Microfluidics
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Lecture - 06
Micro-scale fluid mechanics (Continued.)

Okay so let us continue our discussion on some of the basic flow solutions. Earlier we had looked at how we can obtain an expression for hydro static pressure drop from the Navier stokes equation. And then we also looked at in a flow of thin film of liquid in an inclined plane. Now we will look at it Couette flow situation. In Couette flow we have you know either one or more plate moving related to each other that actually creates flow.

There is no pressure gradient there is no body force and the flow occurs simply because of the movement of the plates. Okay the applications of Couette flow could be in the Rheology where we can you simply determine the viscosity of a fluid. Typically, we have 2 concentric cylinders and would have a fluid present in the annular space whose viscosity we want to determine and if we maintain a constant speed of rotation the torque.

That is required to maintain that constant speed will be related to the viscosity. Okay that is how we would be able to determine the viscosity. So, here we look at simple case flow between 2 parallel plates where you know one or 2 plates are moving okay.

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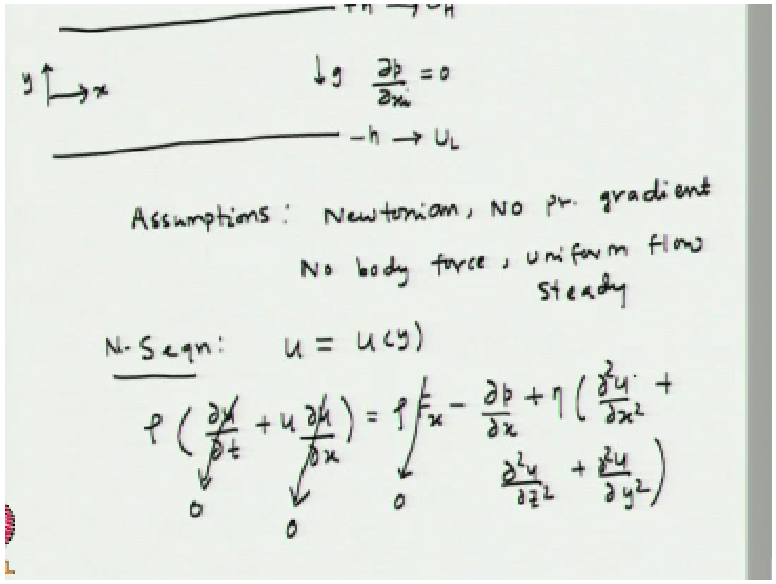

So, we look at Couette flow so here we would have 2 parallel plates so this is the region here x y and this is at +h and this is at-h this plate is moving at a velocity of uh and this plate is moving at velocity of UL and we have gravity acting in this direction and the de1p/del xi=0.So, there is no pressure gradient okay and we make certain assumptions make certain assumptions we assume that the fluid is Newtonian.

There is no pressure gradient acting and there is no body force present okay and flow is uni directional and uniform. So, the velocity component and the gradient of the velocity are orthogonal to each other. So, the convection term drops off from the Navier stokes expression. So, that assumptions we have Newtonian assumption Newtonian no pressure gradient no body force and uniform flow okay.

So, under those assumptions Navier Stokes equation you can write the u will be a function of y only right were talking about infinite parallel plates so the z coordinate will not be important and u is a function of y only it is not changing along x direction. So, under that assumption we can write the Navier Stokes equation rho * del u/del t + u del u/del x=rho*fx-del p/del x + eta*del square u/del x square+del square u/del z square +del square u/del y square.

So, here another assumption is flow is steady. So, this would term would drop off the flow is uniform the velocity and the gradient are orthogonal, so this term will drop off there is no body force term and u is only dependent on y so del u/del x=0 also you can consider that the z components is infinite this is also not important okay.

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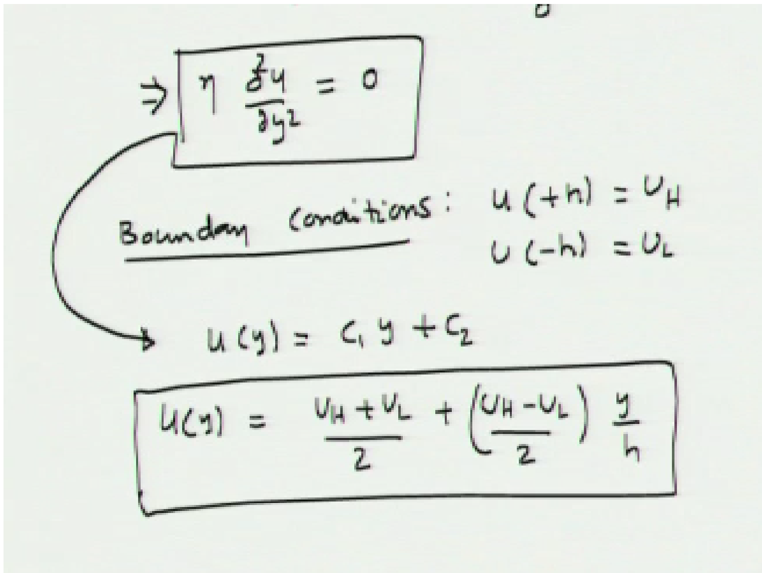

So, the equation we get there is no pressure gradient but this term will be 0 so we say that the viscous force del square u/del y square eta del square u/del y square is going to be 0. So, now if we talk about the boundary conditions the velocity on the top wall +h will be u here if we look at UH okay and will u at -h will be UL okay. Now if this equation if you integrate this equation you get u*y as c1y+c2.

So, if you apply the boundary conditions you would get that u*y will be UH+UL/2+UH-UL/2*y/h okay. So, this is how velocity is going to vary in the y direction. So, this equation satisfies the boundary conditions at both the top and bottom walls where the velocity vanishes and the velocity is linear in linearly dependent in the y direction okay.

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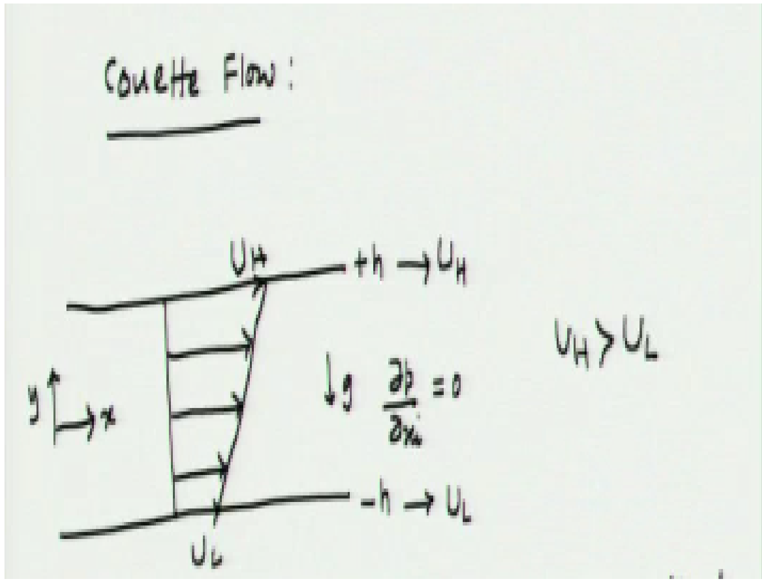

So, if you look at the situation if you try to plot how the velocity is varying it would thus UH is > UL and it will vary something like this okay where this will be UH and this will be UL and this will be very nearly okay. Now from here from the velocity profile we can find out what the value of shear stress is okay. So, if you write the shear stress stanza tau ij = so let us move on to the next page.

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$$\tau_{ij} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix} = 2\eta\,\epsilon_{ij}$$

$$= 2\eta \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix}.$$

Tau ij- the 9 components tau xx, tau xy, tau xz, tau yx, tau yy, tau yz, tau zx, tau zy, and tau zz. So, if you look at the relation between stress and strain this will be 2 eta epsilon ij this we can see from the constitutive relationships. So, this will be 2 eta* the strain components so epsilon xx, epsilon xy, epsilon xz, epsilon yx, epsilon yy, epsilon yz, epsilon zx, epsilon zy, epsilon zz.

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$$= \begin{bmatrix} 2\eta \frac{\partial u}{\partial x} & \eta\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) & \eta\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) \\ \eta\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) & 2\eta\left(\frac{\partial v}{\partial y}\right) & \eta\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right) \\ \eta\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) & \eta\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right) & 2\eta \frac{\partial w}{\partial z} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \eta\left(\frac{U_H-U_L}{2h}\right) & 0 \\ \eta\left(\frac{U_H-U_L}{2h}\right) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Now you should write down the expression for the strength of the metric should look like this it will be 2 eta you write the strain in terms of velocity component so del u/del x and this will be eta*del u/del y+delv/del x and this will be eta*del u/del z+del w/del x and the second one will be eta*del u/del y+del v/del x this will be eta* sorry 2 eta*del v/del y and his will be eta* del v/del z+del w/del y.

And the last row would be eta*del u/del z+del w/del x this will be eta*del v/del z+del w/del y this will be 2 eta del w/del z. Now in this general stress stanza if you put down the values for the velocity gradient then you will see this will be 0 this will be eta*UH-UL/2h this will be 0 this will be eta *UH-UL/2h 0 0 all others will be 0.

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$$\tau_{ij} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix} = 2\eta\, \epsilon_{ij}$$

So what you see here is only tau xy and tau yx exists okay.

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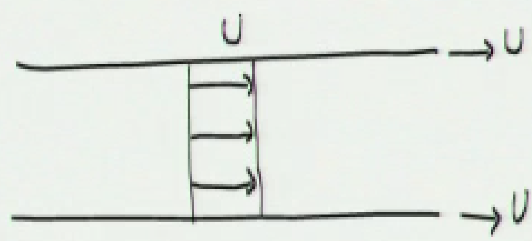
$$\boxed{\tau_{xy} = \tau_{yx} = \eta\left(\frac{U_H - U_L}{2h}\right)}$$



So, you can write tau xy, tau yx=eta*UH-UL/2h okay right. Now from this equation you can see that if the velocity of upper plate and the lower plate are equal if UH=UL then the shear stress is going to vanish okay. So, what that means is that if you have 2 plates here okay and this is moving at U and this is also moving at U. How do you expect the velocity profile to be if you go to this velocity profile equation this term will drop off so this will be 2U/2 so it will be U right?

So, the velocity will stay at U for okay so this your stress is going to drop off. Now how you can define Reynolds number in this case in a you know flow through a pipe of uniform cross-section we know that the Reynolds number will be rho ud/mu right or eta. In this case what should be the velocity scale 1 plate is moving at UH the other plate is moving at UL okay. So, the way we should correctly represent the velocity scale that should be UH- UL the relative velocity should be the scale how the flow velocity need to be correctly represented.

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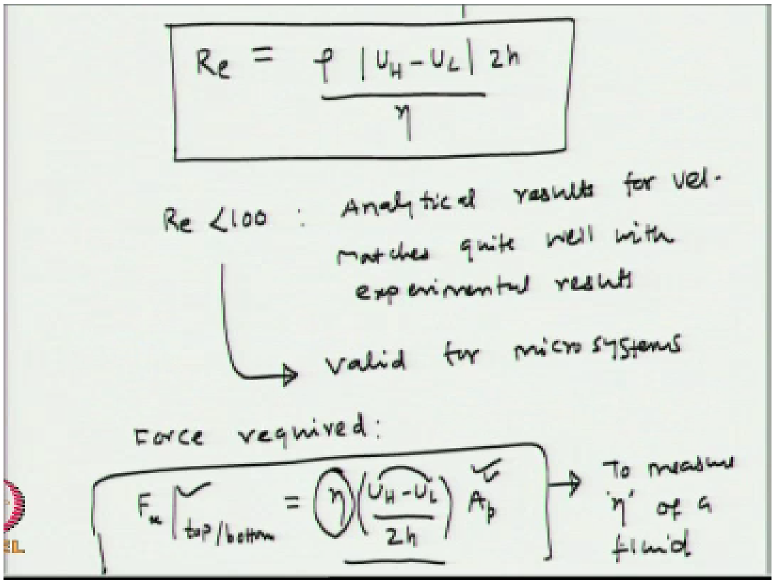

So, the Reynolds number could be correctly defined as so Reynolds number is normally rho uL/eta so that is how Reynolds number is defined it is inner state of viscous. So, in this particular case you can define as rho*u will be UH-UL* so the phishing between the plates this is going to be 2h this plate is located at +h-h right. So, this is going to be 2h/eta so that is how the Reynolds number is going to be defined in case of Coutte flow.

It has been found that Reynolds number < 100 the you know the velocity profile obtained by this model matches very well with that obtained from experiments which is a case in micro scale flow situation you know quite often the Reynolds number would be < 100. So, the analytical research obtained here for the Coutte flow situation is applicable to micro scale flow situation.

So, for Reynolds number < 100 the analytical results for velocity matches quite well with experimental results okay and this is a case which is valid for microsystems. Now if you want to know what should be the value of the force that is applied on the plates in order to move the top plate at UH and bottom plate at UL we can determine that from the shear stress value okay? So we can determine the force required.

The force required can be determined by Fx/top/bottom=eta*UH-UL/2h*Ap. So this is the shear stress eta*UH-UL/2h so shear stress * the area of the plate okay that is the force that you require to move the plates related to each other. Also from this equation knowing the area of the plate knowing the velocity knowing the force that is required we can also estimate eta for practical situation okay. So, this equation can also be used to measure eta or eta of a fluid okay.

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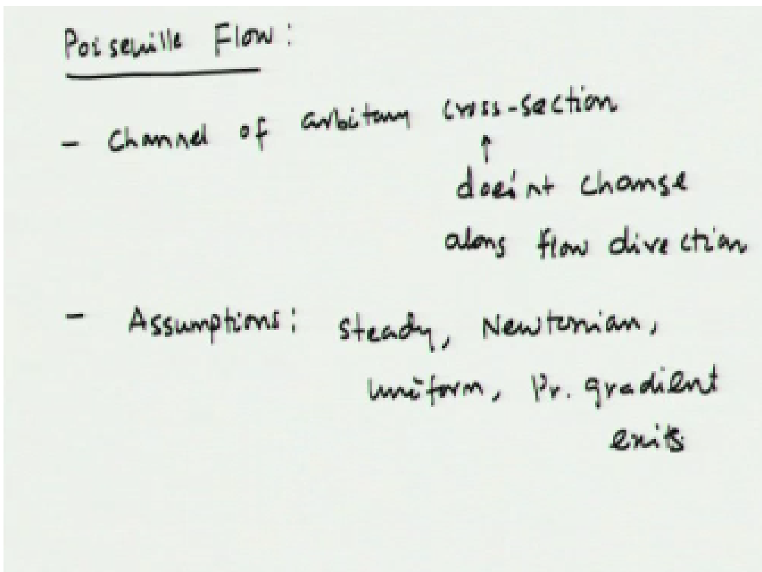

Next we move on to talk about Poiseuille flow okay so in many micro-scale flow situations we have you know flow through channels that have constant cross-sectional area the cross sectional area does not change along the length of the channel okay and that is the case what is discussed in Poiseuille flow and again Poiseuille flow is a special case where the cross sectional area is circular inside.

But in microchannel we have channel cross sections of different shapes we have a rectangular channel, triangular channels elliptical channels so different channel shapes we can encounter in microscope microchannel flow situations. So let us first talk about how we can obtain a flow solution for a channel cross-section of any arbitrary shape so we will not worry about what could be the channel shape it could be any arbitrary shape.

And we will see how far we can push to obtain a analytical solution for that okay. So, let us talk about Poiseuille flow okay so here we talk about a channel of arbitrary cross-section that does not change along flow direction. So the cross-section does not change along flow direction okay and here we make some assumptions we say that flow is uniform, steady and the fluid is Newtonian, under that assumption we can obtain an analytical solution.

We can say steady flow, Newtonian uniform but here pressure gradient exists okay. So, in Poiseuille flow situation we would have the pressure gradient which is non 0 and that actually drives the flow okay. So, the pressure gradient exists in case of Poiseuille flow.

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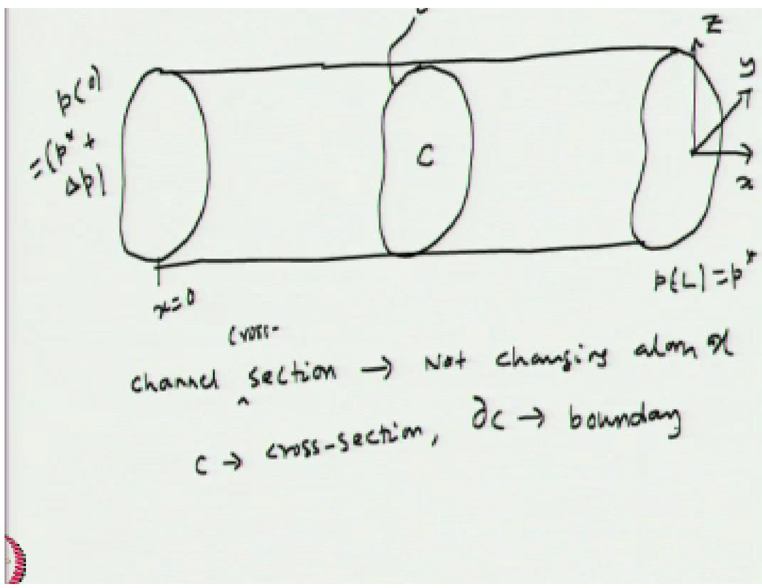

So, here we take an arbitrary channel shape so it could be let us say this shape and we have the channel and the shape does not change cross different section okay. Let us say this is x direction this is z direction and this is y so the cross section is in the yz plane and this is the cross-section denoted by c and the boundary is denoted by del c and here we have p at 0. So, this is let us say x=0 right x=0 p0 which is the inlet pressure=p star+delta p okay.

And here p at L is going to be p star. So, here under that assumption we can say that here the channel section is not changing so the channel section cross-section not changing along x okay. So, c represents the cross section of the channel and delta c represents the boundary right.

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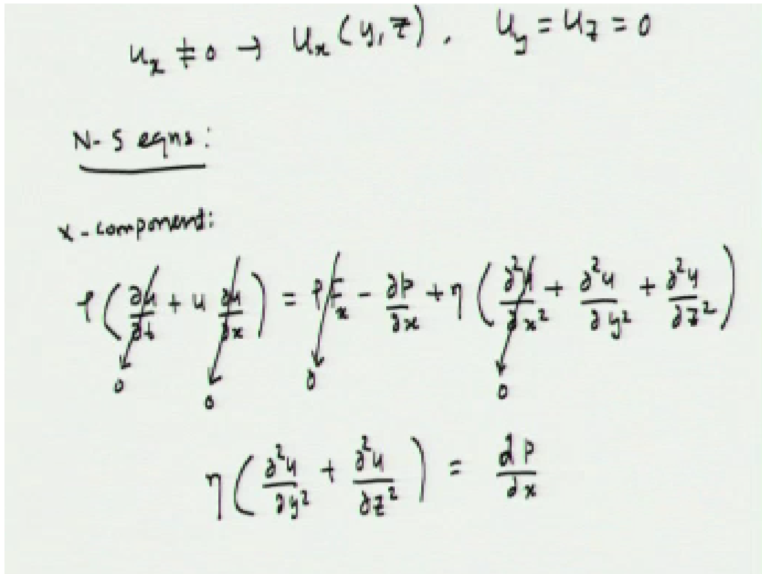

And since the flow is uniform we have ux is != 0 and ux is a function of y and z and the other components ui and uz are 0 okay. So, we can write down the Navier Stokes equation okay so first we write down the x component, so the x component would be rho*del u/del t+u del u/del x=rho Fx-del p/del x+eta*del square u/del x square +del square u/del y square+del square u/del z square.

Now for a steady flow this term would drop to 0 this is uniform so the velocity and the gradient are orthogonal. This term drops off the body force is 0 okay and here you see this u does not vary along x, so this term will drop off so we have an equation which looks like this eta*del square u/del y square+del square u/del z square will be dp/dx you can say del p/del x.

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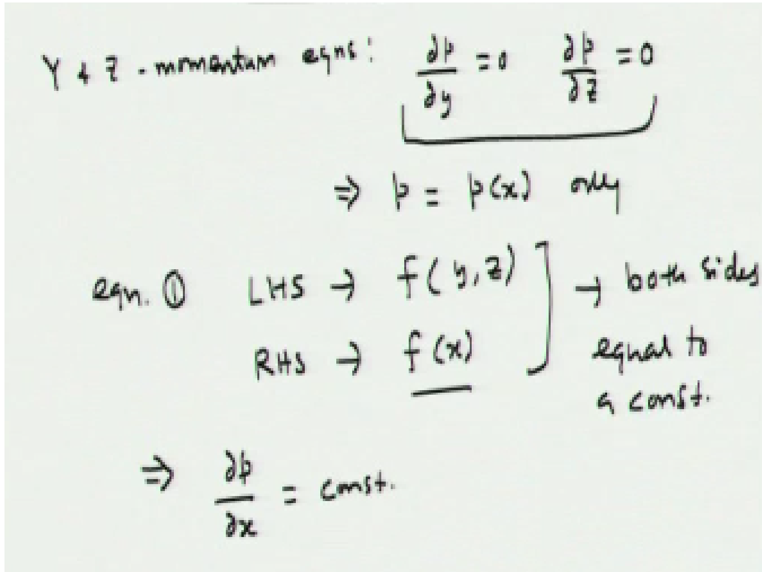

Okay now the x component y and z momentum equation the y and the z momentum equations we would get delta p/delta y=0 delta p/delta z=0 because the other velocity components uy and uz are 0 right. So, what does this mean this means that pressure is a function of x only right pressure is a function of x only right? Now if you look at this equation? Let us call it equation 1 of the left hand side says that the function is a is dependent on y and z.

Here on the right hand side the function is dependent on x okay so in an equation if the left hand side is a is function of y and z and the right hand side is function of x only we can say that both sides can only be equal if they are equal to a constant okay. So, since equation 1 left hand side function of y and z and right hand side function of x we can say that both sides equal to a constant that is the only way this is going to be possible.

So, in that case we can say del p/del x okay which is the right hand side is a constant.

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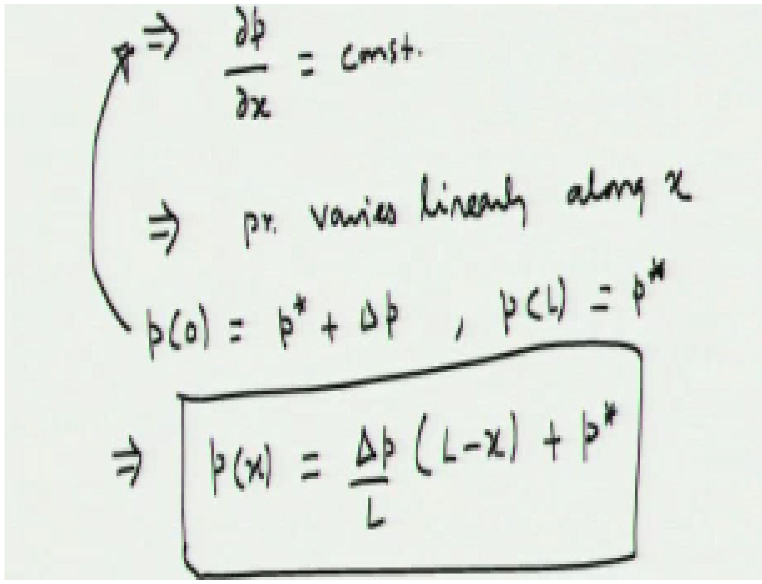

What that means that means that pressure is varyingly nearly along x right. So, pressure varies linearly along x. Now we have the boundary conditions p at 0 is going to be p star+delta p and p *L is going to be p star. So, if you apply that to this equation this delta p/del x=constant what do you get is p*x is going to be delta p/L*L-x+p star okay so this satisfies both this boundary conditions.

So, if you put x=0 the pressure will be p star+delta p and if x=L then pressure is going to p star right.

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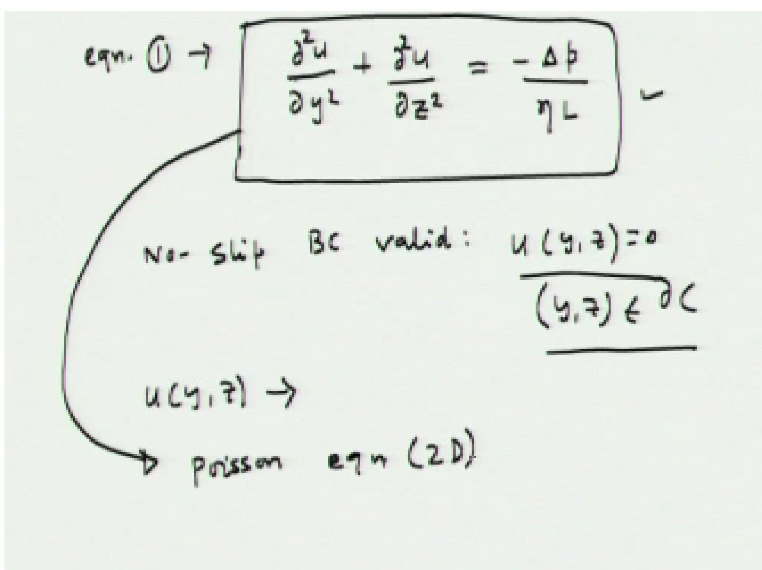

So, in that case our equation the equation 1, this equation would become, so the equation 1 become del square/del y square+del square u/del z square=-delta p/eta L, so this is an equation this equation is going to be valid for different cross sections okay this is independent of the cross-section of the flow okay for any Poiseuille flow right so here we can say that no slip boundary condition is valid for uyz=0 when yz belongs to delta c when yz lie on the boundary.

Which we said here delta c is the boundary here and when yz present on the boundary the velocity is going to be 0 that is the no slip boundary condition. So, in that case we can write the expression for the velocity so uyz can be written so this is basically this equation is basically a Poisson equation right so this is a Poisson equation in 2D.

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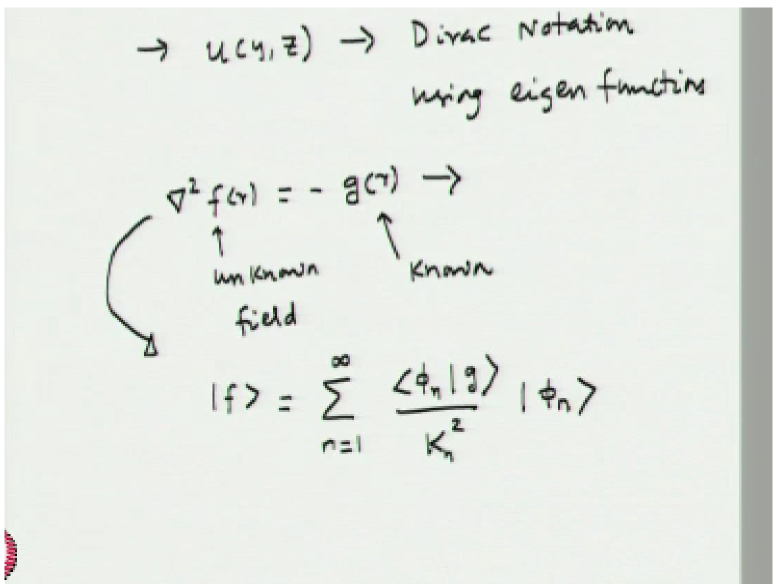

And we can obtain a general solution using the Dirac notation okay so you can write uyz using Dirac notation using Eigen functions. So, in a general sense if you have a Poisson equation which looks like this del square fr= -gr. Okay where this is an unknown field okay and this is known right this equation is similar to the Poisson equation that we see here so this is unknown and the right hand side is known okay.

So, here we can obtain a solution using Eigen function we something looks like this. So, function f this is called the ket of function f which will be summation n=1 to infinity, this is called bracket of phi n*g/kn square* the ket of phi n.

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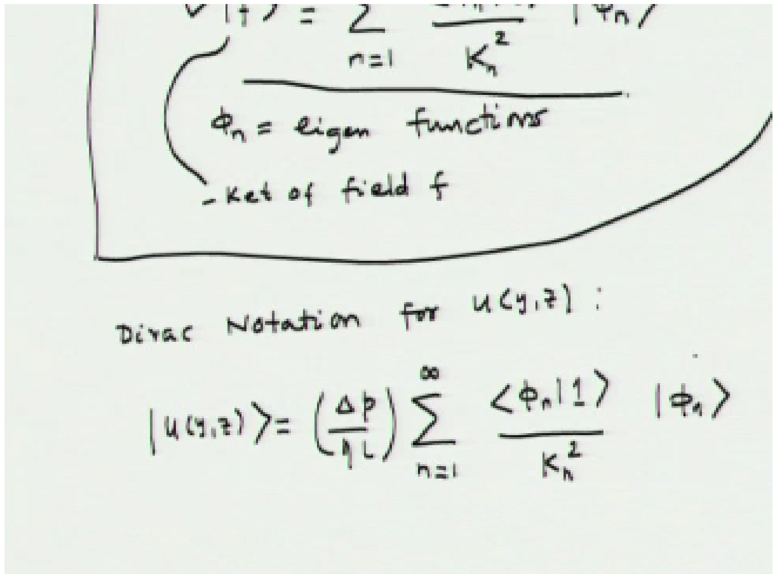

So, this is called is this is the Eigen function bracket notation and this is known as this is known as the ket of phi n and phi n being the Eigen functions Eigen functions and here this one is the ket of the field unknown field f okay. So, that is a standard way how Eigen solutions are obtained okay if you have an equation in this form you can write down the solution Dirac notation of the unknown field in this way okay.

So, similarly if you look at this equation we will be able to write down the direct notation for the velocity field u okay. So, you can write their direct notation for u is the function of y=delta p/eta L*summation n=1 to infinity phi n 1/Kn square* phi n. So, this is the ket of phi okay so this is the bracket of phi n and 1 because your function gr here is delta p/eta L and you have taken delta p/eta L out this function here is 1 this is a known function.

And Kn square are the Eigen values and this is the Eigen function and this is the Dirac notation for velocity fluid okay. So, this is a general solution for the velocity irrespective of the cross-section that were talking about. Now from this we can also determine the expression for the flow rate okay using the Eigen function we can relate the flow rate to the pressure area okay.

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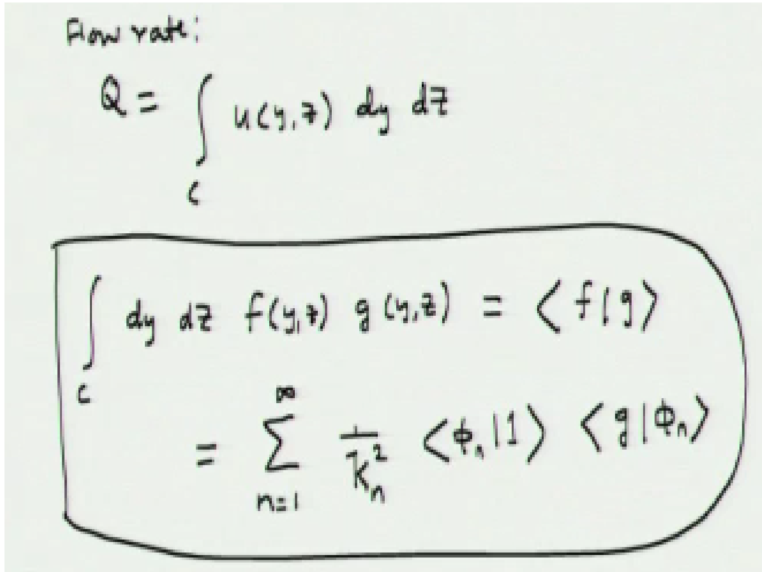

How we can do that so the flow rate Q okay so the flow rate Q can be written as Q will be integration over the cross section uyz*dy dz right this is the expression for the velocity. Now in general sense if you look at Eigen functions the integration of c dy dz*fyz*gyz can be written as the bracket of function f*g which can be represented as this so n=1 to infinity1/Kn square. So, this is the Eigen values Kn right and * the bracket of phi n1* bracket of g phi n.

So this is the general formula that we know from Eigen functions. So, similarly here we can write an expression for Q

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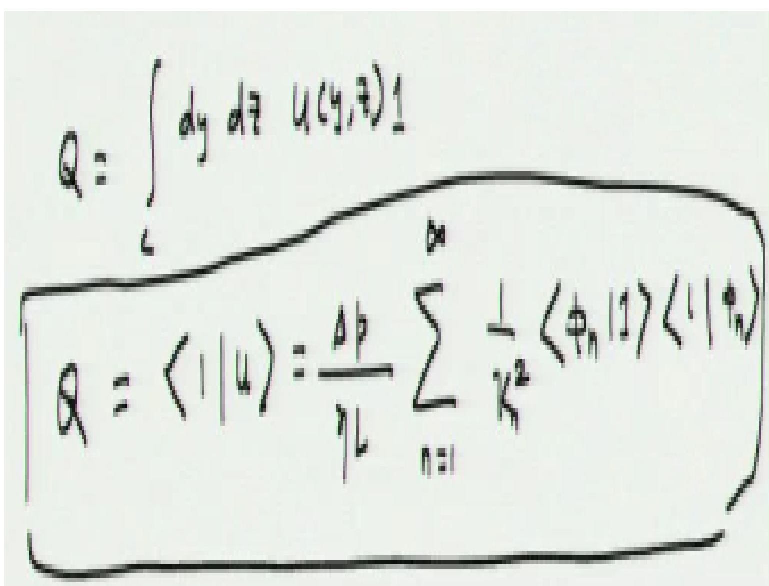

So, Q could be so in this case f is the velocity field and g is going to be 1 right so we can write Q to be dy dz *uyz*1 okay that is the g. So, which will be = the bracket of 1*u which can be written as delta p/eta L*summation n=1 to infinity 1/the Eigen values K square*phi n1*1 phi n okay. So this is the general expression for the flow rate irrespective of the cross section you can write the flow rate in this form right.

Now you try to non dimensionalise this term that is within the summation so you try to non dimensionalise this term. How we can do that okay? So, if you look at this expression here okay the velocity expression here you can find out the units of each of these and it will turn out that the unit of the Eigen value square is of the order of 1/L square so how we can check that we can do it this way.

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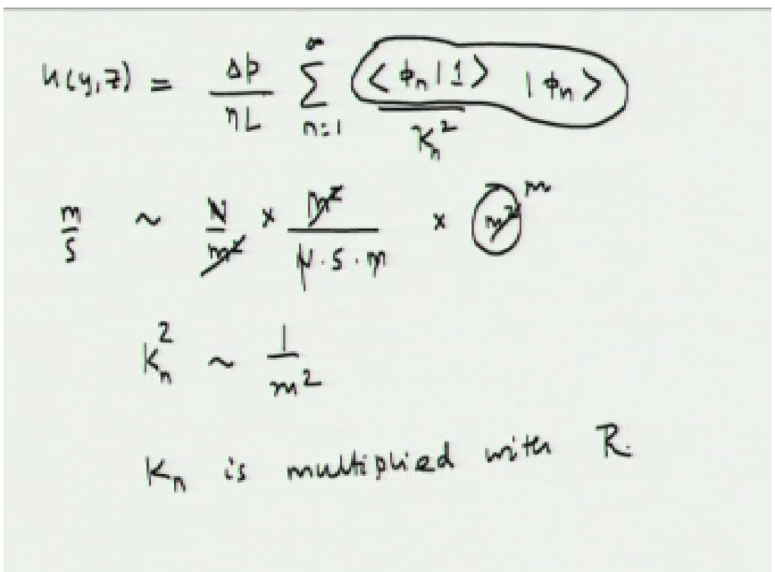

We can write uyz=delta p/eta L summation n=1 to infinity*phi n1 bracket/Kn square* the ket of n right this is what we know from here right. So, if you look at the unit of each of these terms this is meter per second okay this is delta p is newton per meter square and eta is pascal second right. So, newton meter square will go here into second and this will be L will be meter these are the functions okay they are dimensionalised so you would see that Kn square.

This is unit here has to be 1/ meter square right then only it will be dimensionally correct okay. So, the Kn square this has to be meter square so then only it will be valid so this will get cancelled and this newton newton will get cancelled this will cancel with one of the meters this is meter per second right. So, what we get from here is that the Eigen values square of Eigen value okay so the scale as 1/meter square right.

And it turns out that you can multiply this Eigen values Kn is multiplied with another length scale called R okay. So, to make this term non dimensionalised okay so instead of Kn square in the denominator you would have Kn*r square. So that would make this term dimensionalised inside the bracket right so this is important to understand then if you look at the dimension of these 2 terms here right now before we go to that we can you know say that.

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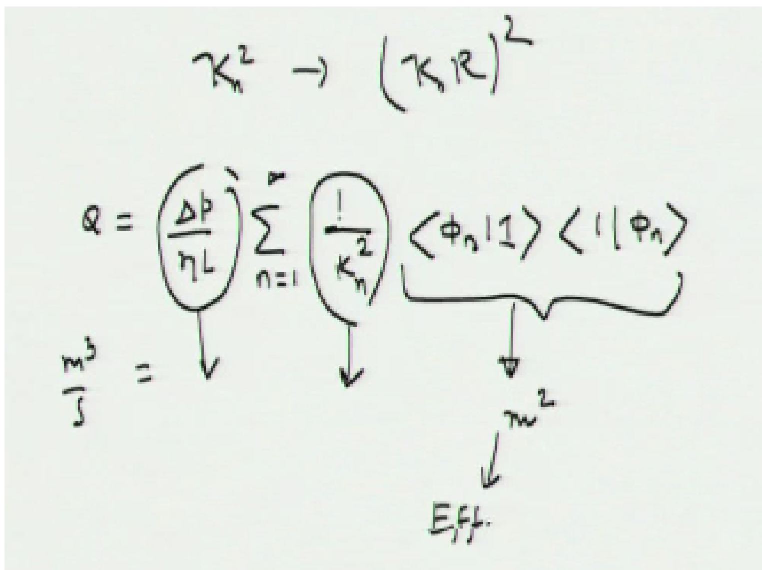

We can write down this equation here Q=delta p/eta L n=1 to infinity 1/the Eigen value square* phi n1*1 phi n okay. So, if you do a similar dimension analysis meter cube per second and if you try to write down the dimension for this what and also the dimension for this what you see is the product of these 2 would scale as the meter square so and this term is call as the effective area which is A right as this term would call as effective radius effective radius ok.

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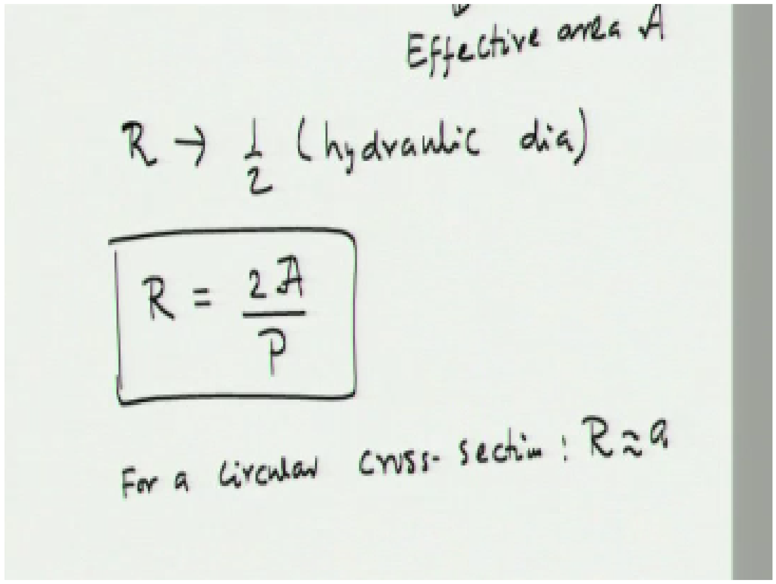

So, you can write you know the R turns out to be 1/2 of the hydraulic diameter. So in that case you can write R = 2* area of cross section/perimeter so you can write the expression for R right. Now for a circular cross section the R=a.

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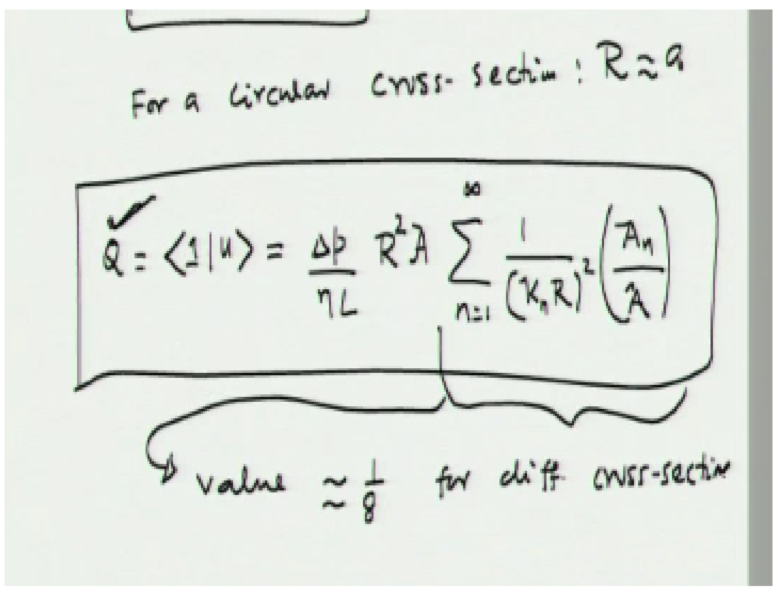

So, now we can write down expression for q as follows so you can write q to be 1*u 1u the bracket of 1 and u which is delta p/eta L*R square A*n=1 to infinity 1/the square of the Eigen values sorry Kn*. We will non dimensionalise the term Kn*R square and here you would have another An/A. So, An is the effective area this is An okay and A is the area of cross section and R is the effective radius okay.

So, this is how you can know non dimensionalise the terms inside the summation okay to obtain a value for an expression for the flow rate okay. So, for most of the micro scale flow situation it has been found that the numerical value of the terms inside the summation is approximately=1/8 okay this has been found whereas for a circular cross-section the value is exactly=1/8 okay. So, this term has been found the value is about 1/8 for different cross sections and=1/8 for circular cross section.

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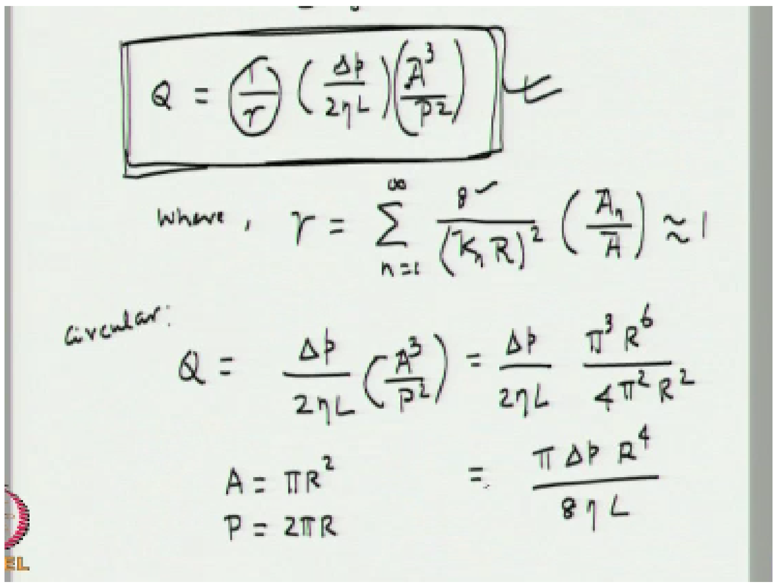

So, we can write the expression q as follows so we can write to 1/gamma*delta p/2 eta*L*A cube/p square cube of the area square of the perimeter where the R where sorry the gamma is given by the summation term n=1 to infinity*8/Kn*R square*An/A and this term is found to be very close to 1. So, since you have multiplied 8 here so that has been taken care of for the value of the summation and for any general cross-section.

You can apply the flow rate expression which would appear like this okay. So, you know you can try it for circular cross section okay R will be exactly=1 so for a circular cross section you can write the expression this will be this will become 1 so delta p/2eta L over circular cross section area will be pi R square and perimeter would be 2 pi*R. So, here you can write delta p*A cube/p square so this will be delta p/2 eta L pi cube R6 because this is Q/4 pi square*r square.

So, you would get pi delta p R4/8 eta L, okay and that is the expression for the flow rate through a tube of a uniform cross-sectional area okay now next move on to you know to discuss the flow between 2 parallel plates of infinite length.

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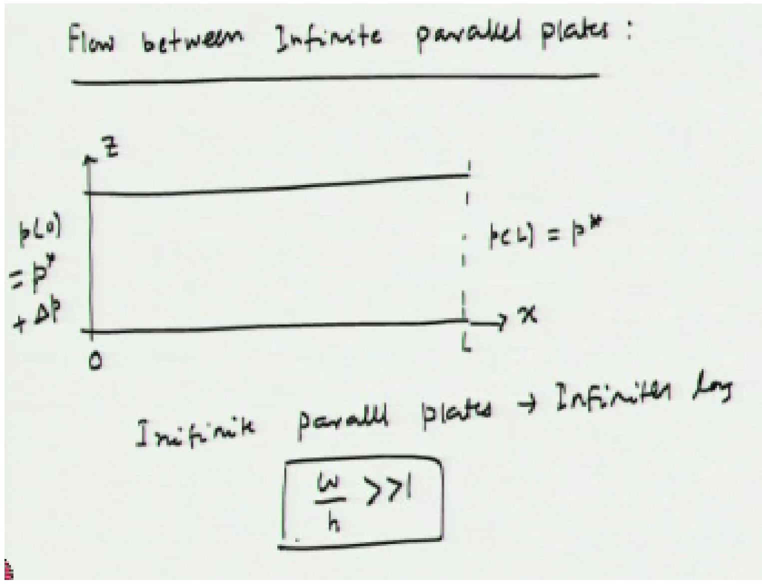

So, let us discuss flow between flow between infinite parallel plates so we have 2 plates here okay and let us is say this direction is good this direction is x this is x=L this is 0 and you know the p at 0 would be p star +delta p and here p at L will be p star and here we are talking about the infinite parallel plates that means the width of the channel is going to be large as compared to the height between 2 plates okay.

So, infinite parallel plates so they are infinitely long long and we consider w/1 to be > 1.

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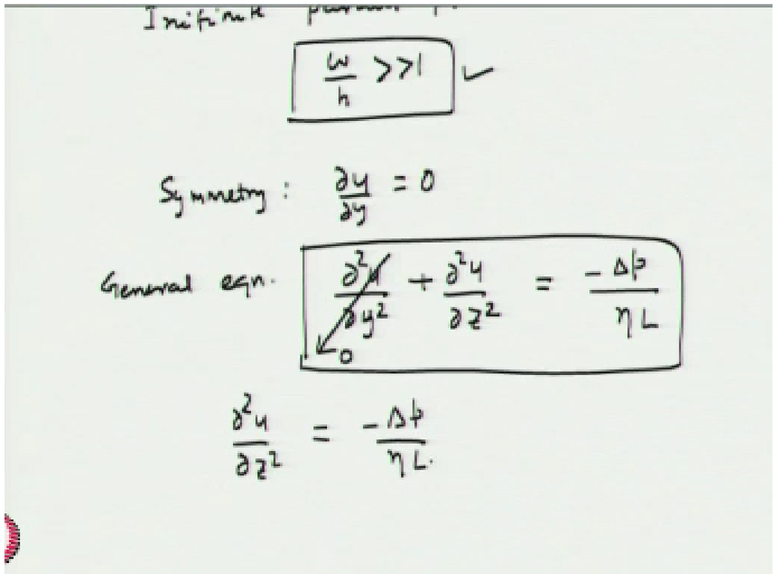

Okay so under that assumption since the width is > h we can write symmetry condition that means del u/del y will vanish okay or y is into the width and z is across the height so this will be vanishing this is h the plate right. So, in that case we can write down the Navier stokes equation. So, let us write down the governing equations what we have found from our analysis here for a general cross-section.

Okay that is valid for every cross section use this general equation here. So, if you use that general equation you have del square u/del y square+del square u/del z square=delta p/eta L okay. So. that is the general equation right here this variation is negligible because we are talking about the width is large compared to the height so there is no variation along y direction so what we see here is del square u/del z square-delta p/eta L.

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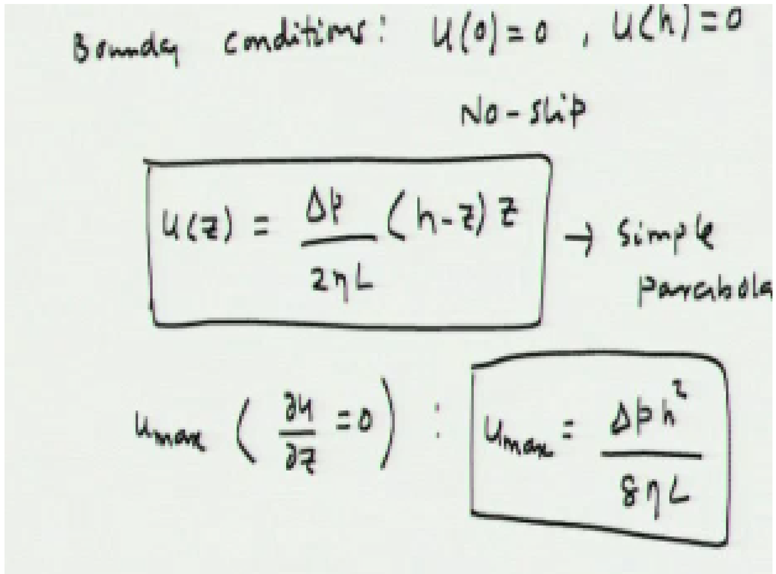

Now if you look at the boundary conditions boundary conditions we can expect that u at 0 is 0 and u at height h=0 this is no slip okay so using that if you apply these boundary conditions on this equation we can get an expression for the velocity which will be a function of the z so delta p/2staL*h-z*z. So, this is a simple parabola okay this is a simple parabola what it says is that the velocity profile would look something like this okay.

So, it will have maximum at the center and it will reduce to 0 at the walls. So, you can find out the maximum velocity u max to find u max you can say that del u/del z will vanish right in that condition you can find u max so u max can be found to be delta p*h square/8 eta L okay.

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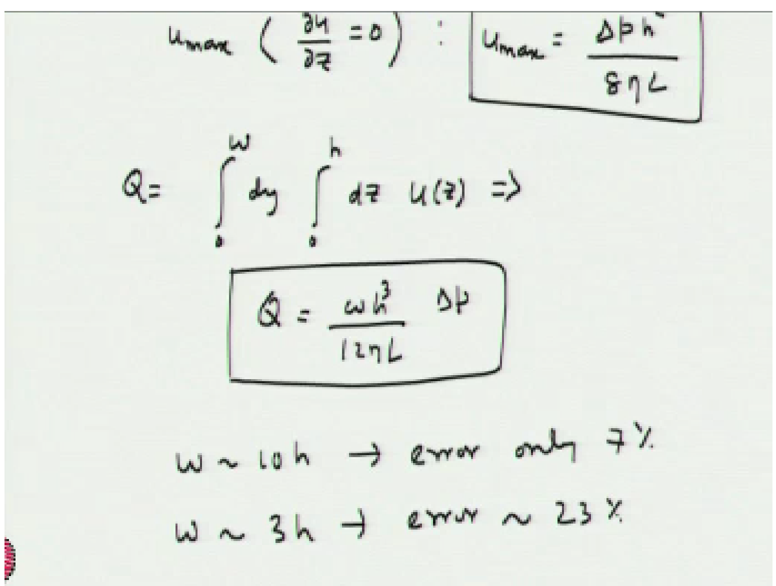

So, from the expression for the velocity you can find out the expression for the flow rate okay so the volume flow rate u can be found us integration of 0 to w dy*0 to h dz* the uz which can be found out be Q would be wh cube/12 eta L*delta p. So, this way we will be able to find out the flow rate between 2 infinitely long parallel plates under the assumption that the width is large compared to the height.

Okay and this expression for the flow rate it can be applied to a no flow through a rectangular channel and the accuracy would be very good if the width of the rectangular channel will be about 10 times the height okay. So, when w is you know when w is about 10 times h the error is only 7% but if w is about 3 times h the error goes to 23%.

So you know of width of channel width greater than you know 10 times the height this expression for the flow rate is good enough. okay similarly you can compare the flow rate found by this formula here with the general formula which you have found out here okay and we see that you can do it yourself.

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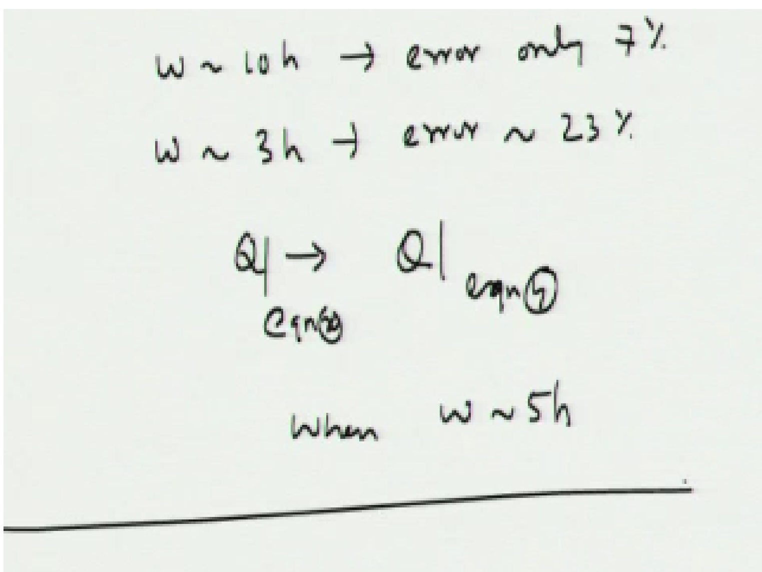

And you can check that flow rates Q can match okay here the next match with some equation x will match Q from equation y this is equation y when w will be about 5 times h. So, with this you can verify with that let us stop this here today.