Gas Dynamics and Propulsion
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Lecture - 25
Prandtl Meyer Waves

In the last class, we looked at the combined velocity triangles for Prandtl-Meyer expansion.

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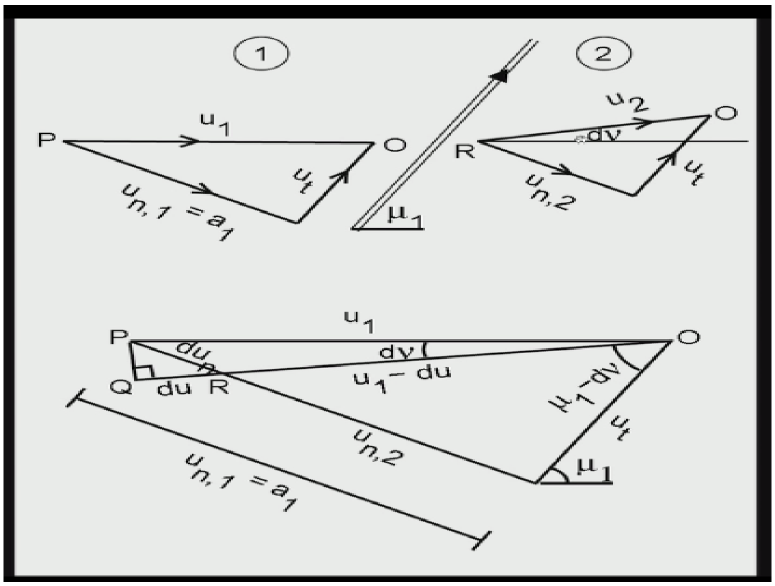

So this velocity triangle that we have here corresponds to the this is the combined velocity triangle for a compressive wave solution because as you can see the flow after passing through the wave is deflected towards the wave. So, this is the compressive solution.

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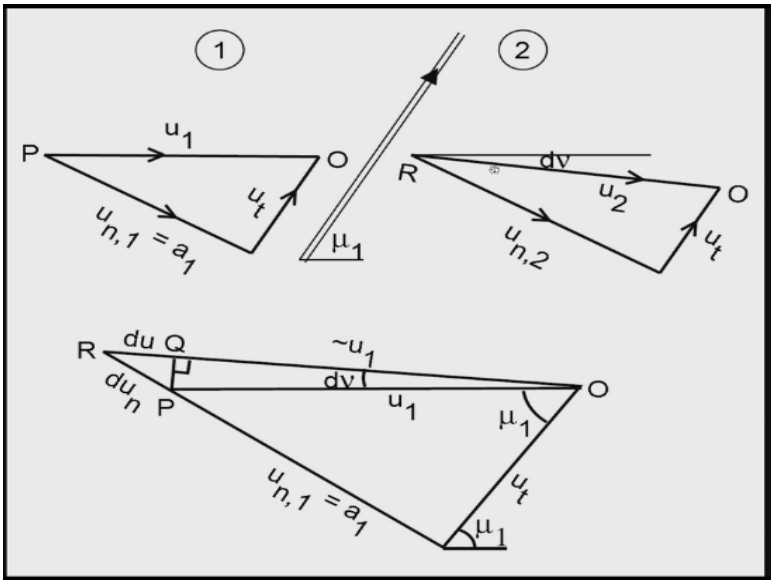

And at this solution also which is the expansive solution. So, here the velocity vector after passing through the wave is deflected away from the wave so this is an expansion solution and based on these 2 triangles we derived some relationship. Let us just recap these things quickly.

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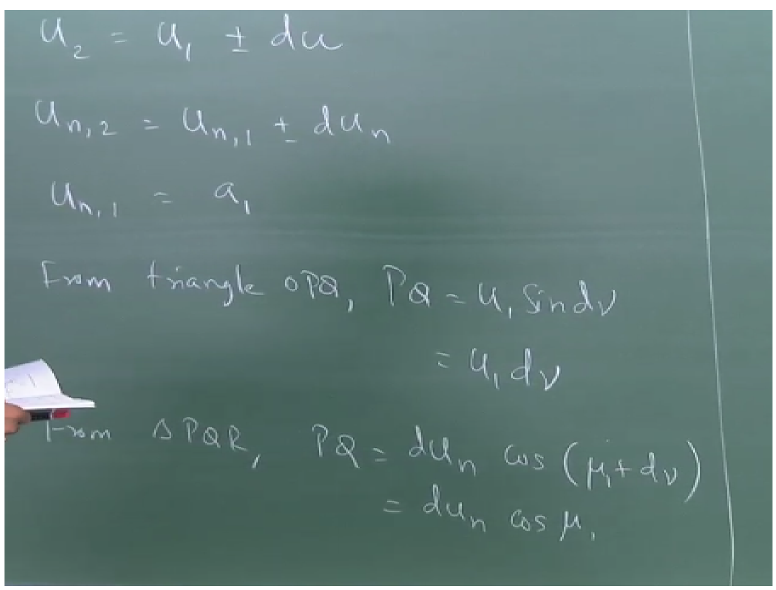

We wrote u2 to be u1+or -du and we also wrote un,2 as u n1+or-dun and if you remember un,1 in both cases is = a1 which is the speed of sound of the flow approaching the wave and based on the triangles for example based on triangle OPQ so from triangle OPQ we wrote PQ = u1 sine d nu and for small values of d nu this can actually be written as u1*d nu and from triangle PQR.

We can write PQ = dun cosine mu+or – I am sorry, mu1+or-d nu and for small values of d nu this can be written as dun cosine mu1 and if we equate these 2 expressions for PQ.

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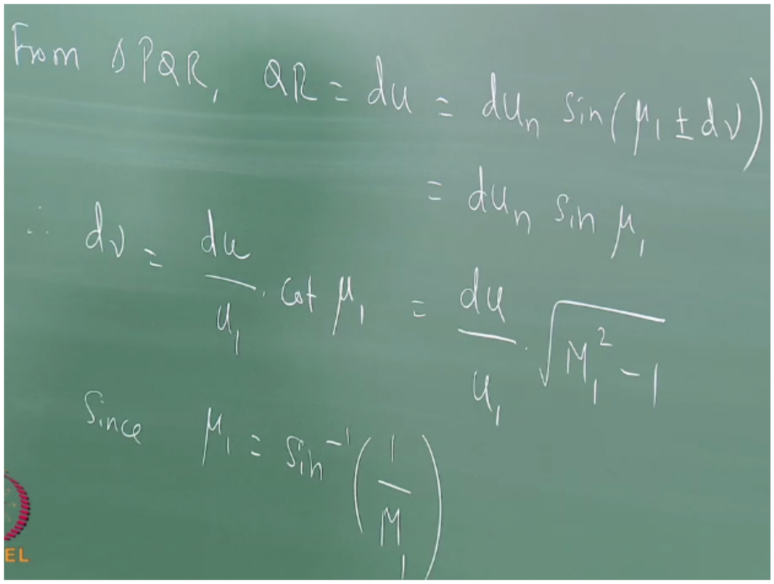

We get dnu = dun/un cosine mu1. Now furthermore from our velocity triangles from triangle PQR, QR can be written as QR = du and that is = dun sin mu1+or-d nu and for small values of d nu this can be written as dun sine mu1. So I can replace the dun here in terms of du and if I do that I get d nu = du/u1*cotangent of mu1 and if you recall the definition of mu1, mu1 is the Mach angle and by definitions sine mu1 = 1/m1.

So if you use that expression that sine mu1 = 1/m1 then I can write this as du/u1*square root of m1 square-1. So here we have used the factor mu1 = arcsin 1/m1.

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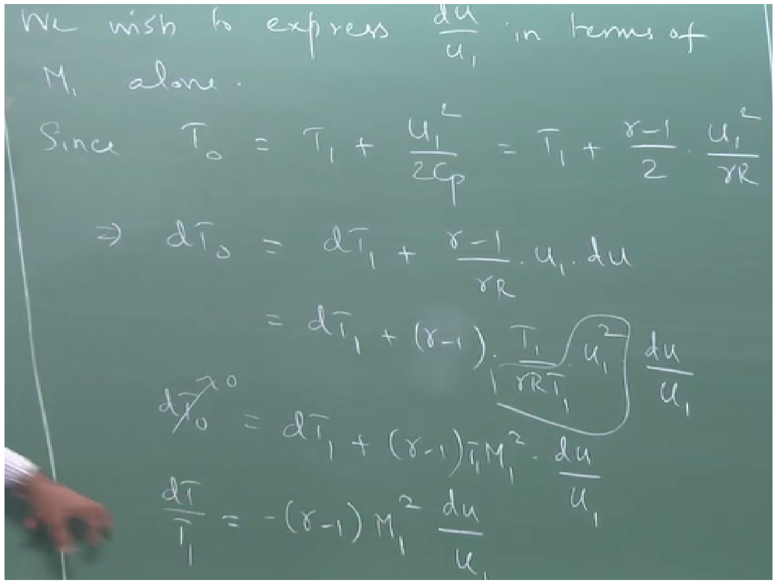

So what we would like to do with this expression is same as what we try to do earlier for Rayleigh flow, Fanno flow, normal shock and so on. We would even cosine 1 dimensional flow, so dnu is the change in the Prandtl-Meyer angle. We want to relate this to m1 alone okay. If you remember for the oblique shock we said that the theta, beta, and m are related. Now we know that the normal velocity approaching this is a1 number 1 and number 2.

The flow turning is also infinitesimal it is an isotropic process so we want to relate the Prandtl-Meyer change in Prandtl-Meyer angle to Mach number alone which means that I want to write du/u1 in terms of m1 alone. So then I will have a relationship which I can integrate and get the close form relationship okay. So, we wish to express du/u1 in terms of m1 alone. So we do this by starting with the definition of the stagnation temperature T0 is = T1+u1 square/2Cp.

Or if I expand for Cp I can write this as gamma – 1/2*u1 square/gamma r. Now if I take the differential on both side I can get dT0 = dT1+gamma – 1/gamma r*u1du. Now if I multiply and divide by a u1 and I multiply and divide by a T1 into this expression I get dT1+gamma-1 I did not like this gamma- 1). So I am going to multiply and divide by a T1 so I have done that and I am going to multiply and divide by a u1 so this becomes u1 square and this becomes du/u1.

So this ratio here you can easily recognize this as Mach number square and this is = m1 square so I can write this as dT1 +gamma-1*M1 square*du/u1. So dT0 is = this and this is an isentropic flow. There is no heat radiation or heat removal which means dT0 is 0. So this is = 0. I am sorry I left out a T1 here, so please make a note of that so there is a T1 here in the numerator. So I can rewrite this expression then as d instead of writing dT1 okay fine if you will allow me I will drop the subscript on the dT1.

And write it as dT/T1 =-gamma-1*M1 square*du/u1. So I have just drop the subscript 1 on the dT1 term which is okay that is alright. So what we have ended up doing is we have try to relate du/u1 to Mach number, but in addition to that we also have a dT/T1 so we need to eliminate this now because you want a relationship for du/u1 in terms of Mach number alone. So what we will try to do next is we will try to use the definitional Mach number to relate this to Mach number along.

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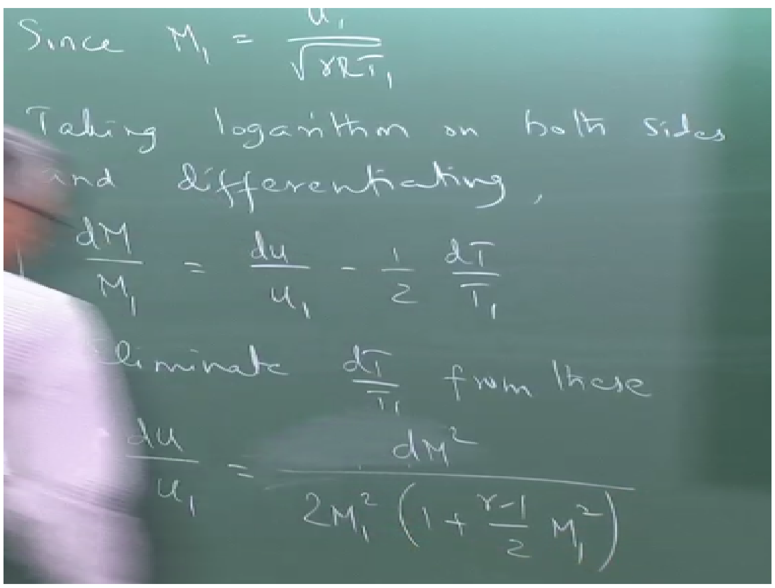

So the definition of Mach number is M1 = u1/square root of gamma RT1. If you take the logarithm of both sides and differentiate, and differentiating we get dM1 or let me write it as dM itself dM/M1 =du/u1 – 1/2 dT/T1. This is very easy to show it is not a problem. So now I have 2 relationships 1 relating du/u and dT/T to dM/M and another one also like this. So I can eliminate dT/T from this and I have the relationship that I am looking for.

So eliminate dT/T to get I can finally write the following. It is convenient to write it like this du/u1 = d of M square divided by 2*M square*1+gamma – 1/2*M1 square. So now I can take this relationship. So finally this is in the form in which I want it. Du/u as a function of M1 alone and so it is in the form in which I want it so I can take that and substitute that into this relationship here. du = du/u*this so I can substitute for du/u from there into this and I will be able to proceed and integrate. Let us do that next.

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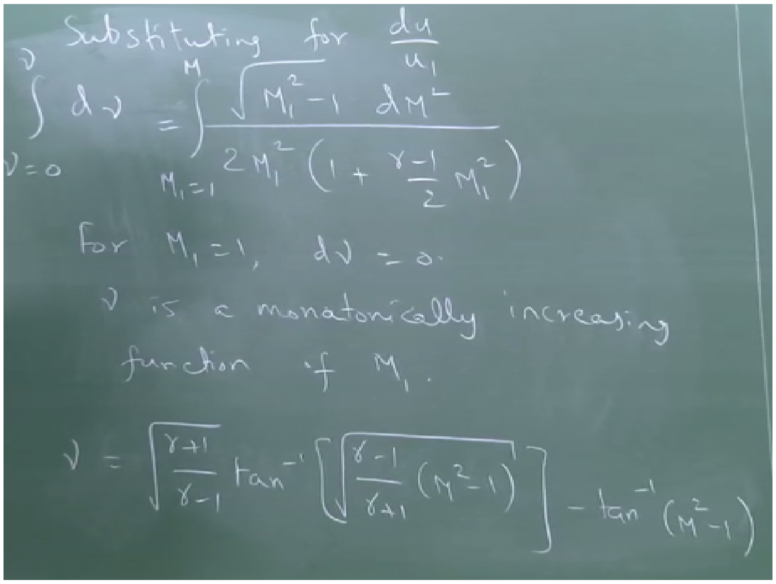

So we substitute for du/u1 into this expression for dnu I get du = square root of M1 square-1 dM square divided by 2M1 square*this quantity within parenthesis. Notice that so there is no flow turning if the flow approaches with the Mach number which is equal to the speed of sound. The Mach angle in this case is 90 degrees and there is an acoustic wave so there is no flow turning in this case. It is an infinitesimally weak acoustic wave.

So nu is monotonically increasing function of M or M1 it does not matter. M1 is the initial Mach number so I can actually integrate this. I can integrate both sides right from say nu = 0 to some value of nu and the left hand side from M1 = 1 to some M. So if I do that what do I get? I get the following messy looking expression M square-1.

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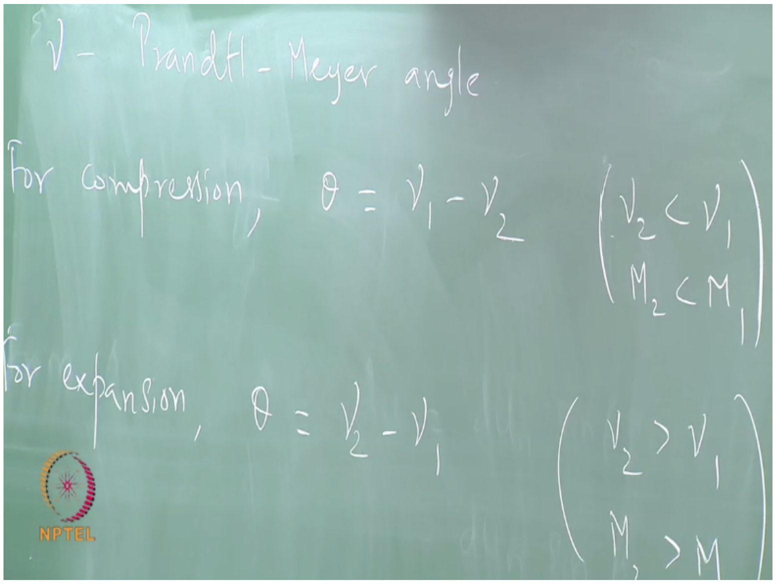

So the angle nu is called the Prandtl Meyer angle. So this is a surprisingly you can actually solve this in closed form and we get this expression. The angle nu is called the Prandtl Meyer angle. So notice that since nu is a monotonically increasing function of M1 you have to make sure that we calculate the angles properly for a compressive and an expansive solution. So if the flow goes through an expansion then nu actually increases and if the flow goes through a compression nu actually decreases.

So, for a compression processes we have to write the flow turning angle theta = nu1-nu2 because nu2 decreases. It is a compression process m2 decreases so nu1-nu2. So let me write it like this nu2 < nu1 because M2 < M1. For an expansion process the flow turning angle theta = nu2-nu1 because nu2 > nu1 because M2 > M1. So you have to be careful about this any questions? So we will do a couple of worked examples to illustrate the concepts and ideas alright.

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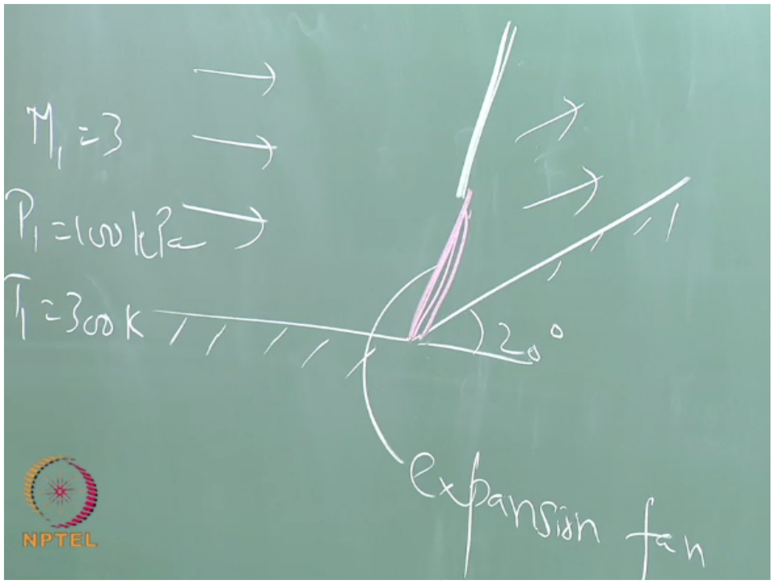

"Professor-student conversation starts" Yes, suppose flow over (()) (16:41) is a supersonic flow when that flow leaves that plate what will be this Prandtl-Meyer angle at that particular point. It is the see the flow turning itself is not defined there. So the flow is going to be much more complex than what we have been doing so far. So it is leaving the flat plate depending upon the ambient conditions.

Then we have to see how the edges the edge is rounded then it is going to go through an expansion processes on both sides. So it will depend upon lot of those things how the edges are okay. So we can really say what it will look for an arbitrary situation. It depends on how the failing edge is and depending on that we can see. If the trailing edge is rounded for example sir, thin flat plate then thin flat plate then I would have to say that you know it is not going to have any effect.

Then you have to use something called linearized airfoil theory. The flow will essentially feel only a very small disturbance you may get an infinitesimally weak disturbance at the trailing edge and may be at the leading edge. **"Professor-student conversation ends"**. The first worked example reads like this supersonic flow at M = 3, P = 100 kappa and T = 300 Kelvin is deflected through 20 degrees at the compression corner.

Determine the flow properties downstream of the corner assuming the process to be isentropic. We did the same example in the previous chapter, but assuming the compression process to be via an oblique shock so we have compression corner like this, this is 20 degrees, so M1 = 3, P1 static pressure is given to be 100 kappa and static temperature is 300 K. So this encounters a compression corner like this.

And as we discussed in our previous class we have series of expansion fans right. Instead of expansion fan is generated from here and this then call us into a normal shock. As you make it smaller and smaller the expansion fans is smaller. So we are going to look at then the flow is deflected like this. So we are going to look at what happens to the flow as it goes through the expansion fan. **"Professor-student conversation starts"** expansion fan or compression fan I am sorry isentropic compression process.

It is a compression process, but you are going to have a as we said as the corner becomes sharper and sharper the fan (()) (19:29) and becomes smaller and smaller so eventually the whole thing will become an oblique shock, but what we are trying to do is to see how the compression process would be had it been an isentropic it is a small radius of curvature instead of being a sharp corner has a small radius of curvature. **"Professor-student conversation ends"**. What is the process going to be like? So we use the tables for this purpose.

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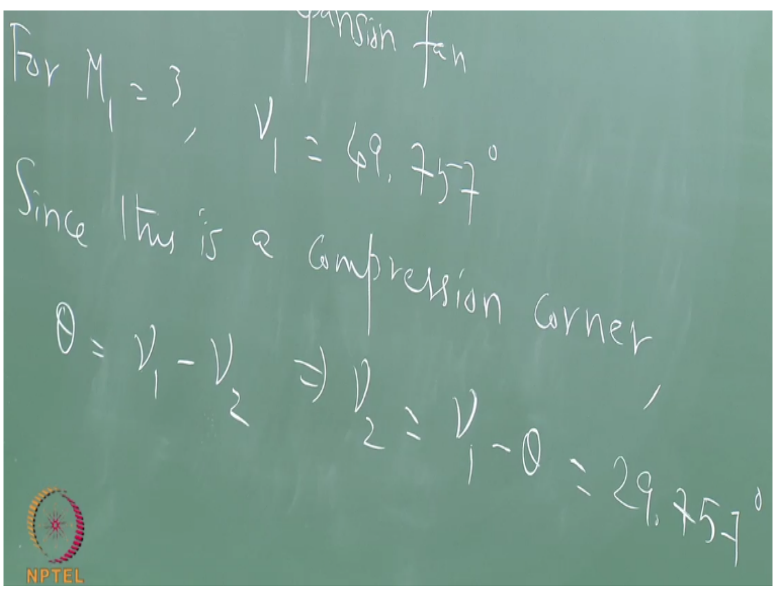

For M1 = 3 the Prandtl-Meyer angles are tabulated in the tabular form so from the table we get for M1 = 3 we get nu1 to be 49.757 degrees. Theta is given to be 20 since this is a compression corner, theta = nu1-nu2 which means that nu2 = nu1-theta and if you substitute the values we get nu2 to be 29.757 degrees.

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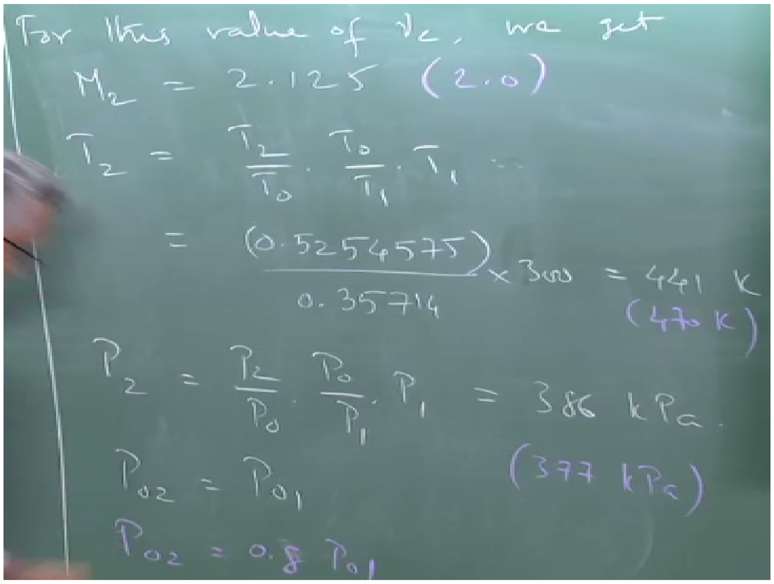

So from the tables for this value of nu2 we get M2 to be 2.125 and the static temperature T2 = T2/T0*T0/T1*and I can get T2/T0 from isentropic table and T0/T1 also from isentropic table and I know the Mach number. So if you substitute the values you get this to be like this 0.5254575 from our isentropic table*T0/T1 sorry I am going to do this divided by 0.35714*300 so this comes out to be 441 Kelvin.

And we also calculate the static pressure down streaming the same way P2 = P2/P0*P0/P1*P1 and if you substitute the numbers from the tables you get this to be 386 kilo Pascal. Of course, there is no loss of stagnation pressure. It is an isentropic compression process so P02 = P01. So the same shock had it being through an oblique shock we worked out this example in the previous chapter.

So as it had been the oblique shock the corresponding numbers would look like this the Mach number for that case comes out to be 2.0 and the static temperature for that case comes out to be or came out to be let me see, did we calculate that, yes 470 Kelvin let me write it down here 470

Kelvin and the static pressure came out to be 377 kilo Pascal and P02 here P02 = P01. In the previous case there was a loss of stagnation pressure to the amount of 20%.

So P02 I write like this*P01 okay. So, now the stagnation pressure is the same for a fully oblique shock there is a 20% loss of stagnation pressure so this is 0.8 I am sorry 0.8*P01. So again you see that both these values are considerably different from what we had seen earlier for the oblique shock. **"Professor-student conversation starts"** see in isentropic compression P2 is 386 and in oblique shock compression it is 377.

But in oblique shock it should be more than the isentropic compression. This is an oblique shock so you have to look at the effective Mach number and then see whether it is going to be greater. See you are comparing normal shock with isentropic compression. Here the normal shock component is happening only for Mn1 and Mn2. So you have to be little bit careful about interpreting these numbers that way, but both the flow rays having same angle unless the deflection angle is the same for isentropic also and also for oblique shock also.

In oblique shock then P2 should be higher than the isentropic, but the no, but you are looking at you know the compression taking place in different frames of reference. The normal shock compression for oblique shock is only for the Mn1 and that is not although M1 = 3 this one is less. So we are considering static it is independent of (()) (25:57) that is correct, no, no; that is correct, but what I am saying is this is fully isentropic, this is not a normal shock.

The curves that we drew earlier where for normal shock so for an oblique shock whether it will hold or not there is something that you know there is not really straight forward so it can happen this way. For normal shock yes, in the normal shock always gives you higher values of pressure and temperature compared to isentropic compression for the same change in specific volume. Remember there are many constrains there for the same change in specific volume this is true.

Here we are not comparing for the same change in specific volume. I am not said what the change in specific volume is, so you cannot directly say that it should be like that okay. When you are comparing 2 curves what are the limits against which we are comparing is also important. Then there will be a slip line definitely there will be a slip line across this because entropy here will be higher. So there is will be a slip line as we drew in the previous class.

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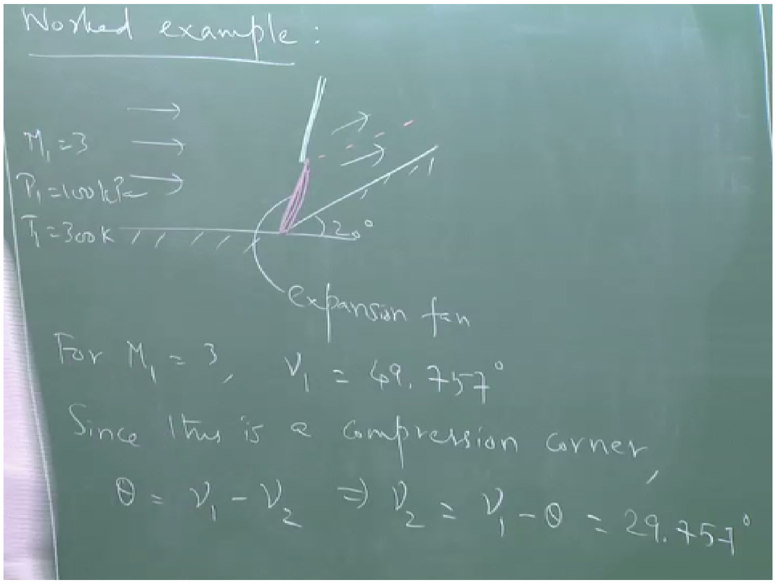

There will be a slip line like that but about slip line pressure should be same, but now pressure above slip line and below slip-line is different. No, the thing is as you bring this closer and closer you are going to have a complex interaction here in this region. The slip line itself may not be just a slip line. So to equalize a pressure here you can have for example an expansion wave which originates from the corner here then goes out.

Slip line need not be just a discontinuity. Slip line can also create additional flow structures to equilibrate the pressure. Our theory is really very simple. We are doing hand calculations you know so many of the complexities we cannot assume or we have to idealize. **“Professor-student conversation ends”**. Okay so that is one worked example. The next worked example that we are going to do is here we looked at a compression corner for 20 degrees. Let us see what happens if this is an expansion corner of 20 degrees.

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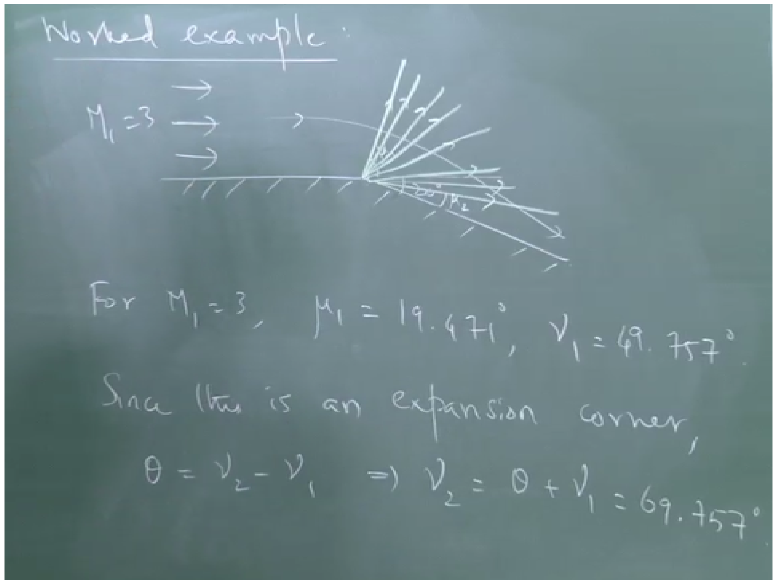

So this is 20 degrees now. So we have flow at the same Mach number M1 = 3 which approaches the expansion corner like this and we are going to generate. So this corner generates and expansion fan the initial angle is let us say this is the first expansion, first Mach wave like this so this has a direction like this. The last one will presumably go something like this and the flow is deflected around this.

So the flow grows through then it becomes parallel to the wall. So this angle is mu1 and this angle with respect to the remember this angle is always measured with respect to the velocity vector that is approaching this line so that means this angle is going to be measured against this velocity vector so this is mu2.

So from the tables for M1 = 3, we have mu = 19.471 degrees and mu1 we had already calculated 49.757 degrees. Now this is an expansion corner so mu increases across the wave so that means theta = mu2-mu1, so which implies that mu2 = theta+mu1 and if I substitute the values I get this to be 69.757 degrees. So you can see that mu increases across the expansion fan because it is an expansion process.

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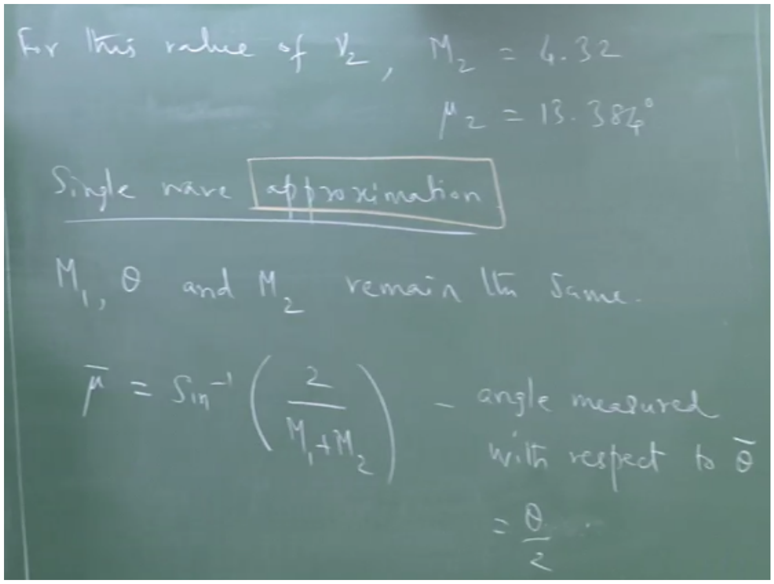

So, for this value of mu2 we can get from the table M2 to be 4.32 and mu2 to be 13.384 degrees. What is that mu1 is measured with respect to u1, mu2 is measured with respect to u2 which is now parallel to this one right. So in between these 2 you have an expansion fan which looks like this. Now having something like this is little bit inconvenient for the kind of analysis that we are doing. So there is an approximation that we can make.

The approximation is for this value of M1 and this value of turning angle and this value of M2 can we replace this expansion fan with the single wave remember single expansion fan for a finite turning angle is prohibited by second law, but for approximation purpose for calculating reflection and so on. See if you want to calculate the reflection of such a fan then you have to look at reflection of each one of these waves it become very cumbersome.

So effectively we can actually approximate this fan with a single wave it is an approximation not realistic, for purposes of hand calculation for the down that is something that is customarily done. So let us just see how that is done just for future use. So single wave approximation remember this is an approximation. So the emphasis is on the word approximation. Such an approximation we will allow us to actually calculate.

I mean do the hand calculations little bit more. We can actually look at expansion and reflection of expansion fan and so on. So this is made under same conditions M1 theta and M2 remain the same as before. So, we do not change any of this. So, what we do is we calculate an average angle for the single wave let us call that mu bar. This mu bar can be calculated in several different ways.

For example, the simplest thing is to take mu bar to be the average of mu1 and mu2 or slightly better approximation is to calculate the mu bar like this take it as the average of the Mach number rather than average of the angle. And this mu bar is measured with respect to this so this angle measured with respect to theta bar which is theta 1+theta/2. So which means that what I am doing is the following let me draw a separate diagram.

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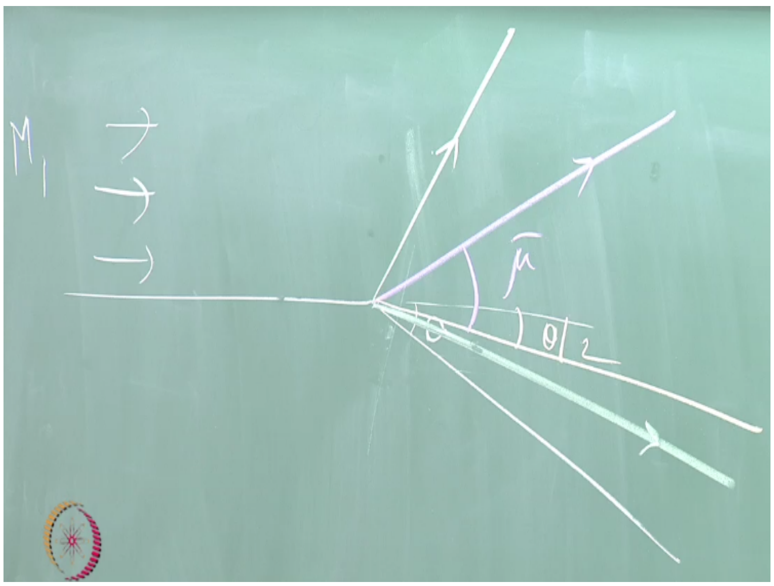

So these were the initial and final Mach waves and let us say that this is my, this angle let us say is theta/2. So the single wave that replaces all this looks like this and this angle is mu bar. So that angle is measured with respect to theta/2 so that is mu bar. So we replace this entire fan with the single wave like this. This is an approximation such a solution is not allowed, but it is an approximation.

It is a very good approximation for the kind of hand calculations that we are doing. We can actually deal with reflection of a single wave much more easily than reflection of an expansion fan. You know which contains an infinite number of waves almost, so very useful approximation for engineering calculations. The usefulness of this approximation will become apparent when we do the next worked example. The next worked example is the continuation of our earlier worked example. If we remember we looked at this example before.

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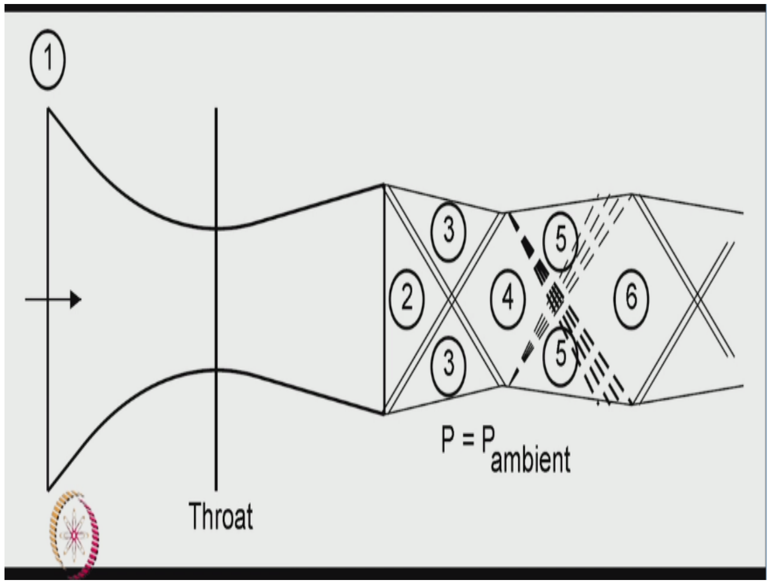

And we had actually calculated flow properties up to section 4 now we are going to actually go ahead and look at flow prosperities in section 5 let us see what happens with this.

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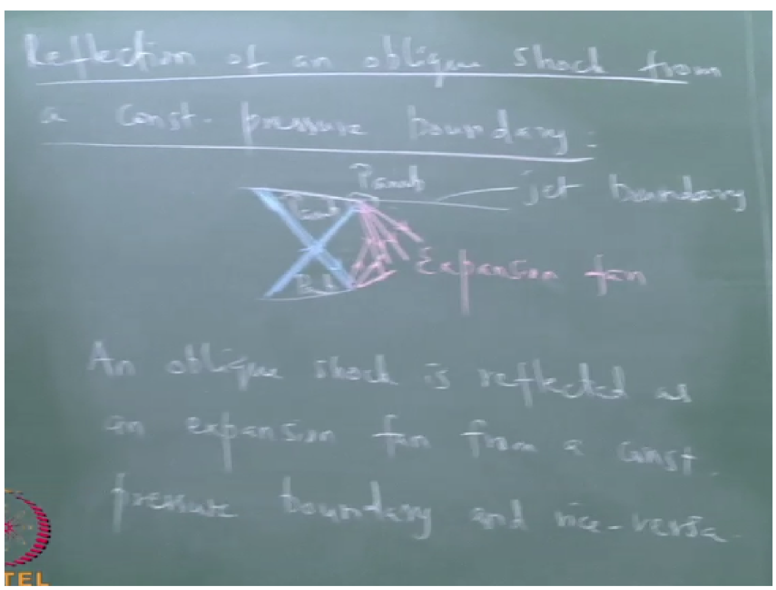

Before we do that we need to look at one concept which is reflection of an oblique shock from a constant pressure boundary. In the previous chapter, we had looked at reflection of an oblique shock from a wall or a solid boundary and we saw that an oblique shock is reflected back as an oblique shock from a solid boundary that was what we looked out in the previous chapter. Now, we are looking a situation where let us say.

We have a jet boundary like this and an oblique shock somehow is triggered and so this is a say jet boundary or a constant pressure boundary. So pressure is same across this. So pressure is P ambient here along the jet boundary and an oblique shock wave is incident. Let us say like this so, that was the situation that we are looking at if you remember our previous solution look like this we had so this is what we were looking at and then we wanted to see what happened after this.

So remember this region this was a P ambient, this was a P ambient and now this is going to be reflected so this was a P ambient this was also at P ambient and now the jet is impingement. The oblique shock is impinging upon the jet boundary we want to see how it comes back. So if I focus my attention on the point of impingement here. So when the jet I am sorry when the oblique shock impinges upon the jet boundary the pressure increases.

Static pressure always increases across the oblique shock wave. However, the point of impingement now is on a boundary which is exposed to the ambient pressure which means the pressure there always has to be the same as the ambient pressure. So the impingement of the oblique shock causes the pressure to increase so the increase in pressure must immediately be relieved by an expansion fan.

So an oblique shock when it impinges upon a constant pressure boundary this is reflected as an expansion fan. So that at this point the pressure is always ambient pressure. We saw in the earlier chapter that oblique shock impinging on a wall is reflected back as an oblique shock and the angle depends upon the angle of the wall or that the point of impingement. In this case here because it is a constant pressure boundary pressure has to remain constant.

The increasing pressure due to the oblique shock must be immediately offset or relieved by an expansion fan which brings it back to which brings this point back to the same ambient pressure. So that means what is going to happen to the pressure in this region pressure in this region is higher right That is what you are looking at and the pressure in this region will continue to be ambient.

Because we are going to have a set of an expansion fan coming from here also and this is what is shown in this figure. So you can see in this figure that the oblique shock impinges at this point and there is a generation of an expansion fan from that point which goes down like this. Similarly, the point of impingement of the oblique shock from above is here and once again we trigger expansion fan from here.

So the pressure in region 5 remember region 5 is now in direct contact with the atmosphere separated only by a jet boundary so that means pressure in region 5 is P ambient same as outside. So an oblique shock impinges on a wall it is reflected back as an oblique shock when it impinges upon a constant pressure boundary it is reflected back as a wave of the opposite kind. Notice that when this expansion fan impinges upon the jet boundary.

Here for the same reason that we mention it will be reflected back as an oblique shock wave because the pressure at the point of impingement tends to be less than atmospheric pressure so we must trigger an oblique shock like this, but the reflection of a fan from a jet boundary is much more complicated so the flow situation becomes more complex for the downstream.

Which is the reason why we discussed replacing a fan with a approximating a fan with a single wave so then the reflection calculations become easy. We can actually proceed a little bit more with hand calculations, but now it if s a fan we cannot do that okay that is the usefulness of the single wave approximation for the expansion fan. Although it is not seen in reality it makes our life easy because we can do hand calculations a little bit more.

So what we are going to do now is the do the calculation continue the worked example. So let me summarize the findings here. An oblique shock is reflected as an expansion fan from a constant pressure boundary and vice-versa. **“Professor-student conversation starts”** This will change flow direction also. Which one? Expansion fan flow will be parallel to the axis. No, flow will not be parallel to the axis. The flow is now going to be deflected away from the wave.

If you remember in the previous case the flow vector was actually like this. It was about 3 degrees above the horizontal on the top side now because it is going to be reflected away it may be brought back to the horizontal, but there is no guarantee then it will be like that. So in the previous case it was not horizontal it was like this. Now we have an expansion fan so the flow is going to be turned further away from this I am sorry.

It is going to be turned further away which means that the jet has to swell now because there is also an expansion process so the jet is going to swell and that is what you are seeing here. So you can see that the jet now begins to swell on both sides and the flow is actually deflected away. Equilibration of the velocity vector will take much longer what is more important is equilibration of the pressure **“Professor-student conversation ends”**.

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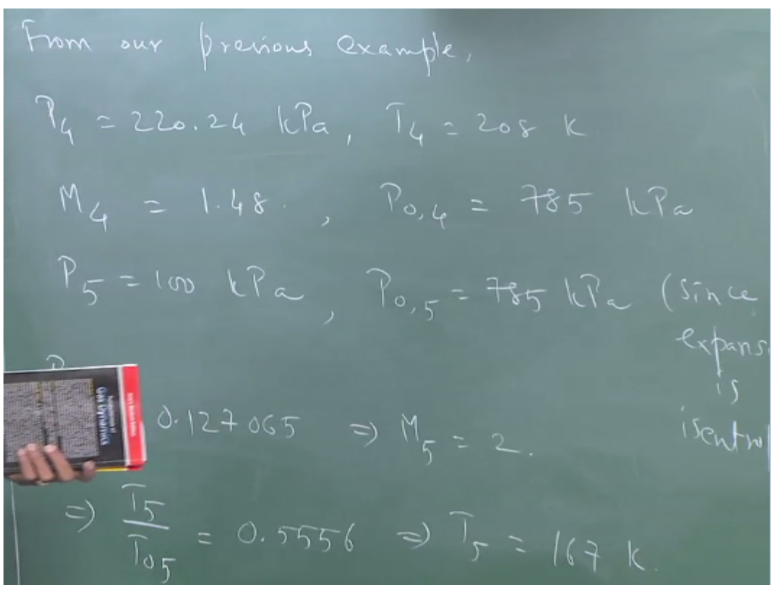

So from our previous example we are going to continue that now and if I remember let me just write down few things just to refresh our memory. We had P4 static pressure in region 4 was 220.24 kilo Pascal, static temperature in region 4 was 208 Kelvin and Mach number can someone let me what the Mach number was M4 was 1.48. So based on our discussion so far P5 = 100 kilo Pascal, P5 is the ambient = the ambient pressure.

So P5 = 100 kilo Pascal and the previous calculation also gave us P0,4 to be = 785 kilo Pascal. The expansion fan is an isentropic process so in this case P0,5 is also = 785 kilo Pascal since the expansion is isentropic. So P5/P 0,5 = 0.127065. So, this implies that M5 is approximately = 2. So, the Mach number increases to 2.

So, T0,5 is 300 Kelvin so from the tables I can get T5/T0,5 is = 0.5556. So this implies that T5 = 167 Kelvin since T0,5 = 300 Kelvin that was given at the beginning of the problem. Now, we are also asked to calculate the angle that the edge of the jet makes with the horizontal. Let us calculate this and finish the problem.

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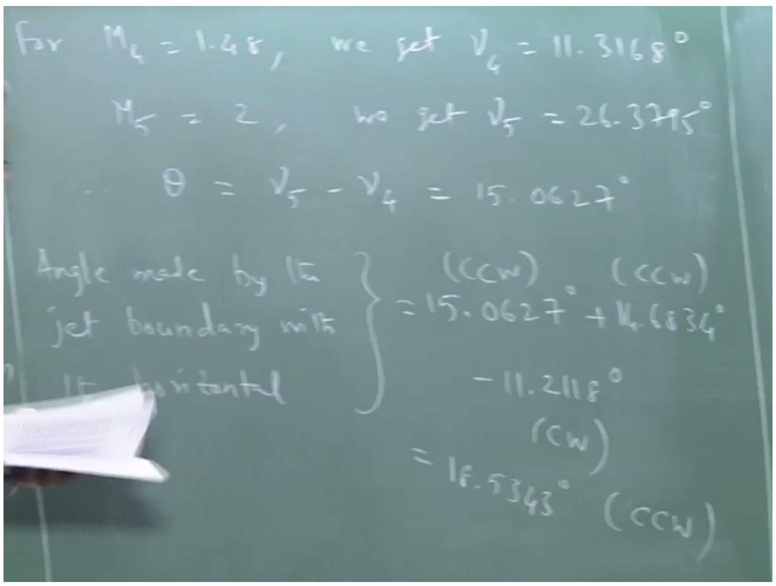

So from the tables for M4 = 1.48 we get mu4 = 11.3168 degrees and for M5 = 2 we get mu5 = 26.3795 degrees is an expansion process so the flow deflection angle in this case is going to be mu5-mu4 so that = 15.0627 degrees. So this 15.0627 remember is the flow deflection angle respect to the previous velocity vector so that means angle made by the jet boundary with the horizontal = 15.0627 degree that is turning in.

This 15.0627 is turning away from the wave in this direction and the previous oblique shock turned it through an angle of 14.6834 degrees-11.2118 degrees. So these are number that we have taken from our previous example. Remember, this is turning in the counterclockwise direction. This is also turning in the counterclockwise direction. This is turning in the clockwise direction.

So let me put it like this. So this is turning in the counter clockwise direction. This is also turning in the counter clockwise direction. This is turning in the clockwise direction so, if I calculate the algebraic sum I get this to be 18.5343 counterclockwise that is the angle that the edge of the jet makes with the horizontal. So this example concludes our discussion of gas dynamics. We have completed all the chapters in gas dynamics. What we will do from the next class onwards is start our discussion on propulsion.