Microfluidics
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Lecture - 10
Micro-scale Fluid Mechanics

Okay, today we will talk about you know hydraulic resistance of 2 channels connected in series and we know that you can apply Poiseuille equation to find the overall pressure drop across 2 channels connected in series and Poiseuille equation is valid when Reynolds number are relatively small, so that the flow is laminar, okay and uniform.

When we have let us say you know (()) (00:43), we have recirculation zones developing in a channel flow situation, you cannot apply Poiseuille flow or Poiseuille equation, okay. So, we consider a case 2 channels connected and find the hydraulic resistance, okay. So, let us consider a case where we have a back step, okay so let us say this height is h1 and this height here is h2 and the resistance of the segment of the channel is R1.

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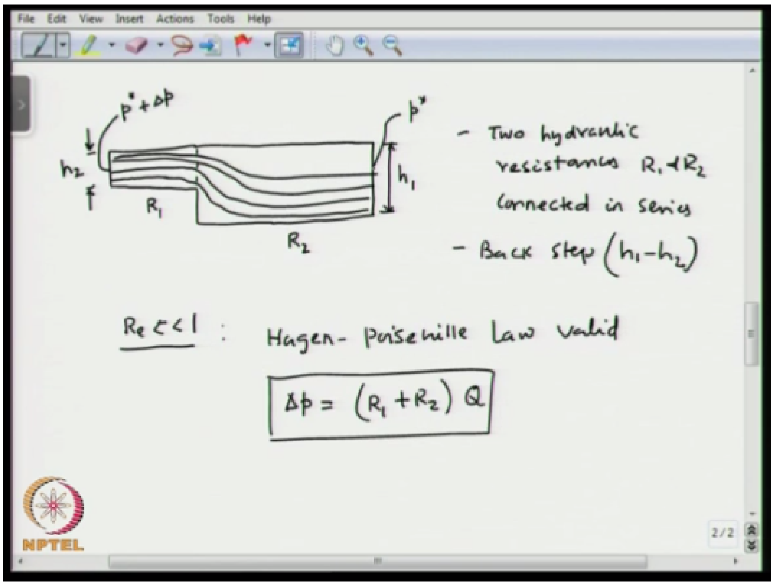

And this segment is R2, okay. So here, the pressure is P* + delta P and here the pressure is P*. So, we have 2 hydraulic resistances R1 and R2 connected in series, right. So and that form say back step, okay. So the back step height is h1-h2, okay. So, we consider the first case, okay where the Reynolds number is << 1, okay. So in that case, the Hagen-Poiseuille law valid.

So, we can write the overall pressure drop delta P will be = R1+R2*Q, okay. Now, so here if the Reynolds number is small, the streamlines will look like this, okay and so on. Now, if we consider the Reynolds number is >> 1, in that situation, we can expect, so this is the back step and Reynolds number is >> 1, what we would expect is a recirculation zone forming here, right.

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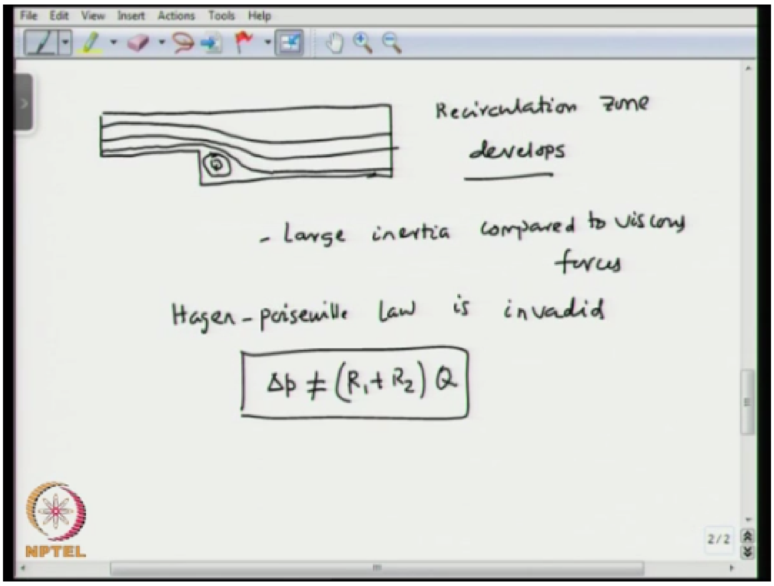

So, L be a recirculation zone forming there and in that case, so here we would have recirculation zone develops and this is because when Reynolds number is >> 1, we have large inertia compared to viscous forces, right. So in that case, we cannot apply Hagen-Poiseuille law. So, Hagen-Poiseuille law is invalid, right. So in that case, we cannot write delta P = R1+R2*Q, okay.

So, you know use of Hagen-Poiseuille law to calculate overall pressure drop across a series of channels is only valid when the flow is laminar and the flow is uniform, okay. There are no vertices presents. So in that case, we can find overall pressure drop by considering the individual resistances, okay. So now, let us look at how the equivalent resistance will be in case of 2 you know hydraulic resisters connected in series and when they are connected in parallel.

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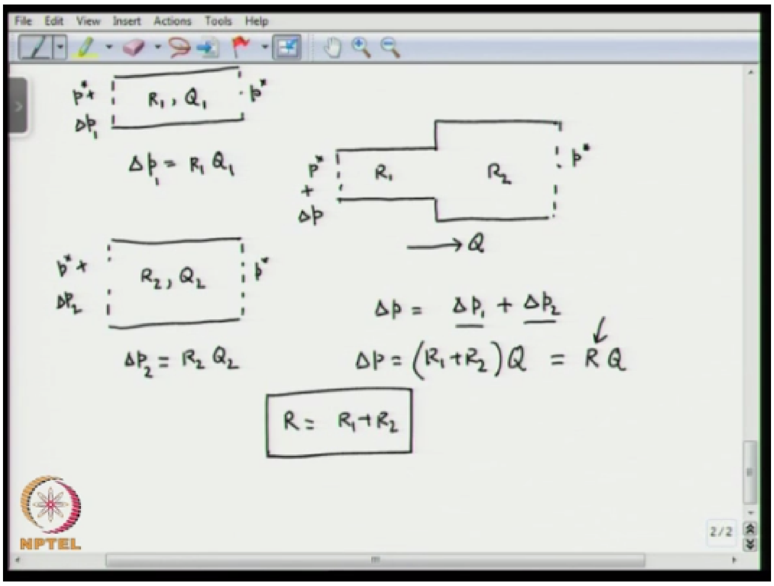

So, we consider hydraulic resisters in series and parallel, okay. So, let us consider 2 channels first, 2 individual channels, so this is let us say channel 1 and here, the pressure drop is P* + delta P and the pressure P*, so the resistance is R1 and flow is Q1, so you can write delta P, let us call it delta P1 = R1*Q1, right. So now, we consider another channel which is little bit wider and here, we have P* + delta P2 and here the pressure is P*.

We have resistance R2 and flow rate Q2, so you can write delta P2 is R2*Q2. Now, if you connect these 2 channels in series. If you connect these 2 channels in series, this is what we will get, right. So here, the pressure will be P* + delta P and here the pressure will be P* and here the resistance is R1, here the resistance is R2 and we have Q constant, okay. So, we can write delta P will be = delta P1 + delta P2, right.

So delta P = R1+R2*Q, so R1*Q is delta P1 and R2*Q is delta P2. So, this will be = R*Q, R is the equivalent resistance of this ne2rk. So, we can write R to be R1+R2, okay. So, when 2 hydraulic resisters are connected in series, the equivalent resistance is the addition of the individual resistances, right. So now, let us look when the hydraulic resistances are connected in parallel.

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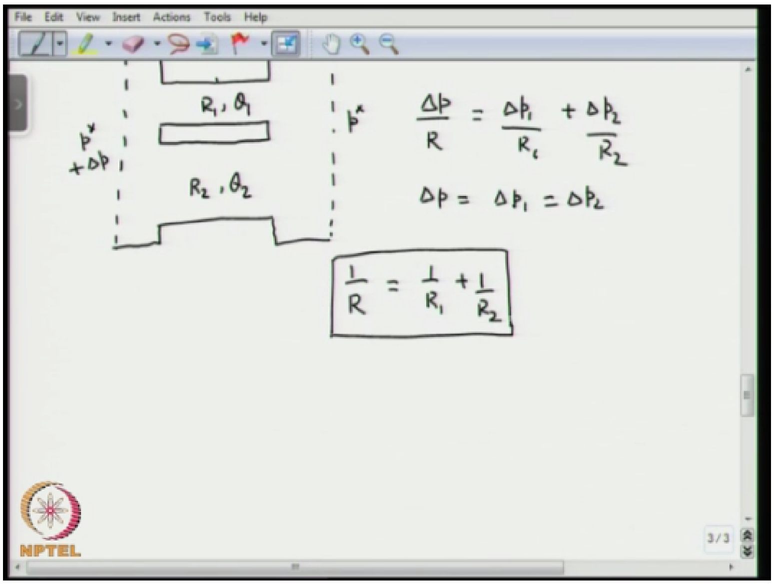

So when they are connected in parallel, let us consider this case here, okay. So, this is P* + delta P, this is P*, so that these are 2 channels, this is channel 1, this is channel 2 they are connected in parallel, this is R1 and R1 is the resistance and Q1 is the flow rate, here is R2 and Q2. So, you can write Q the total flow rate is going to be Q1+Q2, right. So, you can write delta P over R is going to be delta P1 over R1 + delta P2 over R2.

Now, since they are connected in parallel, the delta P are equal. So delta P = delta P1 = delta P2. So you know, the equivalent resistance 1/R is going to be 1 over R1 + 1 over R2, okay. So, the hydraulic resistances behaved the same way as electrical resistances when they are in series, the equivalent resistance is the additional of the individual resistances and when they are in parallel, it can be determined as 1 over R is 1 over R1 + 1 over R2, okay.

This is because the Hagen-Poiseuille law is similar to Ohm's law where the pressure drop is equivalent to the voltage drop. The flow rate is equivalent to the current and we can also say that current is charge per time, so the charge Q is equivalent to volume, right. So, we can define another parameter called hydraulic capacitors, okay which is equivalent to electrical capacitance, which is also known as compliance.

So we discuss a term called compliance, so the Hagen-Poiseuille law is equivalent to Ohm's law and so the pressure drop is equivalent to voltage drop in electrical circuit and the flow rate is equivalent to current which is I, so this is nothing but charge per time, right. So, charge equivalent to the volume, right. So the electrical capacitance c is dq over dv, okay. So, elemental charge divided by the elemental voltage, right.

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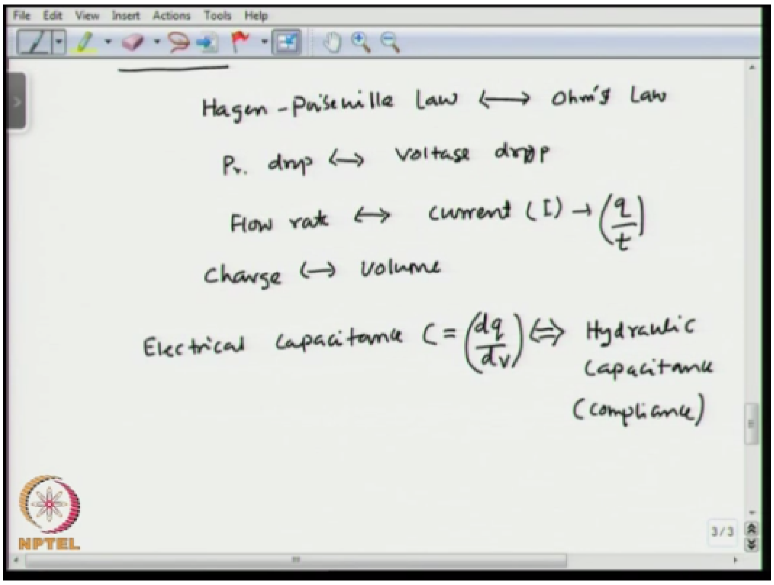

The change in the voltage, which is equivalent to the term called hydraulic capacitance or it is also known as compliance, okay. So, compliance can be defined as, compliance is denoted by c hydraulic is - dv over dp, okay. So, here we notice that there is a negative sign here and the negative sign is because as the pressure increases, the volume diminishes, okay. So, the volume and pressure inversely proportional to each other, so there is a negative sign there.

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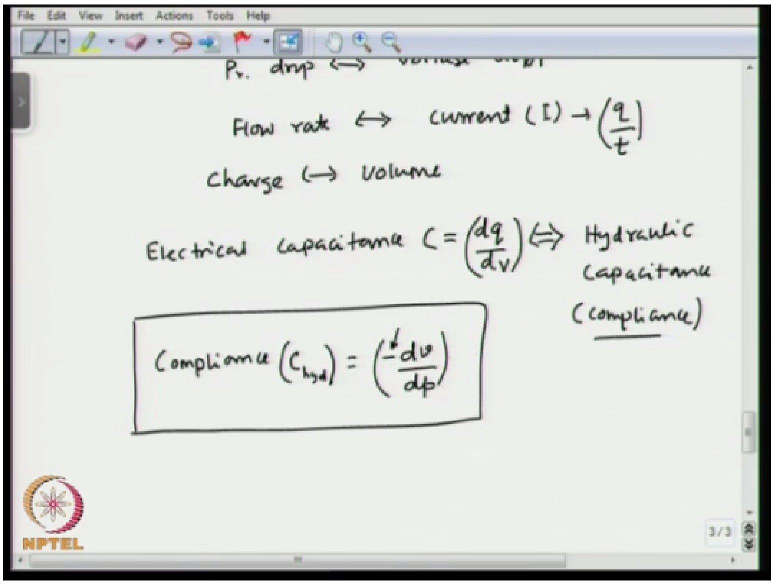

Now, compliance has you know importance in design of microfluidic system, you know if you are not talking about ideal fluids, all real fluids have some degree of you know compliance, so that can be compressed to some degree. Also, if you are talking about in a microfluidic channel with flexible walls, the walls that are not very rigid or we talk about in a microfluidic circuit that I have some kind of membranes in it, so in that case, understanding of compliance is very important, okay.

So here, we will take 2 examples and we look at you know how compliance place a role in microfluidic circuits. So in the first example, we will talk about you know a gas which is compressed by advancing liquid, okay. So, you would have a gas content in a specific volume and which is interfaced by a liquid and as the liquid is advancing at certain flow rate, we would try to find out how the pressure inside the gas would change with time.

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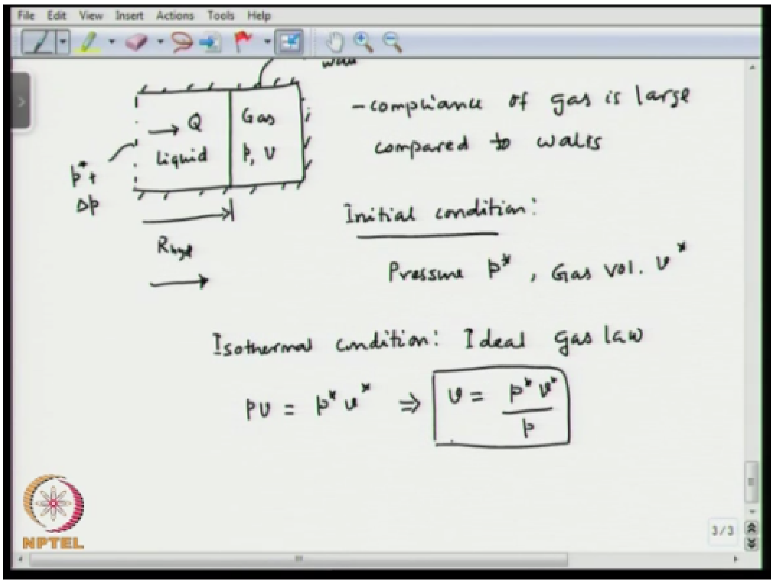

So, we would consider this situation here, okay. This is a rigid wall, so we are not you know talking about deformable walls, we are all interested in how the pressure is varying for the gas. So this is the gas, let us add some instant, the pressure P and the volume is v, this is the liquid, which is moving at flow rate Q and this is that liquid. Here, the pressure is P* + delta P and till here we say the hydraulic resistance is R hydraulic.

Here we talk about rigid wall, so the gas is dropped here in this volume and it is going to be compressed by in advancing liquid that we see here, okay. So, the compliance of gas is large compared to the walls, okay. So that is the assumption that we are making because the walls are rigid. So, let us say initial condition, we have pressure P* everywhere, the liquid as well as the gas, there are pressure P* and the gas volume is v*, okay.

Now, as the liquid is going to advance, the pressure is going to change to PNV, okay and if you assume that isothermal condition, okay, so we can apply the ideal gas law, so you can say

P*V is going to be P* * V*. So, we can find V = P*V* divided by P, okay. Now, we can find compliance c hydraulic is – dv over dp, so we can find it to be P* * V* over P square, okay.

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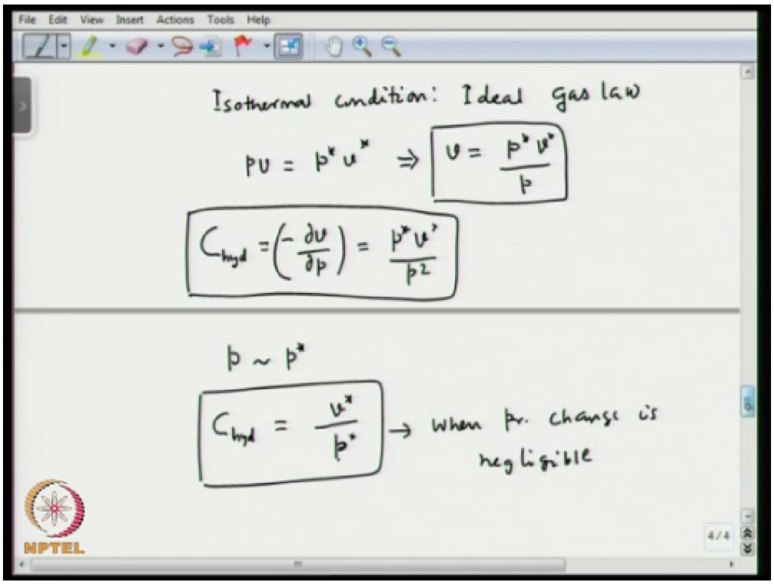

Now, if the change in the pressure is negligible that means let us say P is not very different than P*, we can say the hydraulic capacitance will be V* over P* and this is when pressure change is negligible, right. So at time t = 0, the pressure is P* everywhere as we told in the beginning and as pressure of liquid increases, so the pressure will change from P* to P* + delta P, right.

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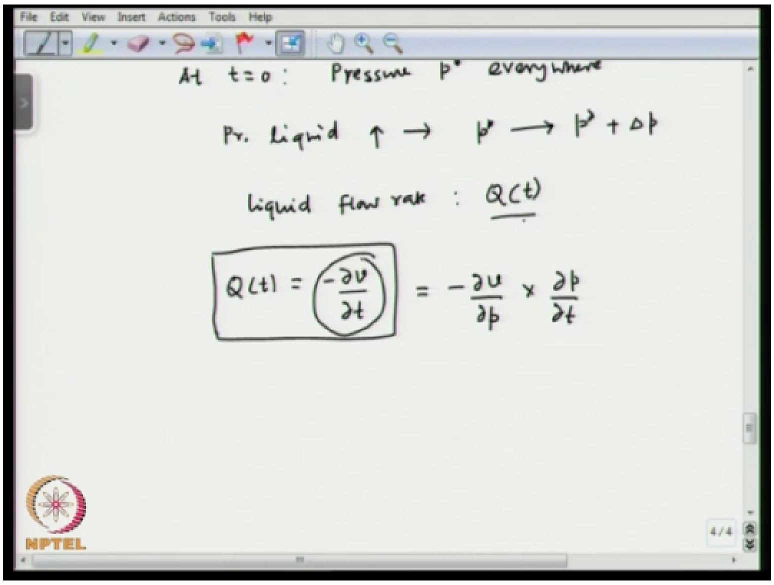

So, the liquid flow rate, as pressure will change, the liquid flow rate would be Qt, right. So, Q will be a function of time and this fluid flow rate would be nothing but the decrease in the gas volume, so as we see here as the liquid is advancing, the gas volume is going to reduce, so the increase in the liquid volume for unit time is going to be the same as the decrease in the gas volume for unit time.

So, the flow rate of the liquid is nothing but the delta V over delta time t, okay. So, the negative sign is because the volume is reducing with time, okay. So now, you can write again Qt is going to be - delta V over delta P * delta P over delta t, so this can be written as - delta v over delta P * delta P over delta t, right, which is = - delta V over delta P is nothing but the hydraulic capacitance, so c hydraulic * delta P over delta t, right.

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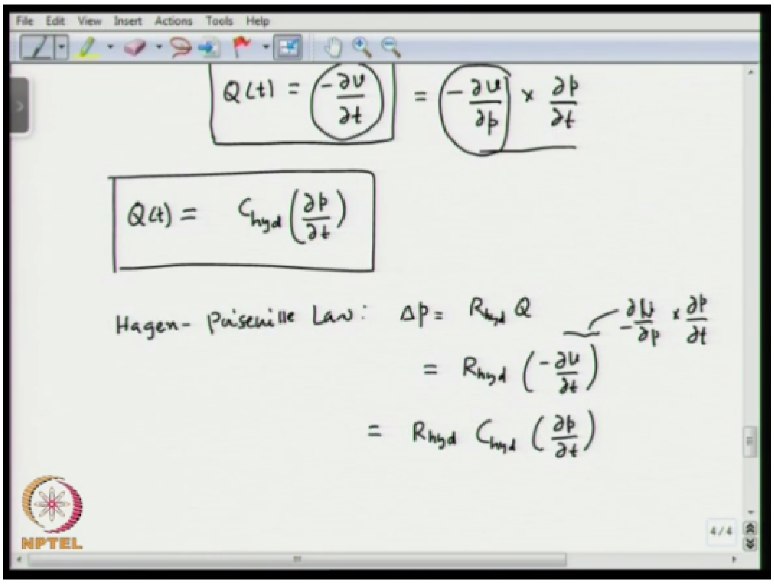

Now, so this is what we found, if you use Hagen-Poiseuille law, you can relate delta P to be R hydraulic * Q, right. So you can write delta P = R hydraulic * Q, which will be = R hydraulic * Q, you can write it as - delta V over delta t and so this you can also write it as = R hydraulic * c hydraulic * delta P over delta t, right. You can write it as delta P over delta t, right.

So this we can expand, delta V over delta P * delta P over delta t, right -, so this becomes c hydraulic and delta P over delta t, right. So, what we see from here is that delta P over delta t = 1 over R hydraulic * c hydraulic * delta P, right. Now, if we integrate, so you first take it to other side, so delta P *, right so now we should do an integration what you would see is P over time is going to change as, sorry this is going to be delta P, right.

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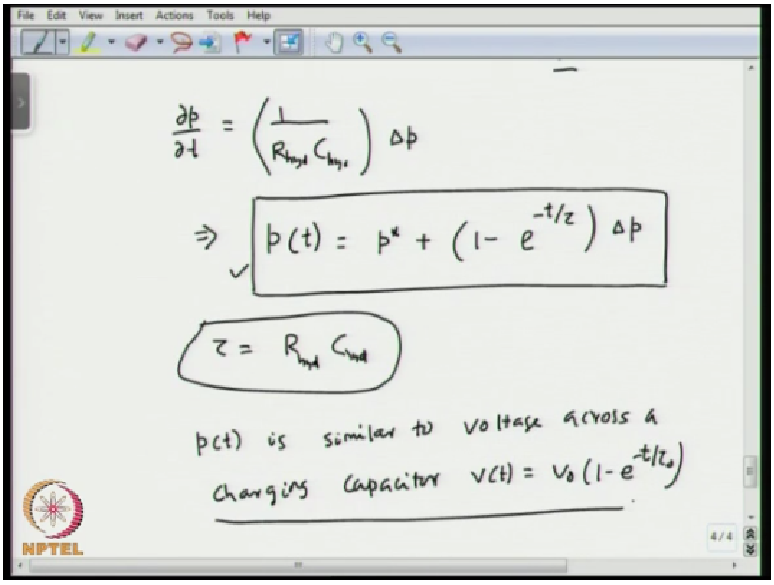

So if you do integration, so you take time that side and delta P here, you get that Pt will be P* + 1 - e to the power -t over tau * delta P. This is what you would get, okay so where the time constant tau is going to be R hydraulic * c hydraulic, okay. So this is the time constant how the pressure is going to change with time and if you look at this term is a very similar to the voltage across a charging capacitor or the voltage changes with time.

In case of a charging capacitor similarly the pressure is going to change with time, okay. So, Pt is similar to the voltage across a charging capacitor where the Vt varies as V0 * 1 - e to the power -t over tau 0, okay. So, if you have an electrical capacitor and you have a source, the charging also occurs in a similar way, okay. So in the same fashion, the pressure of the gas is going to change over time.

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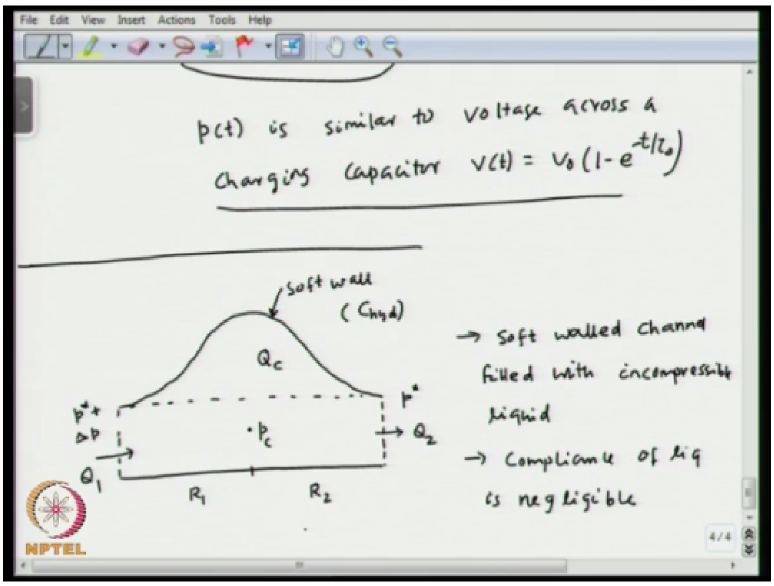

Now, let us look at another example where we talk about a chamber with a flexible membrane, okay and we would see how the pressure of the chamber is going to vary with time, okay. So, we consider another example of a chamber with a flexible membrane, let us say this is the flexible membrane, okay and this is the inlet, this is the outlet of the channel. Now, let us say the pressure here is P* + delta P and the flow rate is Q1.

Here let us say the flow rate is Q2 and pressure is P*, so this is the deformed membrane, let us say initially the membrane is at the shape and let us say the pressure here somewhere at the middle is easy, okay and here, the resistance on this part of the channel is R1 on the left side and this is R2 and the volume rate here is Qc, okay. The rate at which liquid is advancing into the dead volume here and this is a soft wall, c hydraulic, okay.

So, we are considering here a soft walled channel filled with incompressible liquid, okay and so that the compliance of liquid negligible. So what we can write here is we can write the pressure is P* at inlet, let us say for time t < 0 and it gets to P* + delta P for time t > 0. So, the flow rate at the inlet you can find out the flow rate, so when pressure is at t < 0, when pressure is P*, the membrane is on deformed state.

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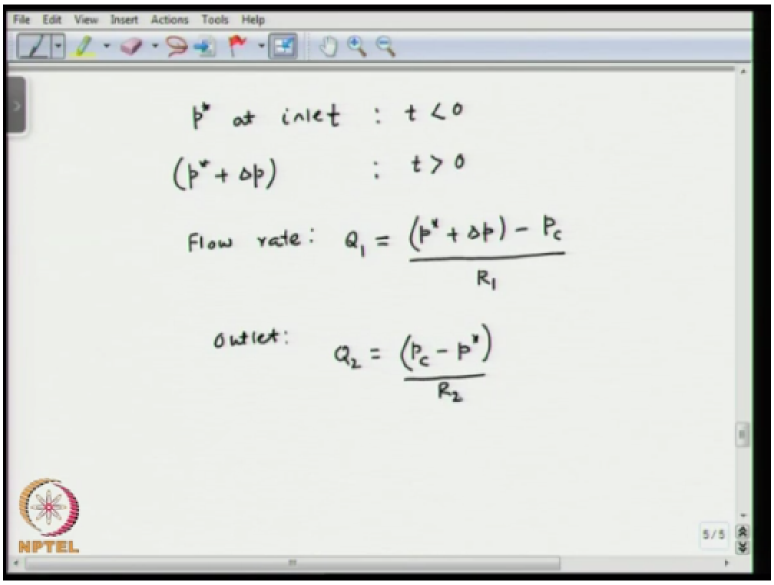

And when the inlet pressure changes to P* + delta P then the membrane is on deforming. So the flow rate Q1 that is coming in at the inlet = P* + delta P - Pc is the central pressure divided by R1, right and similarly, at the outlet, you can write Q2 will be = Pc - P* divided by R2, okay. So from here, you can find the rate of volume expansion which is same as the dead volume created here because of the expansion of the membrane, right.

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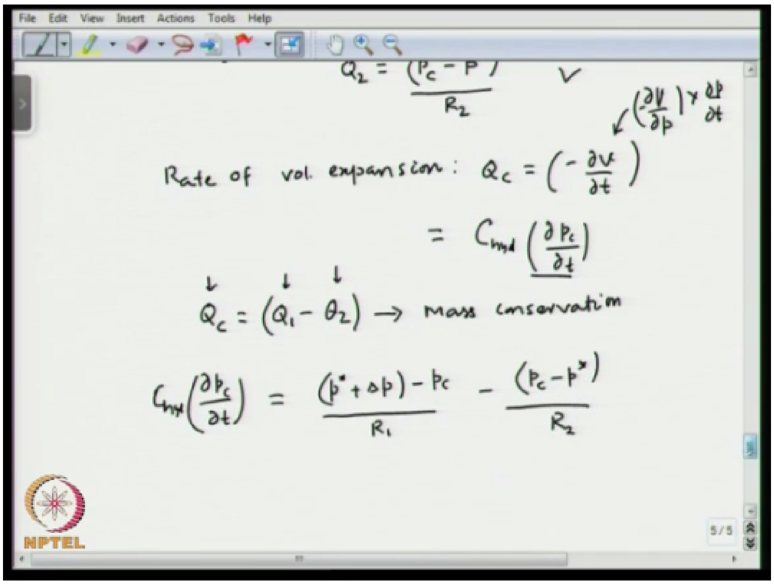

So the rate of volume expansion Qc is - delta v over delta t and you can also write this as c hydraulic capacitance, you can write this as delta v over delta P * delta P over delta t * delta Pc over delta t because the central pressure is Pc, right. So now, you can write Qc = Q1-Q2. So this is nothing but mass conservation, Q1 is what comes in and Q2 what goes of the outlet, so the difference is the volume expansion in terms of expansion of the membrane.

So we can write Qc, we can write this, right. So we can write delta Pc over delta t * c hydraulic will be = Q1, we have an expression for Q1 and also Q2, right. So we can write P* + delta P - Pc divided by R1 - Pc - P* divided by R2. Now, if we rearrange the terms, we will get this delta Pc over delta t will be = - 1 over tau 1 + 1 over tau 2 * Pc + 1 over tau 1 + 1 over tau2 * P* + delta P over tau 1, where tau 1 will be R1 c hydraulic and tau 2 will be R2 c hydraulic.

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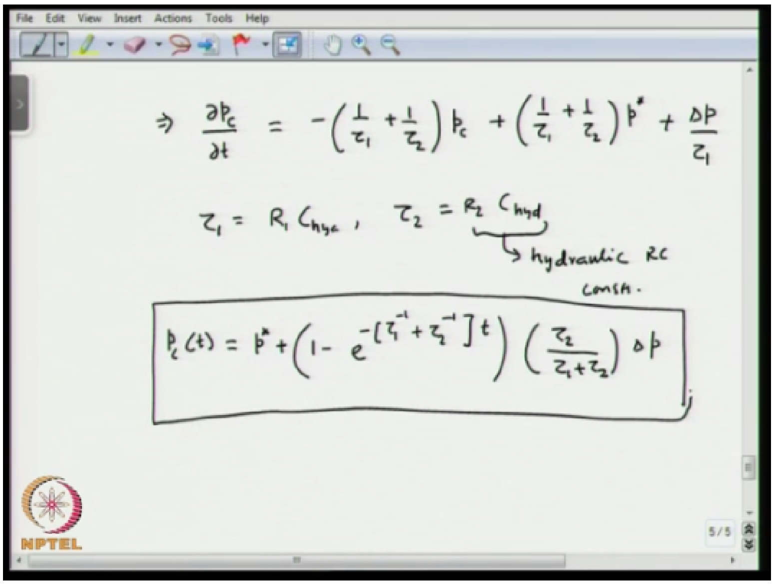

So these are called hydraulic Rc constants, okay. The solution would be Pc t will be P* + 1 - e to the power - tau 1 to the power -1 + tau 2 to the power -1 * t * tau 2 divided by tau 1 + tau 2 * delta P. So that is how the central pressure is going to vary over time, right and this expression is similar to voltage across the charging capacitor which is charged by the voltage divider, okay.

So if we look at an electrical equivalent, this expression looks similar to that, okay. So with that discussion, we move on to talk about you know the drag force on a sphere in a liquid, okay. So we talk about drag force acting on a sphere in infinite domain, okay. So, drag force acting on a sphere has practical importance in microfluidics when we talk about you know movement of particles or cells in microchannels.

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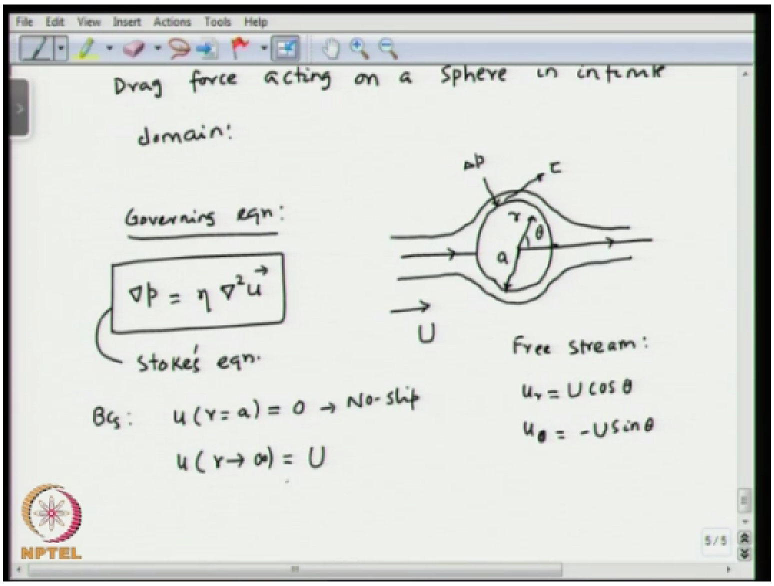

You would like know what is the quantitative expression for the drag force that is acting on microparticles, okay. So let us consider this example, so we have a sphere which is present in an infinite domain, so say this is some r, this is theta, this is radius a, so you would have flow situation something like this. So the pressure force always acts normal to the surface and there will be a viscous component tangency here, right.

And we are talking about uniform velocity U, so we can write the free stream velocities Ur will be U * cosine theta, so this will be along the radius and U theta is going to be - U sine theta, right. So, you know we can write the governing equations, so in this case we are talking about Stokes flow situation. In case of Stokes flow, the Reynolds number is << 1, so the Navier-Stokes equation reduces to something like a Poisson equation, where we have pressure gradient related to the viscous force.

So here, we would have delta P = eta * delta square U, so this is nothing but the Stokes equation, okay. So here, we can write the boundary conditions are U at r = a = 0, so that is the no-slip and the other condition is U at r tends to infinity is going to be U that is the free stream velocity. So you know there are different ways to calculate the velocity fields, okay. So, you know different mathematical techniques can be used to calculate the velocity fields.

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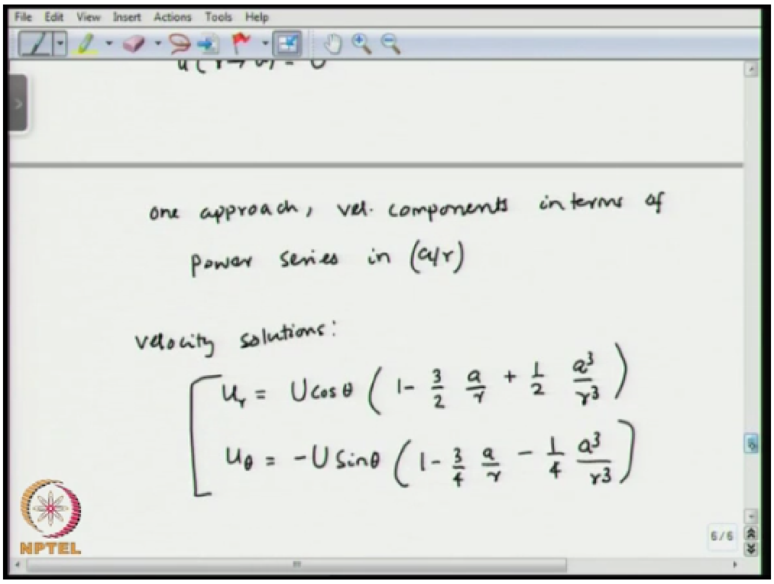

One approach would be to expand the velocity in terms of the ratio between the radius of the circle divided by the radial coordinate, okay. So, one approach so velocity components in terms of power series in a over r. So we can write down the velocity solutions in power series

Ur will be = U cosine theta * 1 - 3/2 a over r + 1/2 a cube over r cube and we can write U theta is going to be -U sine theta * 1 - 3/4 a over r - 1/4 a cube over r cube.

So, if we look at these 2 components and you will see that these satisfy the boundary conditions, okay. When r = a, both these components will vanish and if you use the condition r tends to infinity, the velocity is going to be the free stream velocity, right. So now, if you plug these 2 solutions in the Stokes equation, you can find the pressure expression, so you would find the pressure.

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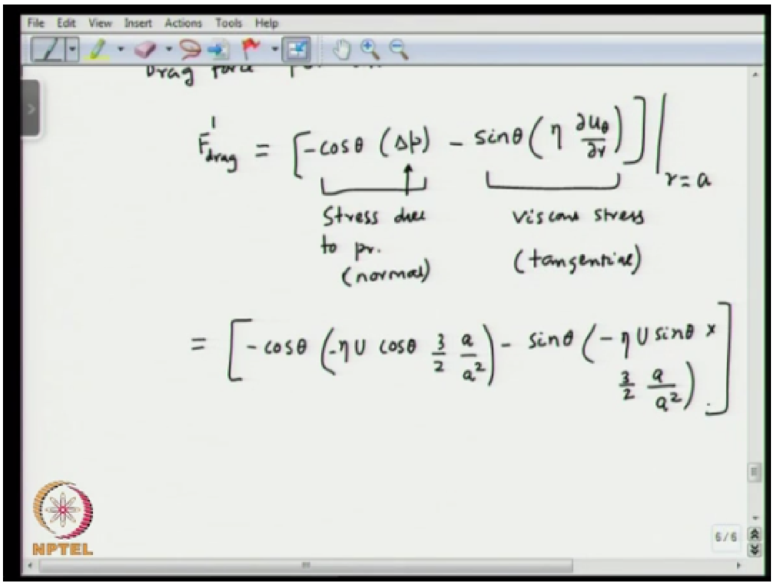

I am writing is as delta P, okay = - eta * u * 3/2 a over r square * cosine theta. So this is the pressure field. So now, if you want to find out drag force per unit area, you can find F drag per unit area use the dash symbol = -cosine theta, so that makes it normal to the surface * delta P, right - sine theta * eta delta U theta over delta r. So, essentially this is the stress that is coming because the fluid pressure.

And if you take a cosine component that becomes normal to the surface, okay. So, this is the stress due to pressure, you know this is the viscous stress. So this is normal to the surface, right, this acts normal and then you make cosine theta, so that makes it the component along the drag and then this is tangencies, okay.

So you can substitute the value for the pressure is - cosine theta * delta P at the see here * -eta U cosine theta * 3/2 a. So this has to be evaluated r = a at the radius. So 3/2 a over r square, we had a over r square so that would be a over a square - sine theta * we have to take the gradient of the theta component of the velocity here, right. So if you do that you will get - eta U sine theta * 3/2 * a over a square, okay.

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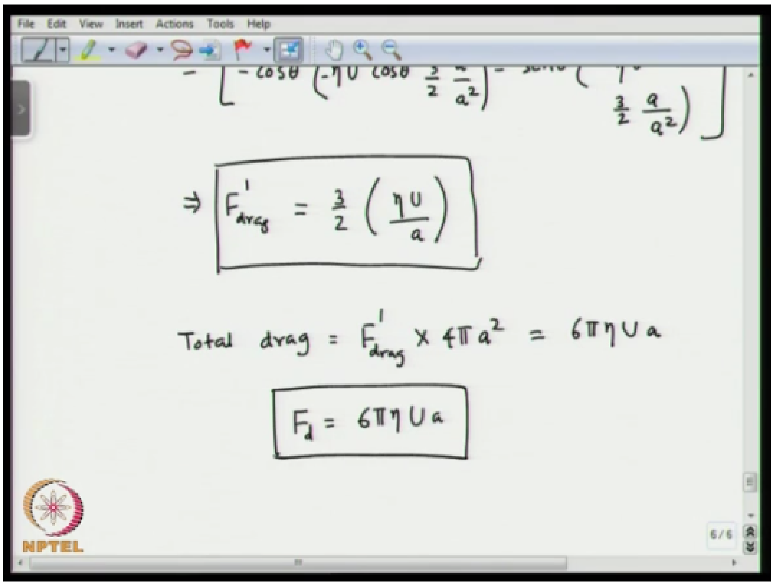

So then if you simplify, you can write this the drag expression will be 3/2 * eta U over a, okay. So that is per unit area. From there, you can find the total drag is going to be drag per unit area * 4 pi a square, okay. So what you will get is 6 pi eta U a. So the drag force 6 pi eta U * a. Now this expression for the drag, you know it is derived from the velocity components which are not valid at last distance from the sphere, okay.

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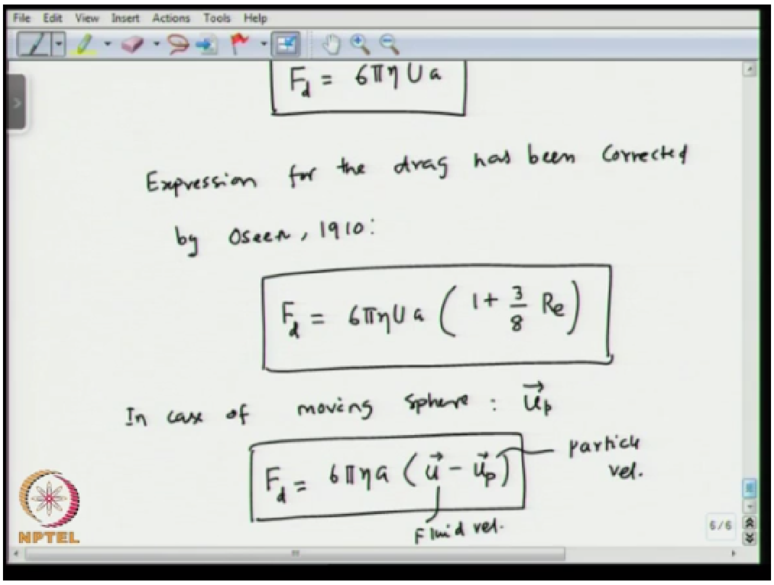

So if you look at this velocity component as r increases, so the velocities ideally should become free stream velocity, so that is not happening so that needs some corrections. So the expression for the drag has been corrected by Oseen in 1910, where he said that the drag is going to be a function of the Reynolds number, okay. So the drag Fd is going to be 6 pi eta U a * 1 + 3 over 8 * Reynolds number, okay.

Now, you have looked at the expression for the drag assuming that the sphere is stationary. If the sphere is moving, then what would be the expression for the drag force, so in case of a moving sphere, let us say it moves at some velocity Up, then the drag force Fd can be written as 6 pi eta a * U – Up, okay. So, this is the velocity of the fluid and this is the particle velocity, okay so with that let us stop here.