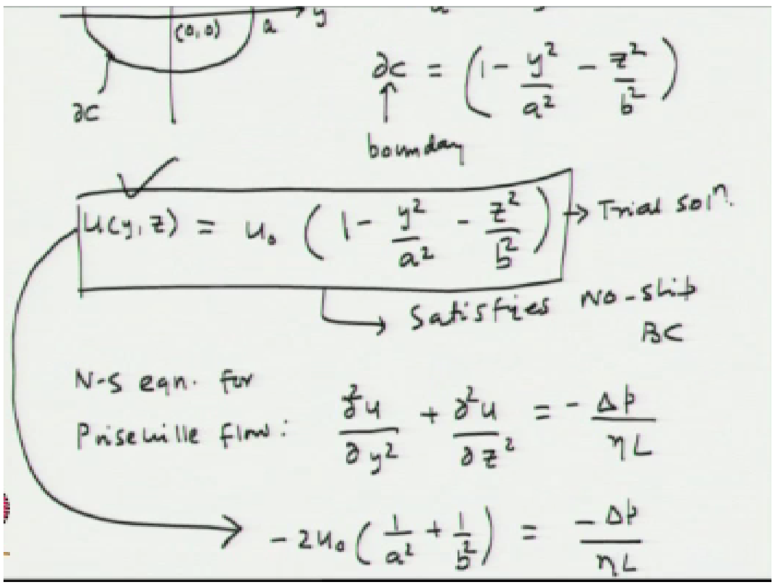

So now if you substitute this in the general equation so the Navier-Stokes equation for Poiseuille flow that we derived so that was del square u over del y square + del square over z square is = -delta p over eta l, so that is the general equation. Now if you substitute this trial solution okay so this is the trial solution, if you substitute there what you get is this you get -2u0 into 1 over a square + 1 over b square = -delta p over eta l.

So you can find u0 to be delta p over 2 eta l*a square b square/a square + b square, okay so that is the expression for u0 and that can be substituted in this equation to get the expression for the velocity field okay which satisfies the boundary conditions.

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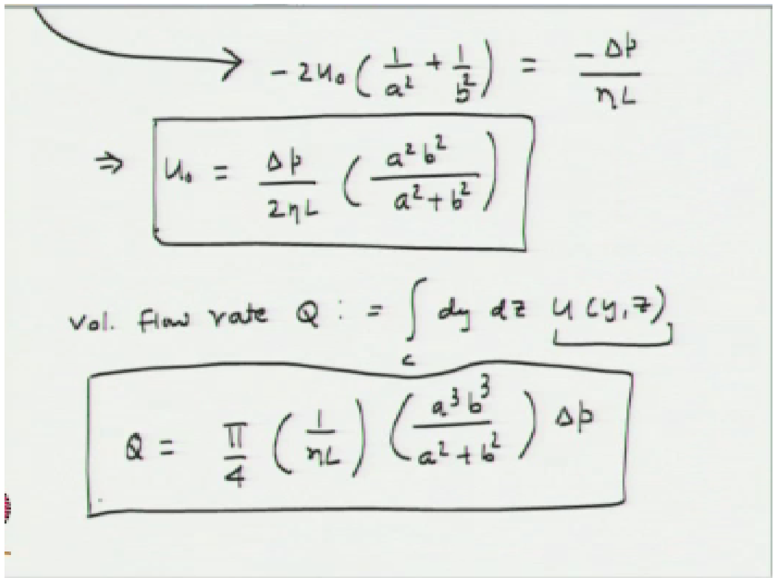

So from there you can find the volume flow rate, the volume flow rate Q can be found so Q will be = integration over the cross-section area dy dz*u (y, z) okay. Now if you substitute the expression for the velocity here, what you would get is Q will be = pi/4*1 over eta l*a cube b cube/ a square + b square*delta p. So that would the expression for the flow rate in an elliptical channel cross-section okay.

So now we move onto talk about channel of circular cross-section and as you know circle is a special case of ellipse when the major access and minor access are equal, okay.

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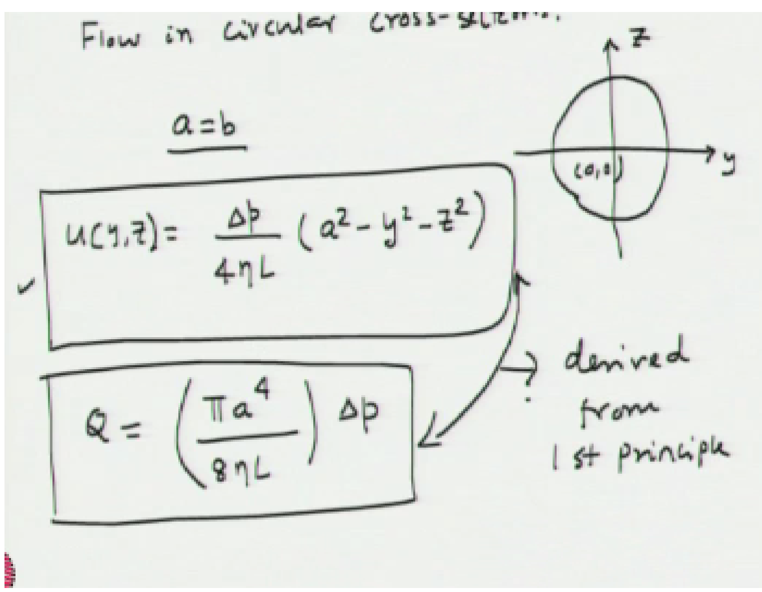

So with that let us talk about in circular cross-sections. So here we say that a=b right for the l is if you write a=b it becomes a circle. So this is a circle; this is let us say y and this is z and origin at 0, 0. For a=b if you substitute in this equation here we can obtain the velocity field. So the velocity field u(y, z) could be obtained as delta p over 4 eta l*a square - y square – z square. Okay.

So basically what you do is you substitute a=b in this situation so get an expression for u0 and then substitute here to get the velocity field, so that is what you would get. Okay, so this is nothing but the expression for velocity field in case of flow through a circular cross-section okay in Cartesian coordinate. So from here you can also find flow rate, flow rate q = pi a 4/8 eta l*delta p. So if you substitute a=b in this equation this is what you would get. Okay.

Now these two expressions can also be obtained by, you know this can be derived from 1st principle, okay. So we can also do that.

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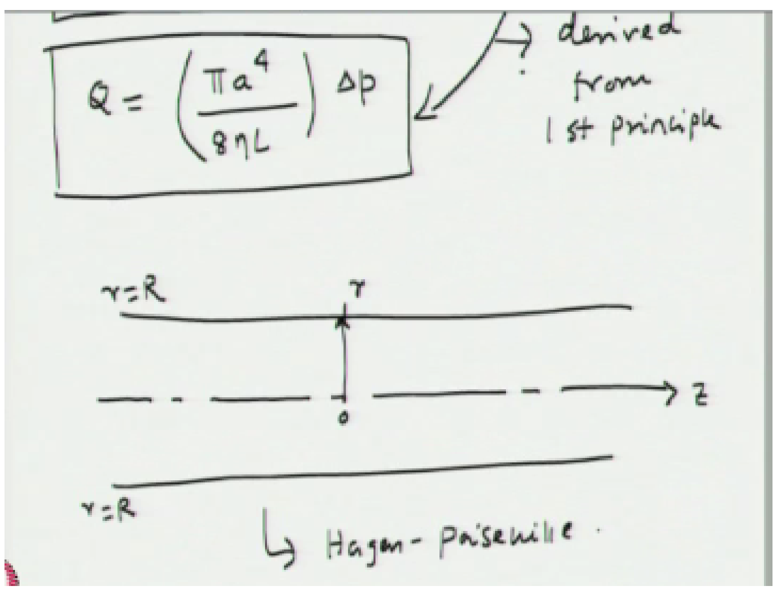

So for that we would consider a tube of circular cross-section okay. So this is a circular tube. Let us say this is the radial direction this is 0 and this is axial direction z, this is r=R, r=R okay. So this is nothing is called Hagen-Poiseuille flow, okay. Hagen-Poiseuille flow is a special case of Poiseuille flow where we are talking about flow in circular channel cross-sections. Okay.

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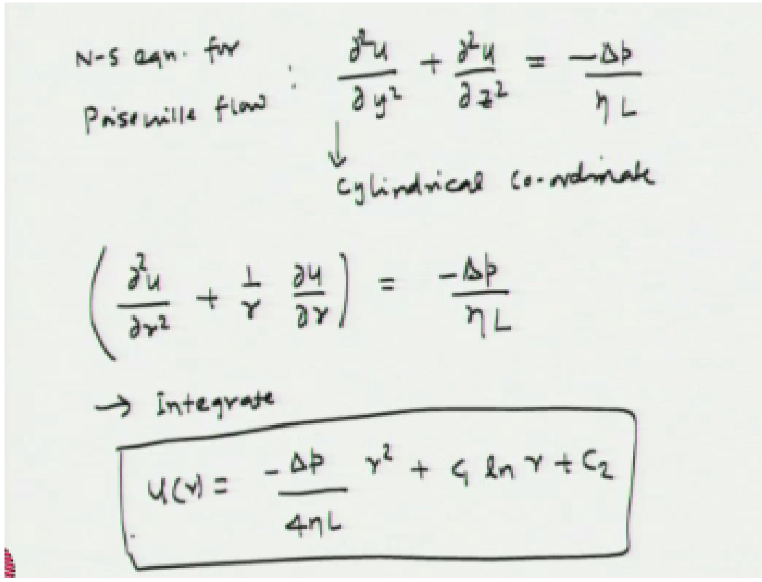

So let us go back to general equation Navier-Stokes equation for the in Poiseuille flow case. So if you write in the same Cartesian coordinate the equation in Cartesian coordinate del square u over del y square + del square u over del z square = –delta p over eta l. If you write in cylindrical coordinate, okay if you write in cylindrical coordinate it will look like this okay del square u over del r square + 1 over r del over r will be equal to –delta p over eta l.

So now if you integrate twice, if you integrate this equation twice what you get is this, you get u(r) will be = -delta p over 4 eta l*r square + c1 learn r + C2, okay. So this is the equation that you get by integrating the Navier equation.

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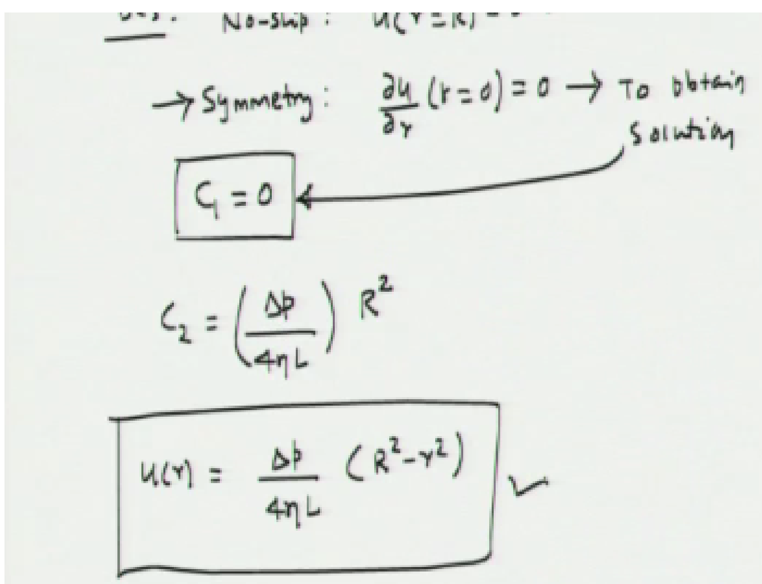

Now if you apply the boundary condition, what are the boundary conditions? One boundary condition is that the velocity on the wall is 0 that is No-Slip boundary condition and the other boundary condition is symmetry condition okay so the velocity gradient at the centre vanishes, okay. So the first one is No-Slip which says that u(r)=r is 0 and the other one is symmetry. Symmetry boundary condition says del u over del r at r=0 = 0, right.

So these are two boundary conditions. Now if you look at this equation and try to apply second boundary condition the symmetry boundary condition so you would see that this should require that C1 will be 0. Okay, since you have a long turn here if you C1 is not 0 then it is not possible to obtain a solution. So to obtain solution the C1 is to be 0. Okay. So then you can apply the No-Slip boundary condition this condition if you apply that you would get C2 to be delta p over 4 eta l*r square.

So you can write down the general expression for the velocity so you u(r) would be delta p - 4 eta l*R square - R square okay, right. So while it is not very obvious to know you know the nature of the curve here the velocity profile here from the cylindrical coordinate you can see that this is parabolic okay. So you can, the velocity profile is parabolic with the maximum velocity occurring at the center okay.

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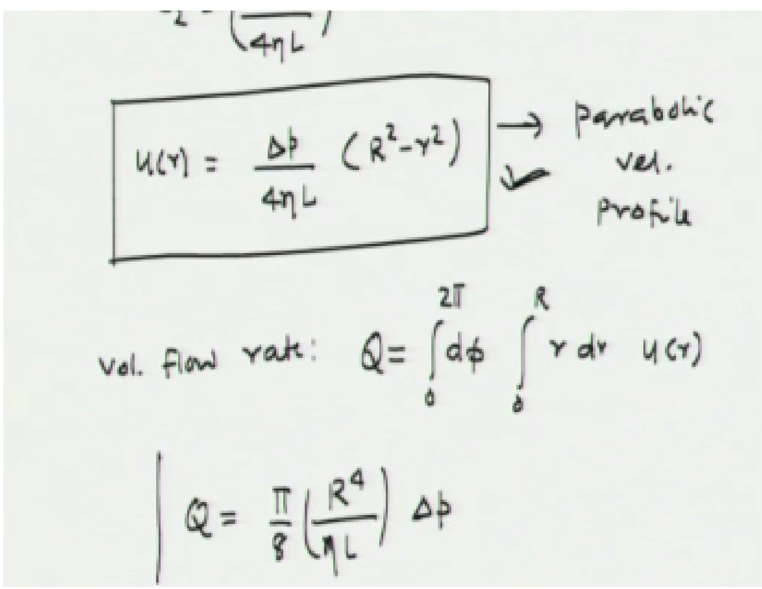

So this is this is a parabolic velocity profile. You can obtain the volume flow rate Q would be 0 to 2 pi over pi coordinate*0 to R, r dr*u(r), okay. So if you do that you will get Q= pi/8 r/4 divided by eta l*delta p, okay. So that is the expression for the flow rate in a circular channel cross-section. Now here we have made one assumption okay, in this equation we have made assumption that u theta is 0 and u-- sorry yeah.

This is the angle pi u along azimuthal coordinate u phi 0 and u of r=0. Okay. So this we have made and this u is only a function of r and this is along z direction okay.

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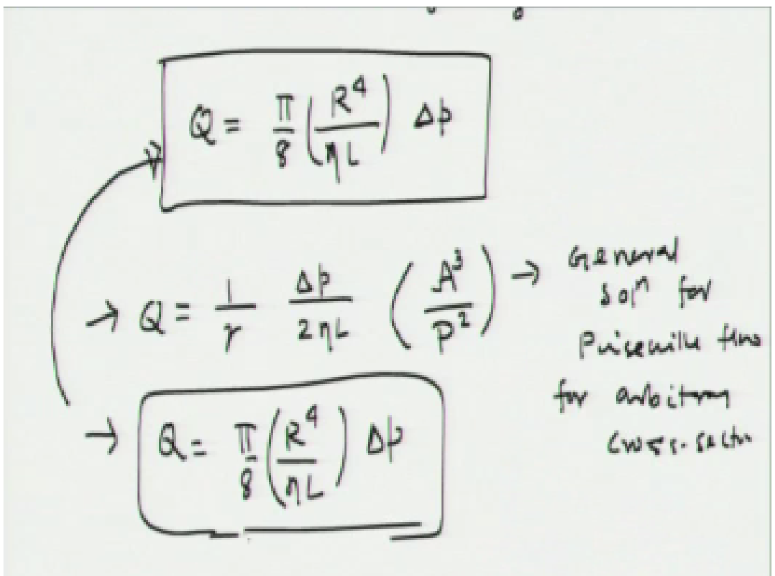

Now if we go back and we see you know general solution that we obtained yesterday for a Poiseuille flow case for an arbitrary cross-section the expression we got was Q=1 over gamma delta p/2 eta l*A cube over perimeter square. So this is the general solution for Poiseuille flow that we obtained earlier, general solution for Poiseuille flow for arbitrary cross-section, okay. Now if you apply this to a circular cross-section you would get exactly the same expression as here. Okay.

So if you apply to the circular cross-section then you would get Q=pi/8 R4 eta l*delta p okay. Right. So you know the circular cross-sections are also have practical relevance in Microfluidics, it can be fabricated in silicon wafer, if you do you know bulk micromachining isotropic etching with solvent agitation you would be able to produce circular channel cross-sections. Okay.

Actually it is a semi-circle that you will be able to produce and if you (()) (16:51) two such semi-circle channels you will be able to generate circular channel cross-sections.

Now another important point I want to discuss here, how do you define Reynolds number in case of Poiseuille flow, Hagen-Poiseuille flow where we have flow through in circular cross-sections and the flow is pressure force, okay. So in this case there is no characteristic velocity that is imposed onto the system. The preserved gradient is actually driving the flow. How do you define Reynolds number in this case?

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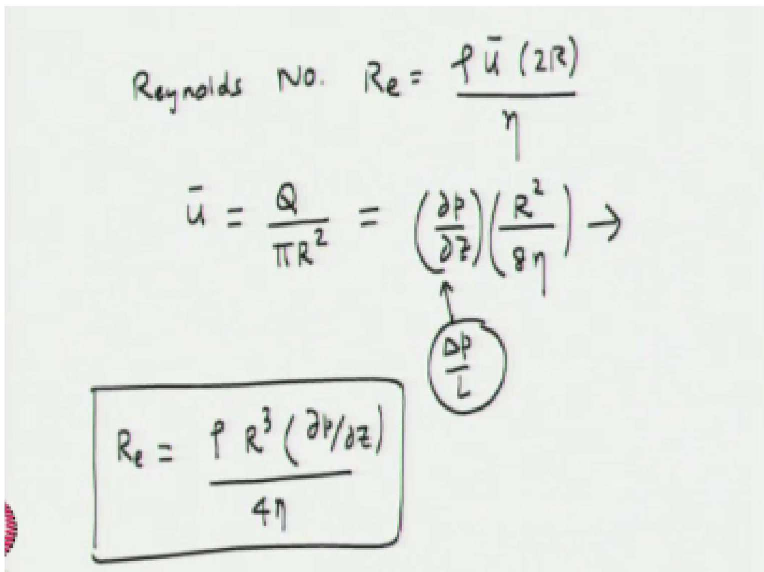

So Reynolds numbers is typically defined as Reynolds number is defined as Re=Rho u lets say here the diameter will be 2r/et la, so that is how you define Reynolds number. Now here the average velocity is a function of a pressure gradient okay. So average velocity u can be written as —if you know the flow rate okay, we know flow rate here and if we divided by the area of cross-section then we would what would be the average velocity.

So average velocity will be Q/pi r square okay, so if you divided pi r square here what you get is you know delta p over delta z okay. This is nothing but del p over L, okay because pressure is varying linearly and into R square and 8 eta l, okay so that is the expression for the average velocity. So if you have to define the Reynolds number or Poiseuille flow the correct way to define would be Rho*you know you will have R cube term appearing here and delta p over delta z/4 eta*eta square.

The flow in triangular channel cross-sections. Triangular channel cross-sections can be obtained again using you know microfluidic devices fabricated from silicon wafer, if you do an isotropic etching you will be able to produce channel cross-sections of triangular shape, okay typically in an isotropic etching you know one who wafer you would get an angle between the you know the inclined wall and horizontal about 54.74 okay.

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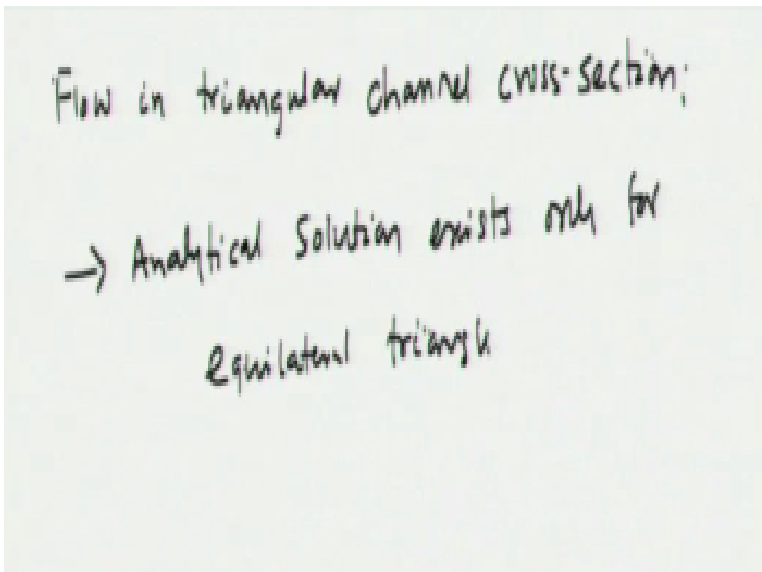

So let us talk about Flow in triangular channel cross-section. So unfortunately we do not have analytical solution for flow through triangular cross-section of any triangular shape only for if it is three sides are equal only if it is equilateral triangular then we will be able to derive an expression for the velocity. Okay. So if the analytical solution exists only for equilateral triangle. Okay?

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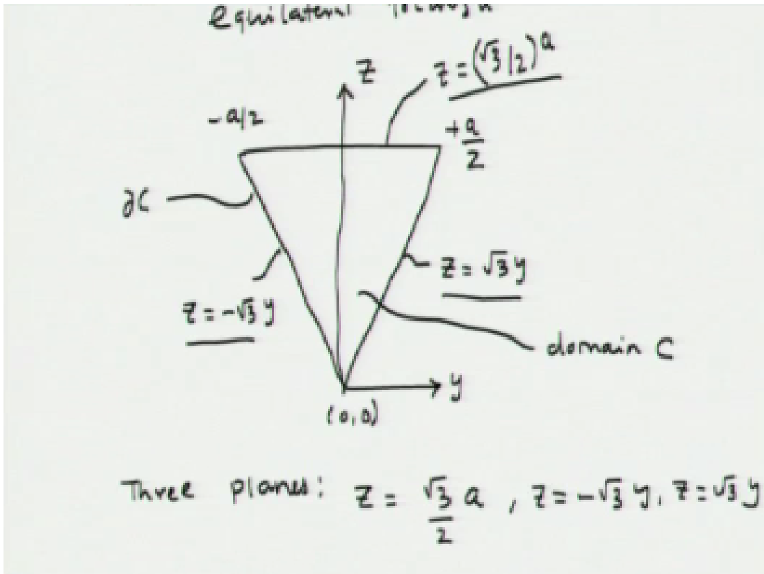

Let us draw an equilateral triangular cross-section which would look like this. So this is our y and this is our z, this is a/2 and this is –z/2 and the equation of this line is going to be z=square root of 3y and this line will be z=-square root of 3*y and this line would be z=square root of 3/2* a and we have the origin here and this is the boundary del c and we are talking about this domain which is c okay.

So in that case we see that the triangle is bounded by 3 lines okay, 3 lines here. So the you know we can come up with so there are three planes that is coming the domain one is you know z=square root of 3/2a okay and the second one is z=-square root of 3*y and z=squarer root of 3*y. So these are three different planes that is forming the domain. Okay. So we can write trial solution similar to the elliptical case and write you know trial solution u(y)z= u(0)/a cube*z and say square root of 3/2 into a-z*z-square root of 3y*z + square root of 3y.

So this is our trial solution okay. So since we are multiplying three different planes here we are dividing the, a is the side length of each side. So a is length of each side of the triangle.

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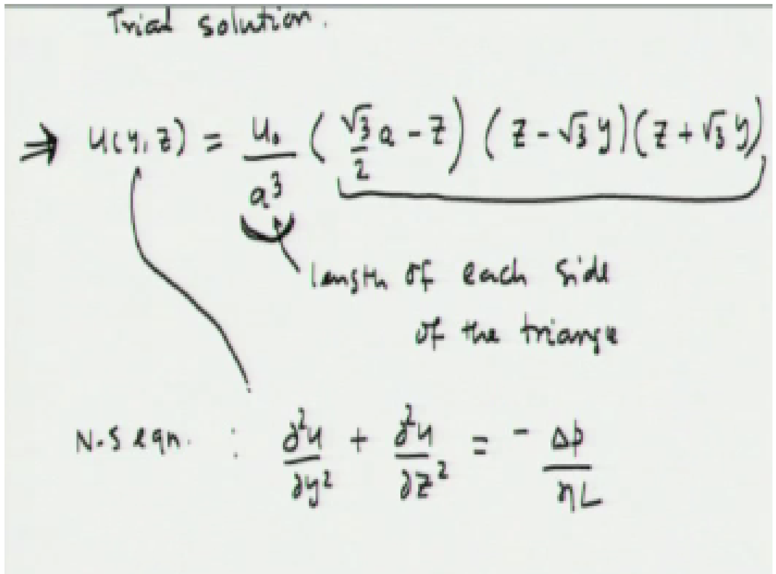

So you know, if you substitute this trial solution in our Poiseuille flow equation so Navier-stoke equation is del square u over del y square + del square u over del z square is –delta p over eta l. If you substitute in this equation what you would get is this, you would get is this.

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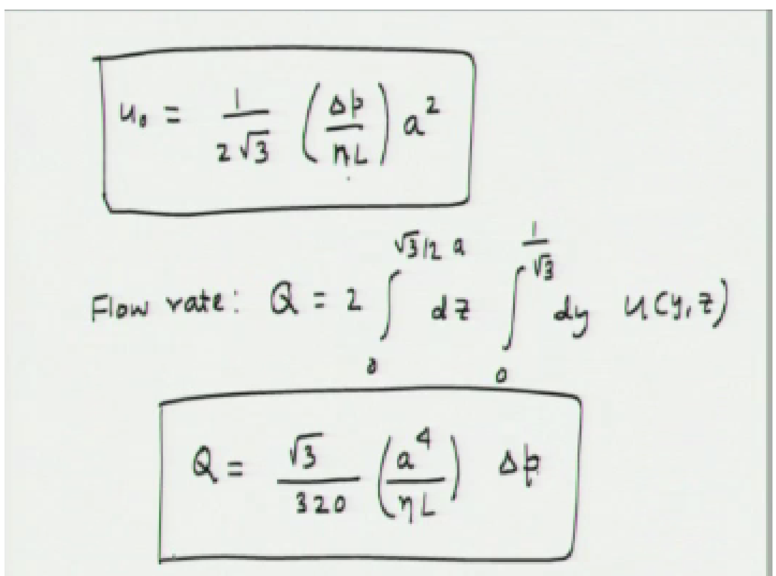

You would get u0= 1 over 2 square root of 2 into delta p over eta l okay, eta*l* a square. So that is the expression for u(0) which can be in this equation to get the velocity field. So from there you can find the flow rate, Q is 2*0 cube square root of 3/2a dz 0 to 1 over square root of 3 dy*u(y,z) okay. So if you do that we can get an expression for Q which looks this square root of 3 or 320 into a4/eta l*delta p. So that is the expression for the flow rate through a triangular channel. Okay.

So next move on to talk about flow through a rectangular cross-section. Flow through rectangular cross-sections is very important in Microfluidic is very widely used especially for polymer best microfluidic devices you know the methods that we fabrication methods that we normally use in (()) (26:45) to channels that are of rectangular cross-sections. Okay.

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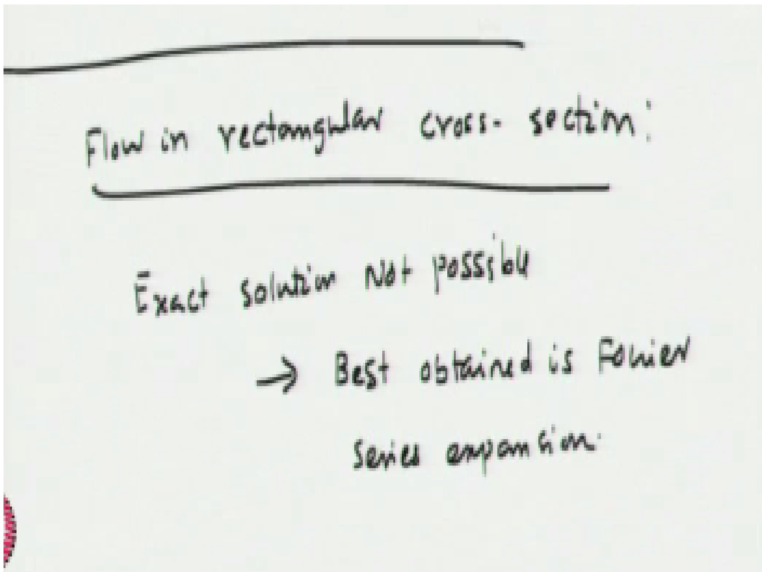

So now let us talk about flow in rectangular cross-section, okay. Unfortunately, in rectangular cross-section we do not have any exact solution, okay. The best obtained is a Fourier series expansion to approximate for the velocity profile, okay. So exact solution not possible here. Best obtained is Fourier series expansion. So let us look at a rectangular channel. Let us draw rectangular channel here.

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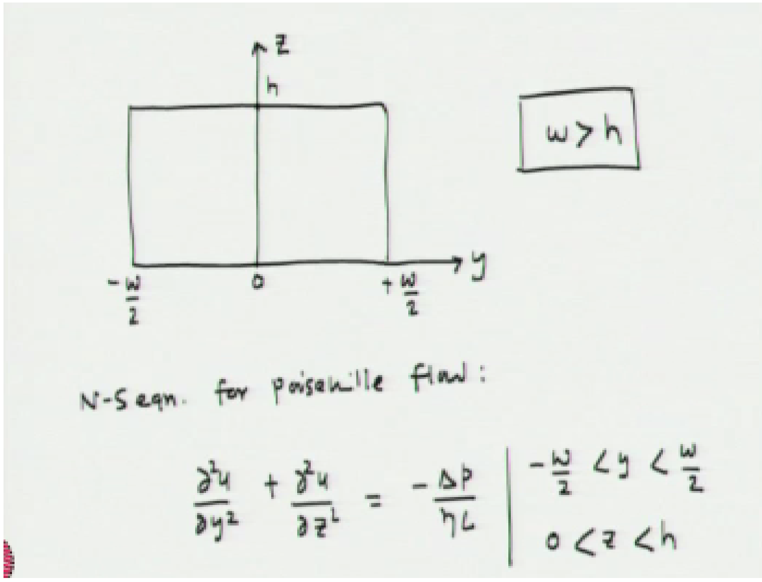

So this is let us say z and this is y 0 is here –w/2+w/2 and the height is h, so we are talking about a rectangular channel which has width w and height h and we assume that the height h is smaller than w. okay this can always be attained in a case where height is more than w then this is possible to rotate the channel and obtain the height < w. okay. So we say that w is greater than h. So this is always possible to maintain. Okay.

So we again go to the Navier-Stokes equation for Poiseuille flow okay which is L square u over del y square + del square u over del z square = –delta p over eta l. okay. So this we want to apply to this domain where –w/2 is < y < w/2 and 0 is < z < z okay so this is the domain that we want to where we want to apply this equation okay.

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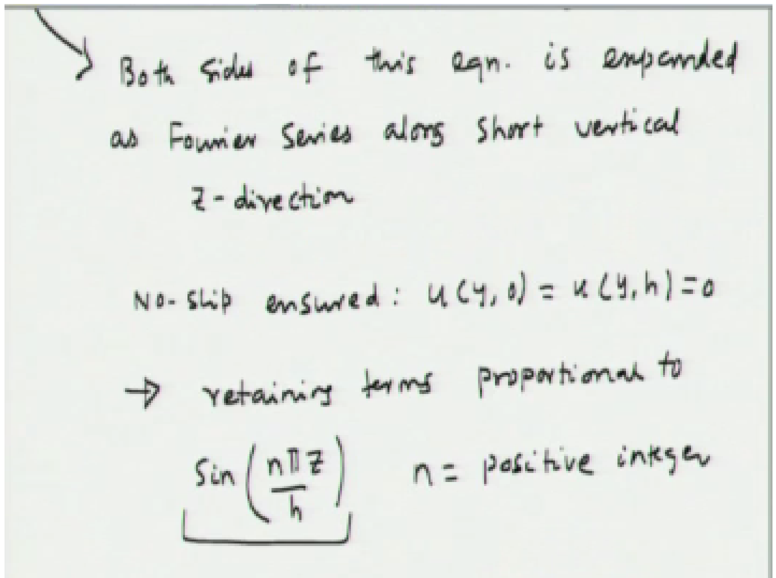

So and here as you know since it is parallel flow u is a functional y and z only and the boundary conditions, you can write down the boundary conditions u(y, z) is going to 0 or y=+- w/2 and z=h. okay. So that is the boundary conditions. Now we will take an approach where you know the term in that general equation for the Navier-Stokes equation for Poiseuille flow will try to express both sides in Fourier series okay.

And we will write Fourier series in the direction of z okay, in the z direction which is the shortest dimension of the channel okay. So both sides of this equation both sides of this equation is expanded as Fourier series along the short vertical z direction. And since we have to ensure No-Slip ensured that means u(y, 0) =u(y, h)=0.

So the velocity is vanishing both at the bottom and top walls you know, you ensured that by you know retaining terms proportional to Sin n pi z over h, so these terms are used where n is positive integer. Okay. So now let us first do the Fourier series expansion for right hand side, okay.

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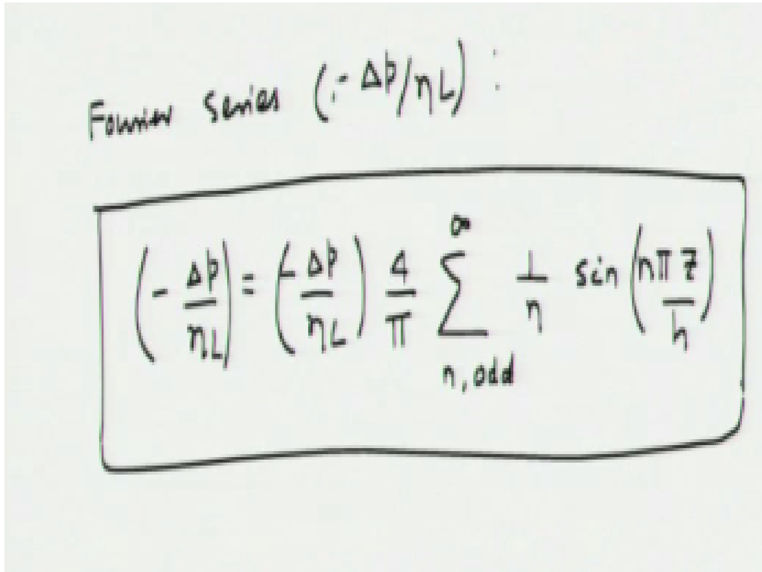

So Fourier series for –delta p over eta l, okay. So if you write –delta p into eta l in Fourier series, we can write -delta p over eta l will be –delta p over eta l*4/pi summation n, odd, only odd values of n used infinite, 1 over n*sin n pi z over h. okay. So this is how we can express right hand side minus delta p over in Fourier series okay, so this summation term will be pi/4 which will get cancel so you have the same term –delta p over eta l. okay.

So now let us try to write the Fourier series expansion for the left hand side. Okay.

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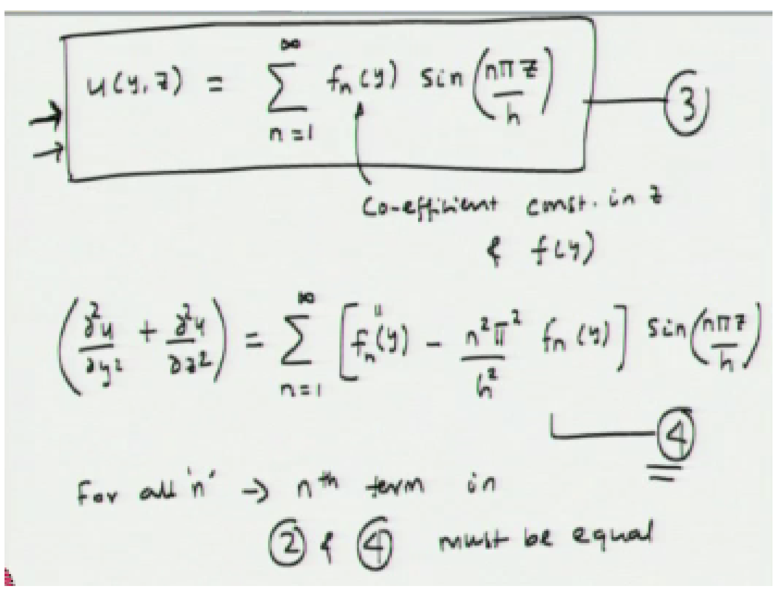

So Fourier expansion of u(y, z) okay. So we are not expanding the gradient term but we are writing for the velocity field. Okay. So the velocity field u(y, z) can be written in Fourier series as this, so n=1 to infinite fn(y) and into sign n pi z over h. Okay. So this is how we can write velocity field in Fourier series. So this is a coefficient which is constant in z and function of y. Okay. Now if you substitute this and this in this equation let us call it equation 1 okay.

So what we would is this, we would get del square u over del y square + del square u over del z square, so if you substitute this equation here on the left hand side this is how it will expand. This will be equal to summation n=1 infinite fn double dash y –n square pi square over h square*fn(y) okay, so just we are substituting this in the left hand side, this is what we would get*sin n pi z over h. right. So let us call it equation 3, let us call this equation 2.

So now if you compare you know equation 3 and equation 1 okay each and every term must be equal. Okay. So you know for all values of n the nth term in 1 and equation 3 must be equal okay. So they must be equal. Now and so what we see here this is the-- we are comparing with sorry—let us call this equation 2, equation 3 and this to be equation 4. Now this is the right hand side and that we are comparing with equation 4.

So we are in fact comparing equation 2 and 4 okay. So equation 2 is the Fourier series for delta p over eta l and equation 4 is the expression the velocity gradient okay, viscous term. So if you know-- here n is odd only okay. So what we learn from there is—

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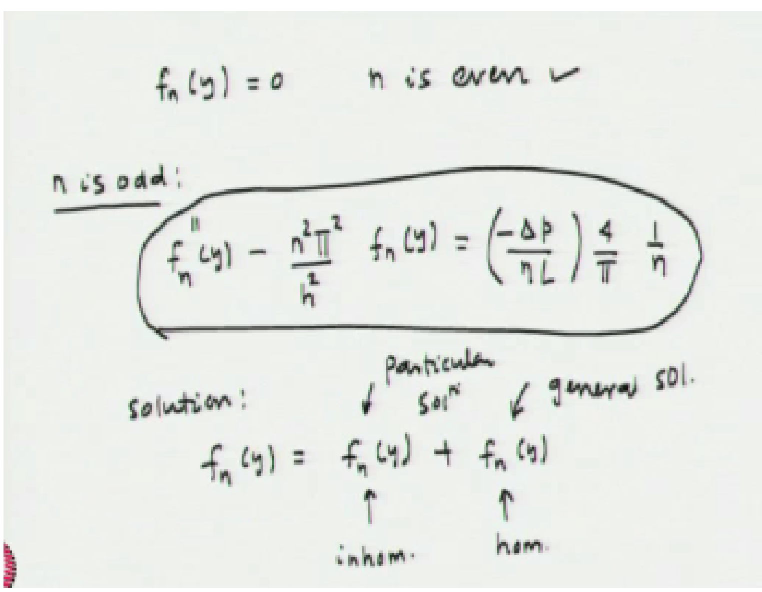

Fn(y) must be 0 if n is even, okay. For any even this term is 0 right so here only n is odd, so in this equation if n is even since here we are not talking odd or even, if n is even here then the function must vanish, so fn must be 0 if n is even and if n is odd n is odd then we can say that fn double dash y- n square y square over h square fn(y) = –delta p over eta l*4/pi*1 over n and this is when n is odd okay.

Now if you want to solve this equation this differential equation it is a seconded differential equation and it is an equation because the constant part is not 0 so this is an inhomogeneous equation. To solve an inhomogeneous equation, we need to divide the solution into both homogenous as well as non-homogenous solution okay. So we can write the solution fn(y) to be fn(y) inhomogeneous.

Let us say this is inhomogeneous and fn(y) homogenous, and this is also called the general solution and this is particular solution okay.

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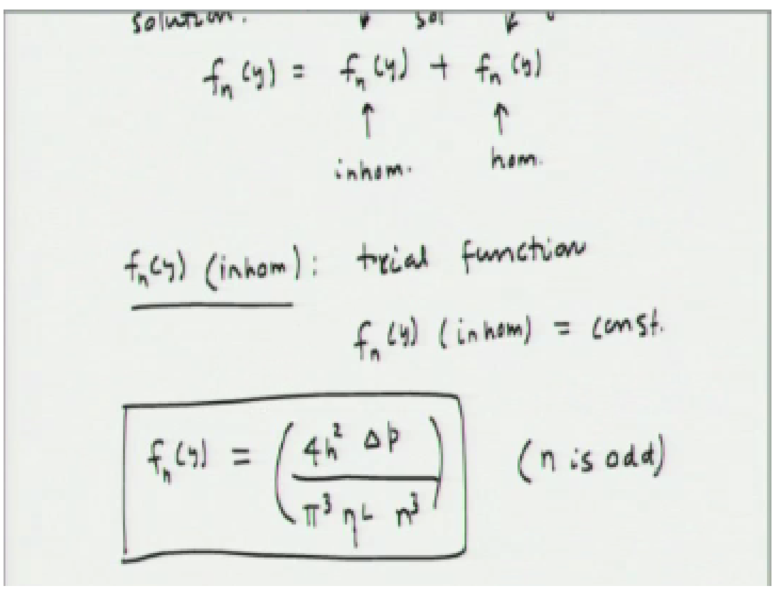

Now to obtain the inhomogeneous solution so fn(y) inhomogeneous obtained that one trial function we can use a trial function you know fn(y) inhomogeneous to be constant. And if this is constant for the inhomogeneous solution then this term will vanished okay this gradient term will vanished so in that case we can find what is fn(y) okay. So the function fn(y) for the inhomogeneous case would be 4h square into delta p/pi cube eta l*n cube so eta must be differentiated from n.

So this is the inhomogeneous solution and here n is odd. Okay. Now to find the homogeneous solution or general solution the right hand side must be made 0.

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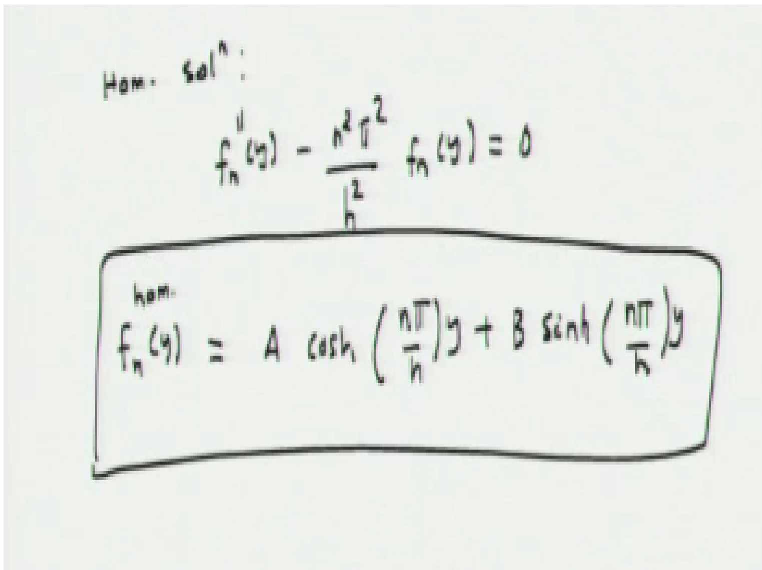

So for homogeneous solution we would have the equation Fn double dash y – n square pi square over h square*fn(y) will be = 0. So we can write the solution fn(y), so this is inhomogeneous and here we can write is homogeneous will be = a cos hyperbolic*n pi over h*y+b sin hyperbolic*n pi over h*y. Okay. So this will be the homogeneous solution, so we can write the total solution here okay.

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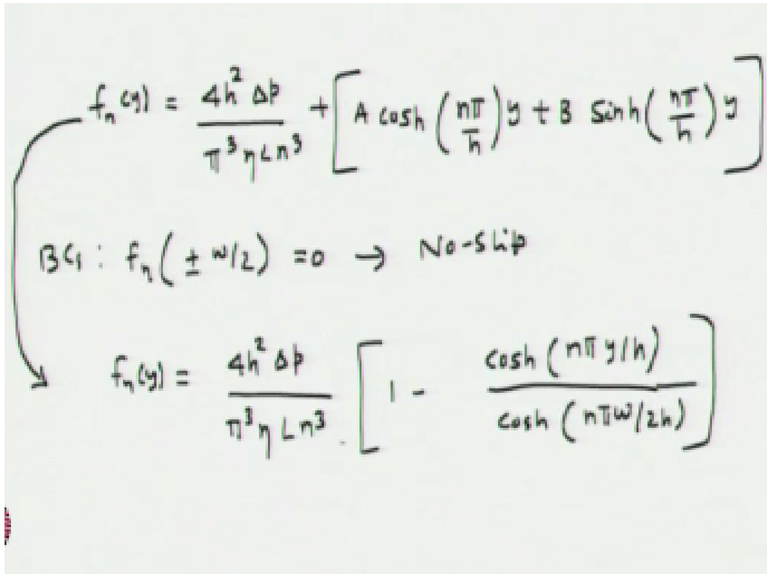

So we can write the total solution fn(y) will be = 4 h square*delta p/pi cube eta l n cube + a cos hyperbolic n pi over h*y+b sin hyperbolic n pi over h*y. So if you substitute in our in a velocity equation but before that lets try to apply the boundary condition our fn will vanish on the walls +- w/2 has to be 0 so this is the boundary condition mostly boundary condition. So if you apply that boundary condition fn can be written as Fn(y) will be = 4h square delta p/pi cube eta l cube*1- cos hyperbolic of n pi y over h/cos hyperbolic n pi w over 2 h.

So that is how we can write for fn(y).

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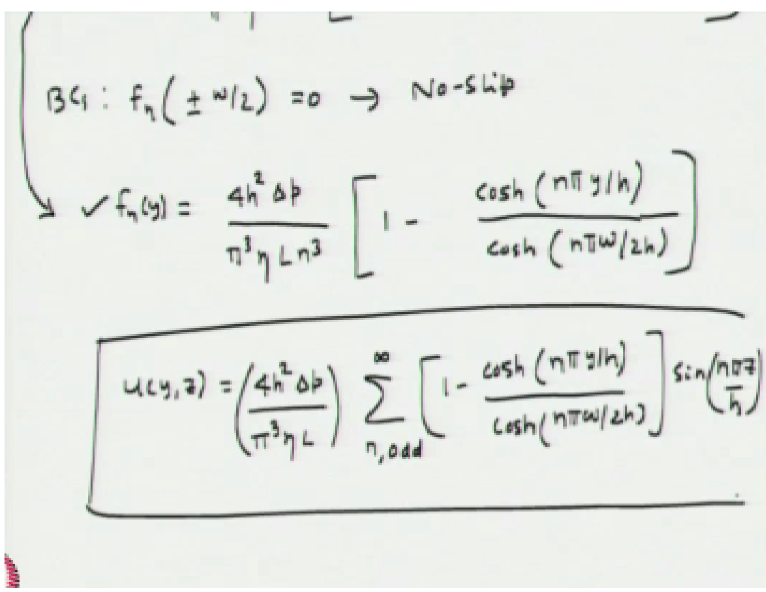

Now if you substitute this in our velocity profile, so u(y,z) can be written as, so we have written how u(y,z) can be written here okay, so this can be written as 4 h square delta p/pi cube eta l*summation n is odd right this is what we have written here, n must be odd right because for n even this is vanishing so only n is odd so n odd to infinite*1- cos hyperbolic n pi y over h/cos hyperbolic n pi w/2h into sin n pi z over h. okay. So this is the expression for the velocity.

Now if you plot this here if you plot here, okay in this domain what you would get is this.

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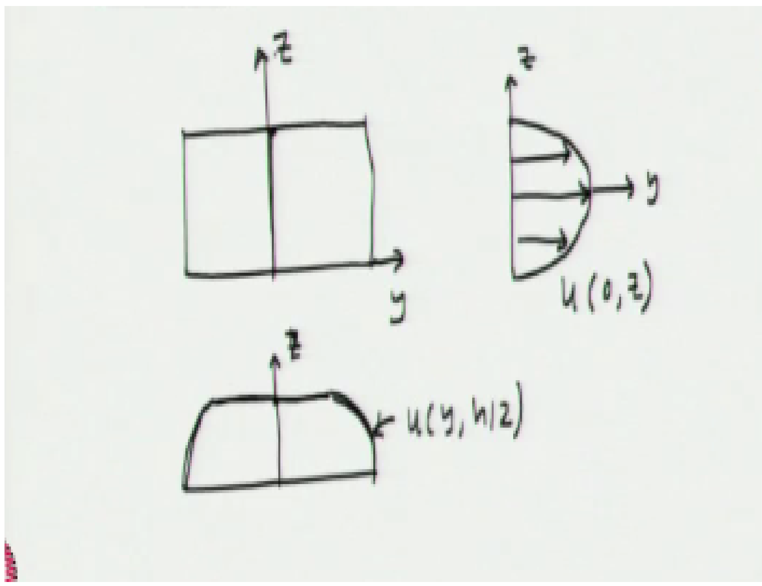

So let us say this is our domain, this is z, this is y, what you get is this so here across this since the walls are closed to each other the viscous affects make this velocity as parabolic, okay. So this is you know this is z direction and this y and this is u at y=0 and with z. Similarly, in this trend if you try to draw the velocity profile it will look like this, it will be parabolic around the edges then it will look flat and then it will become parabolic.

So this or you know this our z direction right and this will be u (y, h/2) so this h/2 okay, it is in the middle plane this becomes flat and this still parabolic around the edges.

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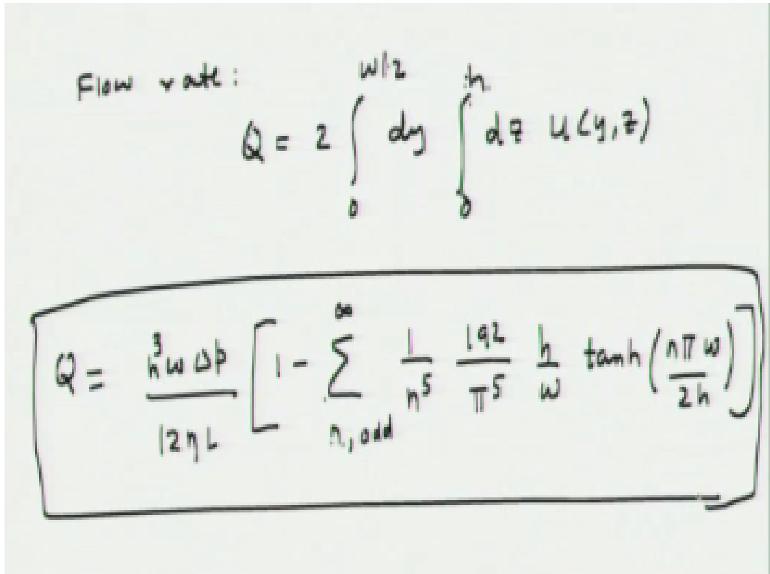

So now we can derive and expression for the flow rate so you can write flow rate Q will be = 2*02 w/2 dy to the 02 h dz*u(y, z) which will be now this, Q h cube w delta p/12 eta l*1- summation n, n is odd infinite 1/n 5, 192/pi 5*h over w*tan h hyperbolic n pi w over 2h, okay. So that is expression for the flow rate that we can calculate during the velocity field. Okay. Now in the limit h/w tending to 0 okay what you get.

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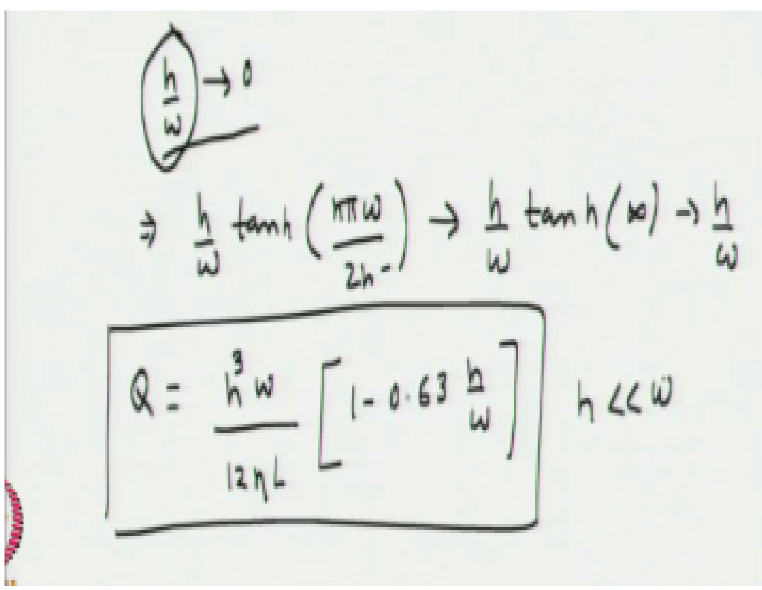

So in that limit what you get is the this term okay h/w tan h h* n pi w/2h would become h/w*tan h hyperbolic infinite. So h/2 is 0, so the height is very small compare to width so the width term will become infinite and so this term will become h over w. So this is the case where the height is very small as compare to the width, so in that case we can write Q=h cube w/12 eta l*1-0.63 h over w. Okay. So this is in the limit where h is less than w.

Now surprisingly this simplified expression for the flow rate where the height is << w is very accurate, okay. When w is,

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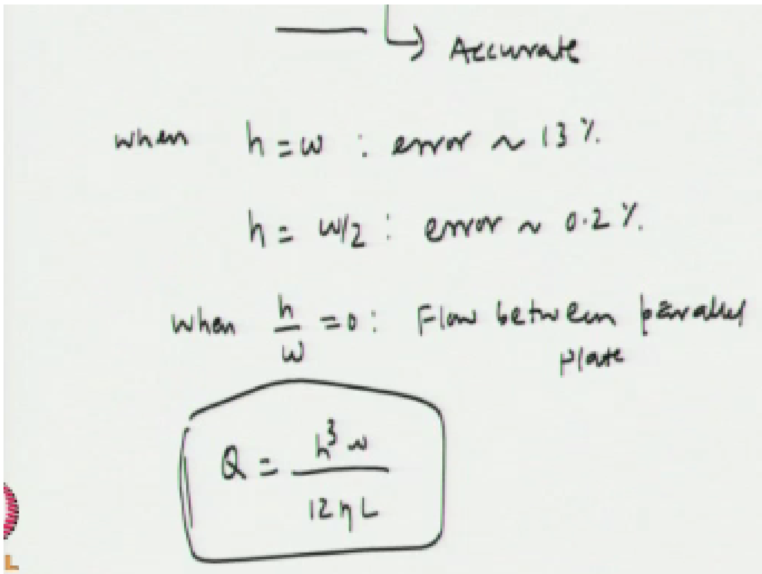

So this equation is accurate when h=w the error is about 13% and when h=w/2 the error goes down to 0.2% okay. So this simplified equation can be used to analyze flow in rectangular cross-sections when you know if h is little bit < w then also it will give you accurate results. And in the worst case when the h=w then also the error is about 13%. Okay. And you know if h is very small compare to w in that limit it becomes like a flow between parallel plates. Okay.

So when h over w become 0 then flow between parallel plates, so in this equation if you put this to be 0 what will happen you get Q to be = h cube w over 12 eta l, so this is the solution we have seen earlier for flow between parallel plates. So let us stop here today.