Microfluidics

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Lecture - 12
Capillary Flows (continued)

Okay, so we have been looking at contact angle and we saw that if you know the surface tension between liquid gas, solid liquid and solid gas interfaces then we will be able to calculate surface tension, right. So, now we will go ahead and talk about in case of capillary flows when the gravity is becoming important, okay when the gravitational force need to be accounted for.

So, if you consider a fluid volume of omega and having the liquid air interface of d omega then we try to see in what situations we need to consider for the gravitational force. So, we need to minimize the equilibrium position of a gas liquid interface can be determined when the Gibb's free energy is minimum, okay and for the Gibb's free energy to be minimum the summation of the surface energy and the gravitational potential energy need to be minimum, okay.

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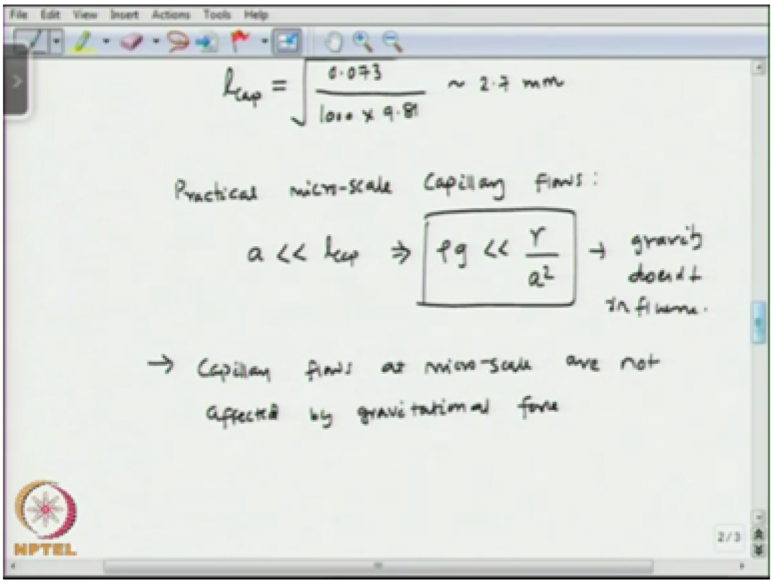

So, let us look at the capillary length and capillary rise and here we consider liquid volume omega with free liquid air interface d omega. Now if you want to determine the equilibrium shape then the free energy needs to be minimized, okay. And this free energy as the surface energy + the gravitational potential energy, okay. So, we can write this the G minimum will be we have to minimize the surface energy gamma*d omega*a.

So, this is the capillary radius, this is surface tension and this is integrated over the interface + rho*G integration dr integrated over the volume*Z, okay. So, this gives you the volume of the fluidic element and this gives you the interface length. So, the entire term is minimized for a fixed volume omega, okay. So, if you talk about a liquid drop present in a free space the liquid drop volume will be spherical, okay.

To minimize energy, the volume has to be spherical because in this case the gravitational potential energy is 0. So, you are basically minimizing the surface area at for a fixed volume. So, the minimum surface area for a fixed volume is a sphere, okay. So, this is why in free space the liquid drop takes a spherical shape. And when we have gravity present to account for the gravitational forces we define a parameter called capillary length, okay.

So, we will define a parameter called characteristic length, okay you can also call it as capillary lengths, l capillary. So, characteristic length for capillary flow is gamma over rho g, okay. So, it is basically the competition between the surface energy and the gravitational potential energy, right. And so, if a, okay so just see here a is << gamma over rho g, okay if this condition is satisfied then we say that gravity can be neglected, okay.

So, the radius of the capillary if it is less than the characteristic capillary length then you can say that gravity is neglected, right. Now we can find for water - air interface at 20 degree centigrade you can find l cap characteristic capillary length would be gamma will be 0.073, density will be 1000*9.81 is Z. So, it is about 2.7 millimeter, okay.

Now in case of capillary flows in micro channels the radius of the micro channel or the capillary is < this value < 2 millimeter, 2.7 millimeter. So, you can say that for all capillary flows in micro channels the gravity force can be neglected, right. So, in practical micro scale capillary flows the radius of the capillary a is << l cap. So, the rho g is << gamma over a square, right.

So, gravity does not influence, okay. So, capillary flows at micro scale are not affected by gravitational force, okay. So, having talked about that let us talk about capillary rise. We know that when you put a small capillary into a liquid container the liquid tries to move up the capillary, okay. Here we try to find out what would be the maximum achievable height up to which the liquid will rise, okay.

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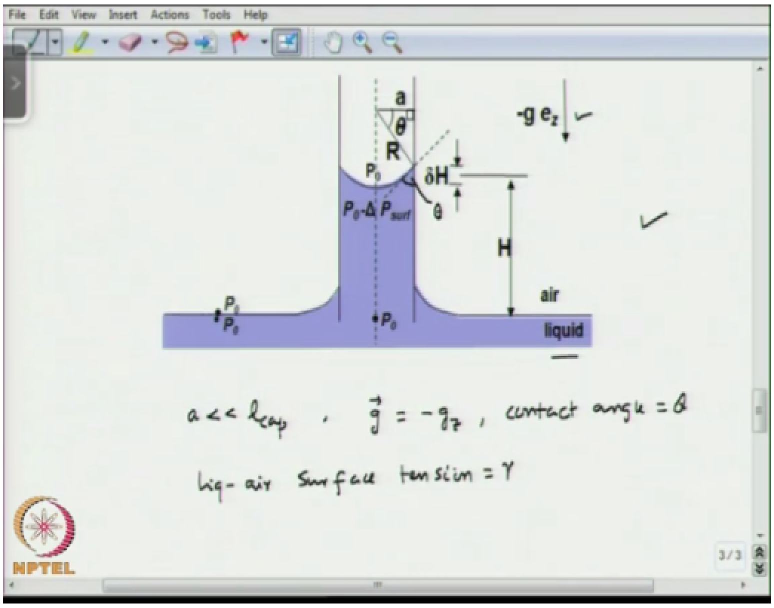

So let us look at, move on to here, so this is the illustrative figure. So, you look at capillary rise height. So, this is the situation as you have see here, okay. For simplicity we are considering a vertical tube into a liquid present here and we are satisfying the condition that the radius of the capillary is << the characteristic capillary length and the gravity is in the vertical direction, okay.

So, g is along the vertical direction along the **g** direction and theta is the contact angle as you see here. So, this angle is going to be theta which we call contact angle is theta, right. So, this is theta and the surface tension liquid air interface liquid air surface tension = gamma. So, here we assume that the radius of the capillary is less compared to the characteristic which mean that gravitational force is negligible and if gravitational force is neglected we can expect the interface to be spherical in shape, okay.

So, that is the reason why the interface in this case will be spherical in shape. So, this is what is shown here. So, this is a spherical interface, okay spherical interface, okay. So, if we look at the geometry this is the radius of this sphere, okay and this is, a is the capillary radius, r is the radius of this interface, okay. So, if you use geometry then this contact angle this angle will be same as the contact angle which is theta, okay.

So, by geometry we can look at cos theta here you can write cos theta will be = a over r, right. So, r will be a over cos theta, right. Now in this particular case, I know for a interface we would have 2 different radii of curvature, okay. Since here in this case the radius of curvature is spherical the 2 radii of curvature are equal, okay. So, you can say that the 2 radii of curvature are equal, so we can say R1 = R2 will be = a over cos theta, okay.

So, now we tried to do a force balance, okay. So, here if you look at the interface, okay if we look at the interface, if the interface is convex in shape. So, the pressure would be higher in the convex side as compared to the concave side, okay. So, in this particular case since at the convex side we would have air present and at the concave side we would have liquid present, okay as you can see here.

At the concave side we have air present here and liquid is present there. So, the pressure on the air side is more as compared to the liquid side, okay.

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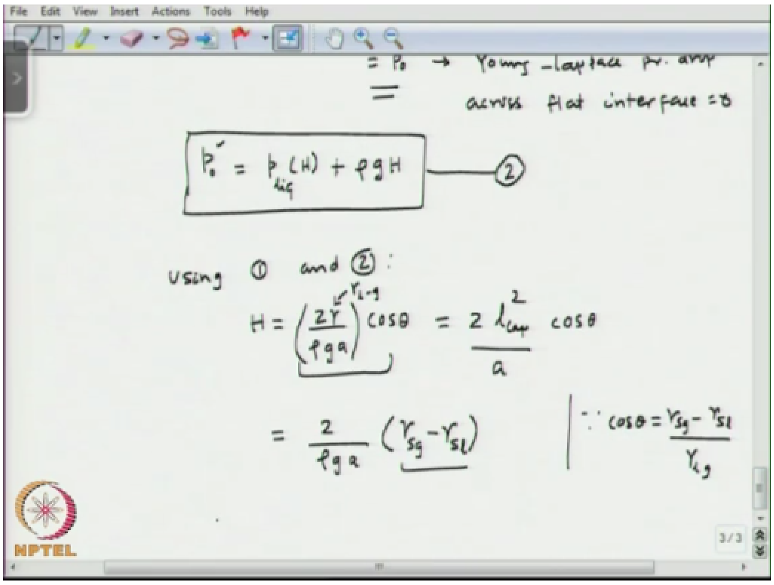

So, we can write here the pressure difference is pressure higher in convex or air side compared to concave or liquid side of the interface. So, if pressure on the liquid side of the interface we call it as p liquid at height h, okay. So, we can say p liquid at height h will be = P 0 is the atmospheric pressure - the Young Laplace pressure delta p surface, okay. So, this is the Young Laplace pressure across curved interface. Whenever the interface is curved we would have Young Laplace pressure present.

So, you can write P0 – 2 gamma over a*cos theta, okay. Because we know that delta p across the surface is 1 over R1 + 1 over R2*gamma. This is what we derived earlier. So, this here R1 = R2 and there = a over cos theta, so you can write 2 gamma over a*cos theta, okay. So, let us call this equation 1. Now at this location here, okay so this is the point which is in the liquid air interface but far away from the capillary.

So, here the pressure will be P0 same as the ambient pressure because this interface is flat and as we know the Young Laplace pressure drop across a flat interface is going to be 0. So, the liquid pressure will be same as the ambient pressure P0, okay. So, here we know that P liquid at 0 is at **Z** = 0 is the pressure away from tube and that will be = P0, okay and this is because Young Laplace pressure drop across flat interface will be 0, right.

So, this is P0, we can write this P0, okay. So, this pressure here is also going to be P0 because they are at the same level, okay. So, you can write P0 is going to be P liquid at H + rho g H, okay. So, you can write P0 will be equal to P liquid at h. So, this is the pressure at the liquid side of the interface + the hydrostatic pressure rho g H, okay. So, let us call this equation 2. Now if you combine equation 1 and 2 if you substitute P0 from here and there and rearrange the term what you get is this.

So, using 1 and 2 what you get is H = 2 gamma divided by rho g a*cos theta, okay. And this we can write it as we can write 2 l capillary square, okay gamma over rho g square l caps. So, l caps square divided by a*cos theta. You can also write this as 2 over rho g a*gamma solid gas – gamma solid liquid. Remember this gamma is gamma between liquid and gas, okay.

And we know that cos theta = gamma solid gas – gamma solid liquid divided by gamma liquid gas. So, you can substitute for gamma liquid gas gamma liquid*cos theta is gamma solid gas – gamma solid liquid.

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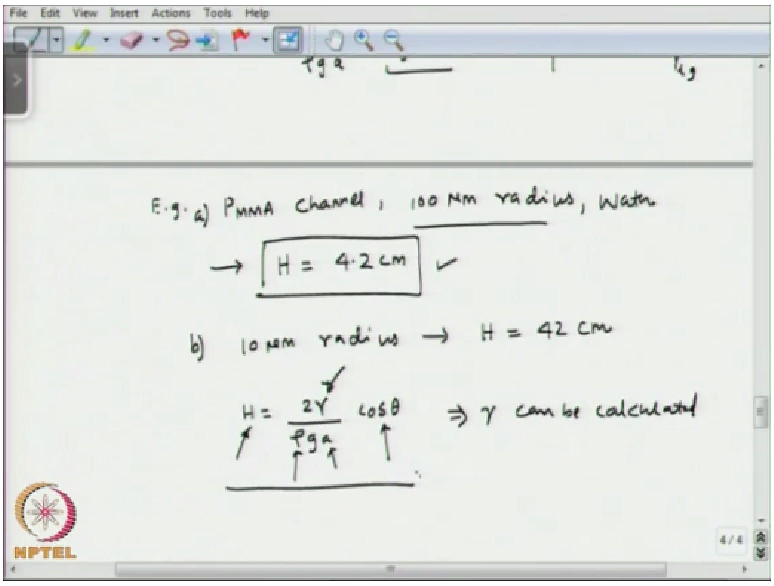

So, now if you take an example, so you consider one example, let us consider a PMMA channel and let us say 100 micron radius you can find that the capillary height is going to be 4.2 centimeter, okay and if you consider water as the liquid. So, you know the density H is going to be 4.2 centimeter. Now if you instead of 100 micron radius, okay if you go for the same material and the liquid water and the radius is 10 micron then H will be 10 times, okay.

H will be 42 centimeter, okay. Now if you look at this equation this equation here H = 2 gamma divided by rho g a cos theta H = 2 gamma over rho g a cos theta. Here it is possible to measure contact angle the capillary height this will be known for a liquid and this will be known for a capillary. So, it is possible to find out gamma. So, gamma can be calculated, okay.

So, it is possible to do an experiment, okay where we would know the capillary radius we can measure the contact angle, we can measure the capillary height to predict what the surface tension would be for a liquid, okay.

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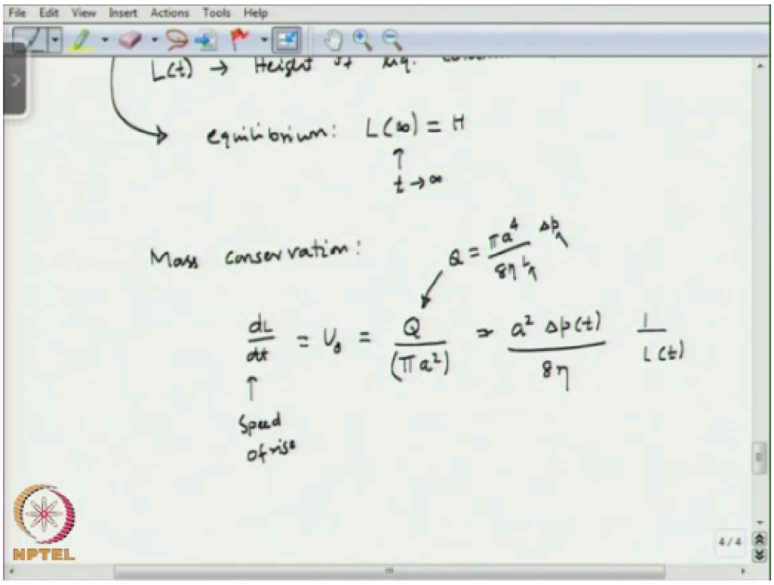

So, next we move on to talk about the capillary rise time, okay. So, we know that if you put a capillary or a micro channel we can expose to some liquid because of the capillary action the liquid will move into, okay. And we found an expression for in case of a vertical capillary what is going to be the maximum height to which liquid will move.

Let us look at what is the time scale up to which this maximum capillary height can be attained, okay and how it is going to achieve this maximum height, okay. So, for that we would consider the equilibrium height, so the equilibrium height H and so that we would find the rise time required, okay. Let us say L t is the height of liquid column at time t, okay and this equilibrium can be attained.

So, equilibrium L infinity will be = H. So, as time would tend to infinity the liquid would reach the equilibrium height. First let us do mass conservation, okay. If you do mass conservation we can write d L over d t, okay the change in the height or this is the speed of rise will be equal to the velocity, rise.

U 0 which will be equal to the flow rate divided by phi a square, a is the capillary radius and we can substitute for Q from where Q can be written as phi a 4 divided by 8*delta p. So, we can write it as a square delta p which is a function of time*8 eta*1 over L t, okay. So, all our time depended here, right. Now we can do energy balance.

So, when you put a capillary in liquid the liquid tries to pull up and this is because of the Young Laplace pressure that is present across the interface and this pressure is utilized to overcome the viscous drop and also to overcome the potential energy because of any gravitational present, okay however small that might be. So, you can write the energy balance.

You can write the Young Laplace pressure drop will be = the viscous pressure drop + the hydrostatic pressure of the potential energy of liquid column, okay. So, you can write delta p surface will be = delta p at, so this is the viscous + rho*g*L t. So, this is the hydrostatic pressure and this is the viscous pressure drop, right. Let us call this equation 2 and let us call this equation 1, okay.

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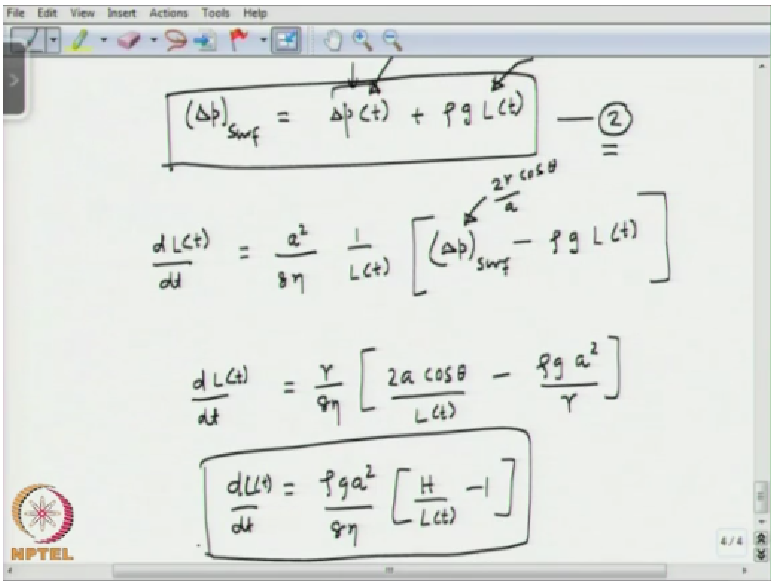

Now if you insert this equation 2 expression for the Young Laplace pressure here, okay sorry not Young Laplace pressure but the viscous pressure drop, okay which can be substituted here if you do that what we get is this. We get d L t over d t will be = a square over 8 eta L t, okay a square over 8 eta*1 over L t*delta p t which will be equal to delta p surface – which is the Young Laplace pressure – rho g*L t.

And we have an expression for this, right which will be 2 gamma cos theta over a. So, if you substitute, now if you simplify you get d L t over d t will be equal to gamma over 8 eta*2 a cos theta 1 over L t, so divided by L t – rho g a square over gamma, okay. So, which can also be written as, rho g a square divided by 8 eta*H over L t – 1, so this is d L t over d t, right.

Now, to precede further let us try to non base this equation, okay.

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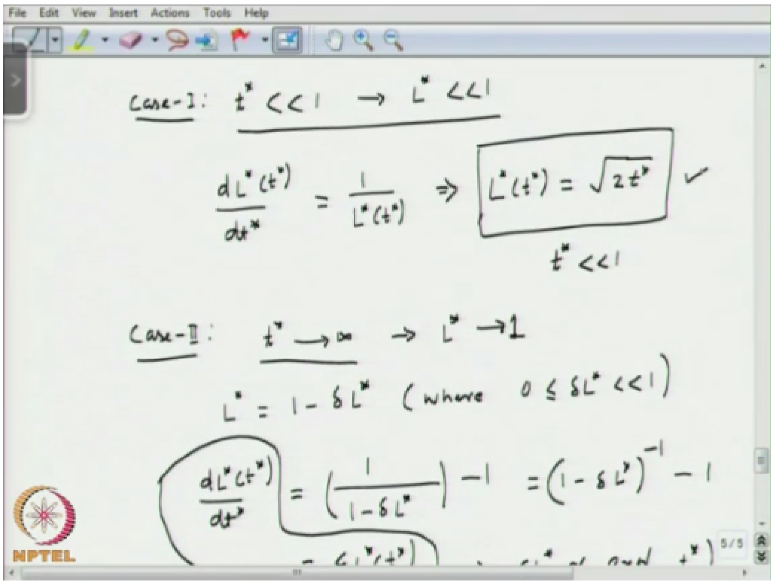

So, use some characteristic scales, we can say t is the characteristic capillary time scale*t star and where zeta capillary will be = 8 eta H over rho g a square and we can say L will be = H*L star, okay. H is the maximum capillary height and L is at time t. Now with that we can write this equation in non-dimensional form, so you can write d L star t star of d t star. So, d L t over d L start t star of d t star = 1 divided by L star t star – 1.

So, this is the equation that we have to solve. So, with the boundary conditions what are the boundary conditions? The boundary conditions are L star at 0 is going to be 0, so when time t = 0 this is going to be 0 and when time t = infinity L star infinity will be = 1, okay. So, because L will be H L star will be 1. With that we can have 2 extreme cases. In case one, we say that the time t star is << 1, okay so L star is << one.

So, we are talking about at the beginning of the capillary rise, okay. So, in that case we can write d L star t star over d t star, you know L star is small compared to 1, so 1 over L star will be very large. So, we can rise, so 1 can be dropped off there, so L star t star. So you can write L star t star will be = square root of 2 t star, okay. This is valid for t star << 1. Now you can consider the second case, case 2 and here the time will be large, okay.

So, t star would tend to infinity, so which means that L star will tend to 1, okay. So, in that case we can write L star = to 1 – delta*L star delta being very small. So, you can say that where 0 < = delta L star << 1. So, you can write d L star t star over d t star will be = 1 divided by 1 – delta L star – 1, right.

So this is the, this equation, okay instead of L star we substitute 1 – delta L star, so this will be = 1 – delta L star – 1, right, - 1 and this will be equal to delta L star or L star t star, okay. So, this is the equation we need to solve, okay. So, what we can see from here is delta L star is going to proportional to exponential – t star, okay. So, we can write L star t star will be = 1 – a exponential t star.

So, this is the solution when t star is large, okay and this is the solution we obtained when t star is small, okay. So, what we see is when you have capillary rise at the beginning of the capillary rise when t star is small it follows of t trend and at large time scale it goes exponentially, okay it is exponentially and achieves a steady value, okay. Asymptotically it reaches the steady value.

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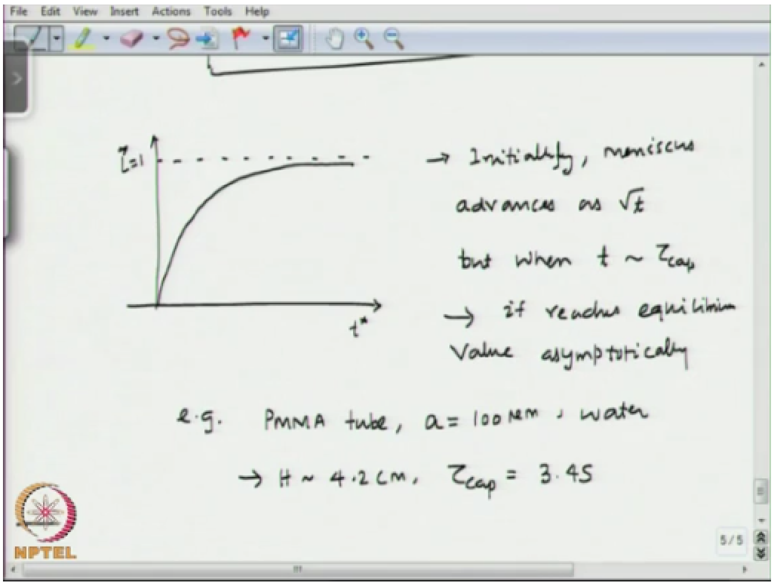

So, this is what we see, we let us say this is L star = 1, okay and this is let us say t star and so this is going to vary something like this, okay. So, this is the trend it is going to follow what we conclude from here is initially meniscus advances as square root of t and this is but when time t is the characteristic time scale then it reaches equilibrium value asymptotically, okay.

So you take an example, if you take an example with the same PMMA channel case, PMMA tube and radius is 100 micron and we have water as the liquid and we saw that H will be about 4.2 centimeter and the time scale will be about 3.4 second, okay. So, in this case we can expect that the capillary rise is going to follow the square root of t trend up to about 3.4 second and then asymptotically it will reach 4.2 centimeter, okay.

So, now having talked about that let us talked about few dimensionless numbers that are important in capillary flows and the first number that will talk about is known as bond number which is the ratio between gravitational force to surface tension force, okay.

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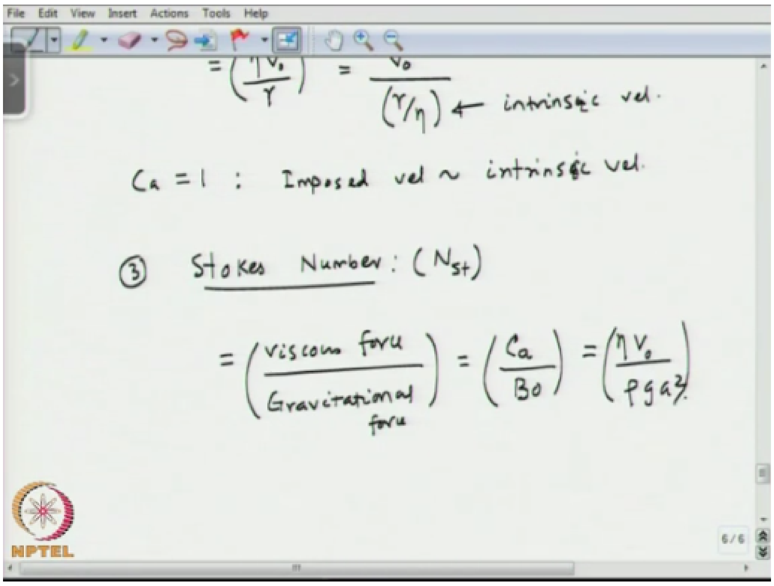

Dimensionless numbers, the first number we talk about is bond number, is the ratio of gravitational force divided by surface tension force, okay. So, we can write this, so bond number is specified by Bo, gravitational force is rho g a divided by gamma over a. So, gamma rho g a square divided by gamma will be = a square divided by L capillary square, okay.

So, that is how you can mathematically express the bond number in terms of the square of the capillary radius and the square of the characteristic length capillary length scale. If bond number is of the order of 1 we can say that the radius of the capillary and the characteristic capillary length they are of the same order and if bond number is < 1 and we can say that surface tension dominates.

Surface tension dominates over the gravitational force which is the case in micro scale. Gravitational force as a result a is << l cap, okay. Now the next dimensionless number that we talk about is capillary number, so we talk about capillary number. Capillary number is defined as the ration between viscous force to the surface tension force. So, capillary number is defined as the ratio of viscous force to surface tension force, okay.

So, we can write this as eta V0 over gamma, okay and we can write this as some velocity divided by gamma over eta. Gamma over eta is known as the intrinsic velocity, okay because this capillary number and if capillary number = 1 the imposed velocity, so this is the imposed velocity = the intrinsic velocity, okay. That next number we talk about is Stokes number represented by Nst.

So, Stokes number is given by the viscous force by the gravitational force which is capillary number divided by the bond number, okay. So, Stokes number is the ration of capillary number to bond number which you can write as eta V0 divided by rho g a square, okay. So, with that let us talk about the capillary advancement in a capillary vest pump, okay. So, we talk about here capillary pump advancement times.

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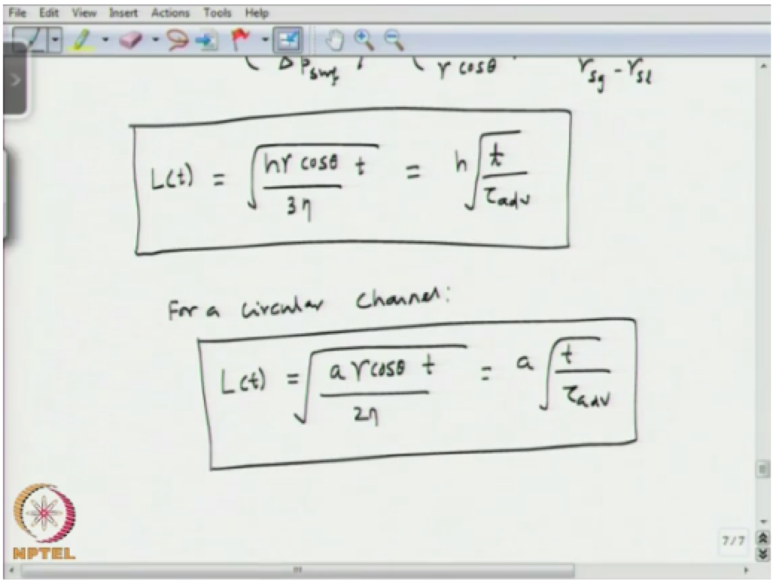

So, we consider a flat rectangular channel, okay. So you consider a flat rectangular channel it looks like this and we have this is the top channel wall. The inlet of the channel is typically larger in cross-section, okay. So, this is the channel wall here we consider this to be the interface at the inlet which is flat, okay let us consider the pressure here is P0 is the atmospheric pressure since the interface is flat here the pressure let us consider this to be the interface, okay.

So, pressure her also be P0 because this is the flat interface big interface. Let us consider this to be the Z axis and this is X axis, this height here Z = H, okay. Now here pressure will be again P0 because this is ambient this is the liquid and on this side the pressure will be P0 + delta P, okay. So, this is the pressure that is pushing the liquid in this side, okay.

So, you know here we assume that flat rectangular channel cross section is w*h and we say h is << w, right. So, we can apply Hagen–Poiseuille beta used. So, if you do that you can write Q = h cube w in delta p over 12 eta l. So, here it is a horizontal channel, right it is a horizontal channel. So, the Young Laplace pressure drop will balance the viscous pressure drop.

Because there is no gravitational force involve because of the horizontal channel. So, these 2 are equal, okay. So, you can write delta p surface is going to be 2 gamma over h*cos theta, okay which will be = 2 over h gamma solid gas minus solid liquid, okay. So, this is the expression for the Young Laplace pressure drop, similarly you can write down the expression for the viscous pressure drop.

So, if you do mass conservation you can write d L t over d t will be = V0 = Q divided by W h, right and which will be = h square delta p surface divided by 12 eta. So, here you substitute for the expression for Q that we got here for a Hagen–Poiseuille between 2 flat plates. So, this is what you will get into 1 over L t. Now, if you introduce a characteristic time scale tau advance as 6 zeta over delta p surface.

So, this will be 3 eta h divided by gamma cos theta or you can write it as 3 eta h divided by gamma solid gas – gamma solid liquid. You can obtain a solution for this, okay. You can obtain a solution for this equation, okay. You can write L t = h gamma cos theta divided by 3 eta*t square root which we can also write it as h*t divided by tau advance*square root. So, here you see that the advancement takes place as square root of t.

So, in case of a horizontal channel, okay if you have capillary flow the pumping is going to depend or the L t is going to depend as square root of t. So, for a circular channel you can write L t as a*gamma cos theta*t divide by 2 eta square root which is a*t over tau advance, okay. So, with that lets us stop here.