Gas Dynamics and Propulsion
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Lecture - 24
Prandtl Meyer Waves

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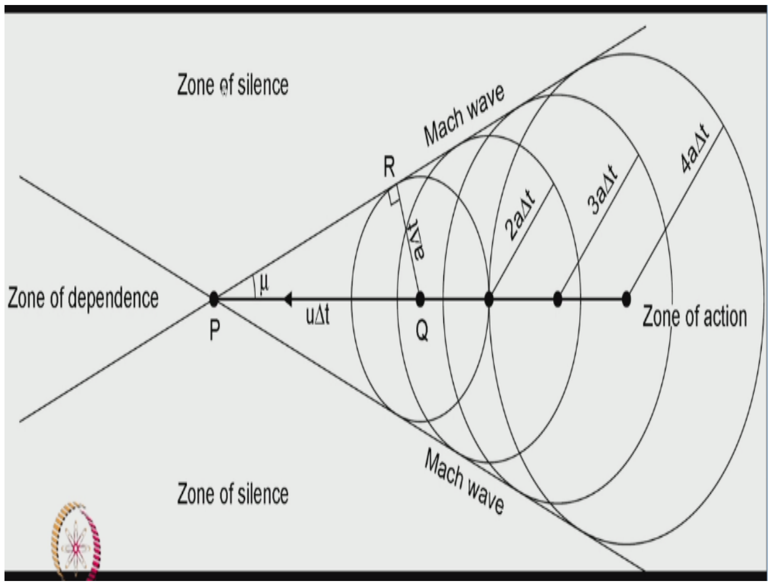

In the last class, we looked at the Mach cone and the propagation of disturbances from an object which was moving with speed greater than speed of sound. So this is the point disturbance which moves with the speed which is greater than the speed of sound and these are the spherical wave fronts of sound waves which were generated at an earlier incident and time, and as you can see from this triangle PQR you see that the radius of this triangle.

So, this is drawn after an instant delta t after time interval of delta t. So in a time interval of delta t this disturbance propagates the distance a *delta t as you can see from here and in that same interval of time the object would have travelled a distance equal to u*delta t, so that the angle semi vertex angle of the cone mu is going to be nothing but a/u which is then equal to sine inverse 1/m.

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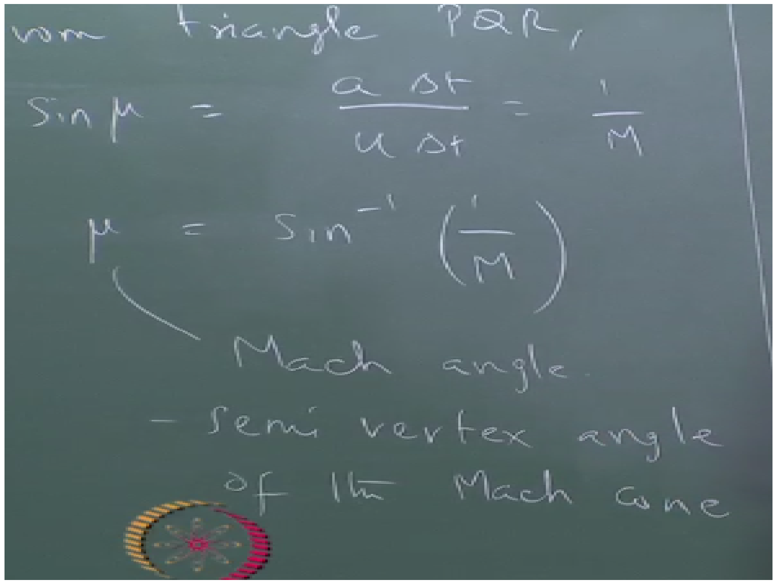

So from the triangle PQR in this figure, we can see that sine mu = a delta t divided by u delta t that is 1/m so that mu is arc sin 1/the Mach number and this mu is called the Mach angle. Note that mu is not the flow deflection in this case. The flow is deflected through a very small value, much smaller than mu. So mu is the semi vertex angle of the Mach cone. So this is the. The other point that I want you to note from this diagram is that the point R.

If you look at point R what is that the wave front here any point here where the surface of the cone is tangential to this noticed that the surface of the cone is tangential to this spherical wave front at any point as you can see from here so the angle here is 90 degrees. We have already made use of that in defining or in evaluating the Mach angle. So this is 90 degree so which means that the spherical wave front expands this wave with velocity = a.

So that means if you look at the Mach wave, the Mach wave is the surface of the cone. So, the velocity component normal to the surface of the cone is equal to speed of sound a, and we can also see that there is a flow which approaches this wave. If you look at an observer sitting at point R. He perceives the flow to approach him with the supersonic speed in this direction from left to right because the object itself is moving right to left and it is dragging the cone along with it.

The object is dragging this cone along with it with the supersonic speed in this direction so if you put an observer here he will perceive the flow to approach with a supersonic speed from left to right. In addition, you will also see a component of the flow which is approaching him with a velocity = a in this direction, because this cone itself is moving the observer this way normal to the direction.

So he will observe the flow to be approaching him in this direction with a velocity a. So, the observer sees 2 components, one velocity u in the horizontal direction, another one velocity a in this direction. We will make use of this information later on when we look at flow deflection as the flow passes through the Mach wave. So, the normal component of velocity is a, the speed of sound of the approaching flow. Now if I look at situation like this.

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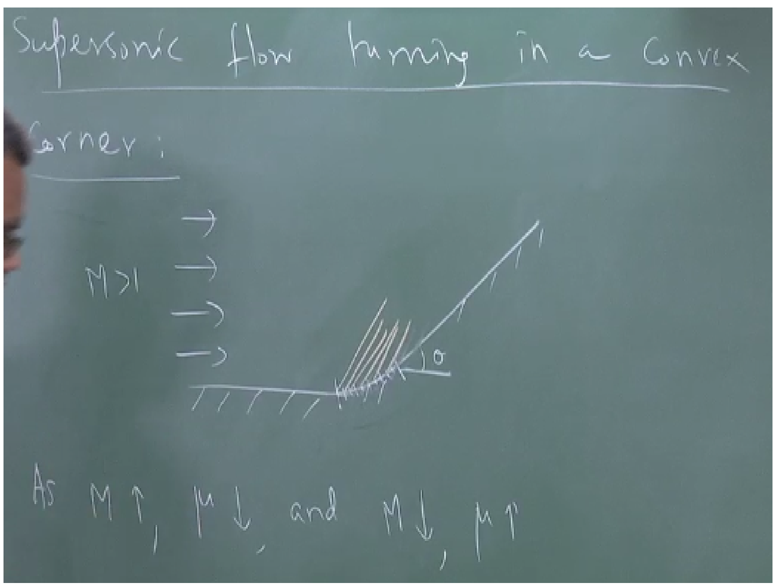

So, let us look at a convex corner. We have already seen a shock corner. Now we are going to see a smooth corner. So let us say that we have a flow situation like this. So we have supersonic flow approaching a smooth corner like this. This is a smooth convex corner. Earlier we saw a shock concave corner and now we are looking at a smooth concave corner. Now, as the flow goes through this corner what happens because we have a smooth corner is that I can imagine.

Let us say that the turning starts from here to here let us say this is the corner turning starts from this point and ends at the other point. Then I can actually imagine the corner being divided into many small infinite number of pieces, many small pieces. And each of this little tiny piece deflects the flow by a small amount and acts as the disturbance that we saw in the previous case okay.

So each of those tiny piece of the wall is what we have described here as the disturbance. So each of the tiny piece causes a disturbance in the flow and deflects the flow by a small angle which means that I should be able to see a Mach cone from each one of this tiny piece. So I am going to draw starting from here. So, each one should generate a Mach wave like this.

Now the question is are these waves going to be parallel to each other or will the waves intersect or will the waves diverge from each other. That is the next question that we must answer. I have just drawn it without looking into that affect. Now if you see we define the Mach angle to be arc sin 1/M. So, when M = 1 the Mach angle is 90. So as M increases what happens to the Mach angle?

This angle as M increases the Mach angle decreases as M increases the Mach angle decreases. So as M increases the Mach angle decreases correct and as M decreases the Mach angle increases correct. So this is the cave corner, this is the compression corner which means the Mach number is decreasing. So that means the Mach angle from each of this piece is going to increase. So that means the waves are actually going to it is not going to look like this.

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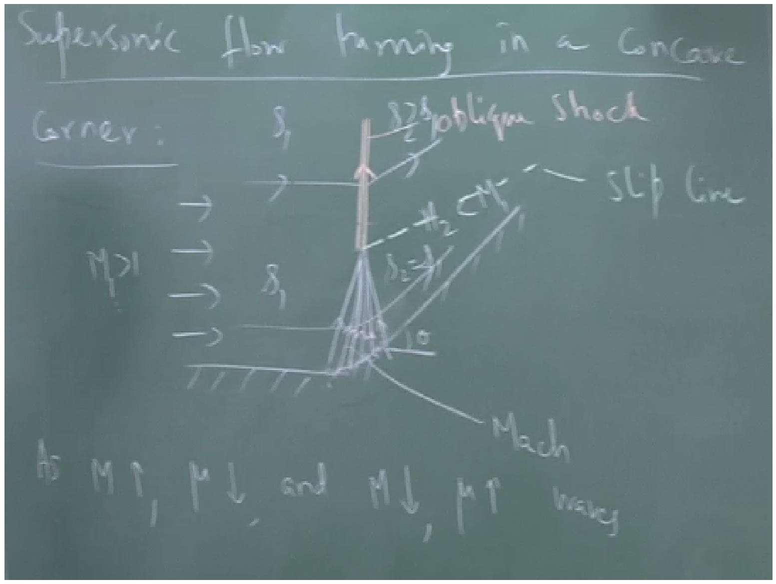

The waves are actually going to intersect and normally the way this happens is so here is our first wave, here is our next wave so this is how the waves normally do. So they actually intersect at a point. I am not going to show this, but that is what happens. So these are Mach waves. I have shown only a finite number, but what you must remember is the reason for discussing the earlier theory.

The reason for discussing this theory, propagation of a disturbance is to bring out the fact that any small point disturbance will do this in this supersonic flow and we are actually saying that when you are want to turn each one of this little surface it is going to act like a point disturbance that is why I keep emphasizing point disturbance. So the flow if you look at a stream line so the flow grows through then it is deflected through an angle theta.

So here, if I call this M1 so M2 is going to be <M1 probably > 1 in a corner like this, but the interesting point is as this Mach wave focus at a certain point they all call us and from this point onwards we have so the wave coalesce and turn into an oblique shock. So flow that goes through like this will go like this. So, at this point each one of this wave turns a flow through an infinitesimal angle.

But when they focus and coalesce like this then the turning is actually through a finite angle that is why coalesce into an oblique shock wave because oblique shock wave represents turning through a finite angle whereas Mach wave represents turning through an infinitesimal angle so the coalesce and then do this, but notice that the process that the fluid undergoes when it goes through an oblique shock wave it is an irreversible process.

So in this case, S2 is if I say this is S1 then S2 is > S1 whereas here the Mach wave compression through a Mach wave represents an isentropic process so here S2 = S1 so if this is S1, S2 = S1 which means that in this case we are going to get a slip line which represents a discontinuity in some other properties that we talked about earlier entropy particularly and other properties.

Now if I actually decrease the radius of curvature of this corner as I keep reducing the radius of curvature of this corner this point keeps moving towards the corner and eventually when it becomes a shock corner we will have an attached oblique shock we will have an attached oblique shock.

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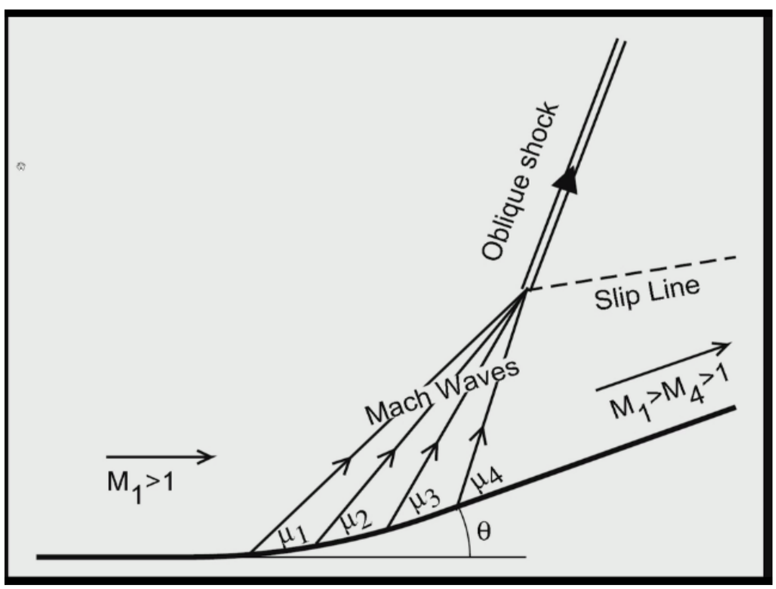

Let us look at this picture. So you can see the situations sketched here. Notice that this is the supersonic flow here we have a smooth corner and you see the coalescence of the Mach waves into an oblique shock with a slip line. Remember the Mach angle again like the flow deflection angle, and the wave angle in an oblique shock, the Mach angle is also measured with respect to velocity vector u1 or the velocity vector before the shock wave.

So that this angle here is mu1, this angle is mu2 not the angle with the horizontal, but this is mu2 now. This is mu3, this is mu4 and given the fact that the angle increases with Mach number and the flow itself is continuously being deflected that is why these waves focus to a point okay. So this is flow turning through a convex corner let us look at flow turning. I am sorry this is the concave corner please make this change. This is a concave corner not a convex corner so this is supersonic flow turning in a concave corner.

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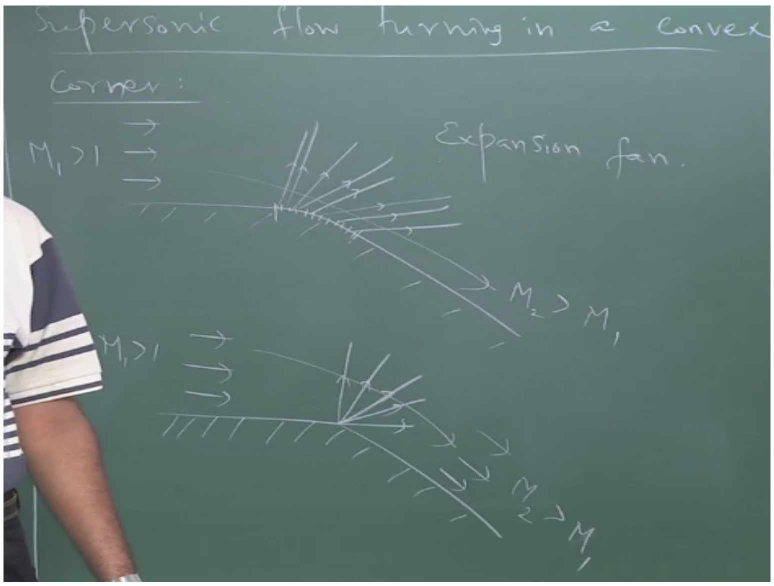

Next we are going to look at supersonic flow turning in a convex corner. "Professor - student conversation starts" Sir, flow passing through an oblique shock will have a higher pressure downstream of shock and flow passing through isentropic compression will have less pressure, but about the slip line the pressure should be same. No, here in this case, you cannot say that the pressure after the oblique shock is higher.

Because the oblique shock is a coalescence of all these Mach waves so whatever pressure rise you are seeing across all the Mach waves at the end will be the same pressure rises. Strength of the oblique shock is such that that is what you are going to see. So if you look at this you can see that the wave angle here the wave angle is such that the pressure here and the pressure here are the same. The Mach number is the same.

Flow deflection is through the same theta, but then you are actually adjusting this to get a shock wave whose strength is such that the pressures are the same. How can we make sure that deflection angle is also the same and pressure rises also same. We cannot make sure of that. We cannot make sure of that, but the flow that is solution that the flow will always towards that you are going to get a flow where the pressure is the same across this, but not necessarily the other flow properties.

So, there will be some mismatch. So, the mismatch shows up in the other properties, but not in pressure. The flow will always evolve in such a way that the pressures are matched. There will be a mismatch in velocity for example. Mach number see Mach number downstream of this need not be the same as Mach number downstream of this. We are not trying to say that everything will be the same.

We cannot say everything will be the same obviously entropy is not the same. So there will be some other quantities which are different. For example, there is a lots of stagnation pressure across the wave. There is no loss of stagnation pressure here which means that if the static pressures are the same here and here that tells me that the velocities have to be different they cannot be the same so the mismatch will show not in pressure, but in other quantities.

“Professor - student conversation ends". So here we are looking at flow turning through a convex corner and once again if this is the part where the flow turning takes place where the flow is continuous being turned so there is no turning in this section and there is no turning in this section we have flow turning only between these 2 points. So each one of this is going to act as an infinitesimal disturbance in the flow and it is going to generate a Mach wave.

Each one of this will generate a Mach wave, but in this case, because we are turning through a convex corner the Mach number continuously increases. So given the variation of the Mach angle with the Mach number, this tells me that the Mach waves are actually going to the angle is going to reduce so that means if this wave from this part looks like this away from the next part would actually look like this.

So there is no possibility of the waves focusing on each other and coalescing into an expansion shock, so each one of this will generate a wave which looks like this. So this is an expansion fan and the flow is expanded through this and the Mach number here if you call this M2, M2 > M1 in this case. This is an expansion process and you may recall that whatever we are doing now is actually comforting to know that this trend is consistent with what we discussed earlier.

And we said that there is no counter part of an oblique shock wave. In other words, you do not have an oblique expansion wave. Their flow is turned through a finite angle. Because you said that would result in reduction in entropy or decrease in entropy which is again second law of thermodynamics. What is interesting is this trend also tells us the same this. It is this trend where you know we say Mach number as.

Mach number increases mu decreases it is that trend which tells me that these waves cannot coalesce into expansion shock like this. So, we are now looking at the same phenomenon, but from 2 different prospective. In the first case, we said that there can be no expansion shock from second law of thermodynamics arguments. Now, we are saying that there can be no expansion shock based on considerations of isentropic expansion.

So it is very nice when looking from different aspects the concepts that we are discussing are consistent that shows that everything hangs together nicely, everything works nicely. So in this case, we said that as I reduce the radius of curvature the oblique shock keeps moving towards the corner and eventually when we have 0 radius of curvature for the corner it will anchor itself at the corner here.

A similar kind of situation here is something like this, so you have a shock corner. So I have a flow coming in like this M1 > 1 and I have a shock corner M2 > M1 then what happens in this case is this corner actually triggers an expansion fan so we have an expansion fan like this and then the flow is deflected through this fan as it goes around the corner. So even this expansion process it is not accomplish with the single expansion fan, but with multiple expansion fans.

Again that is consistent because if I accomplish this expansion with the single expansion fan that means I am turning the flow through a finite angle which means this would have been an expansion shock which is not permitted. So even this kind of turning has to be accomplished through a series of expansion fan each one turning the flow through an infinitesimal angle. The only difference between this and the oblique shock is that the Prandtl-Meyer wave.

Or the Mach wave can represent both the compressive and expansive solution. So if the flow decelerates, then the Mach waves represent a compressive solution. If the flow accelerates then the Mach waves represent an expansive solution so both are possible. So when we derive expressions for connecting the Mach number and the flow turning, we must take this into account so let us write this down.

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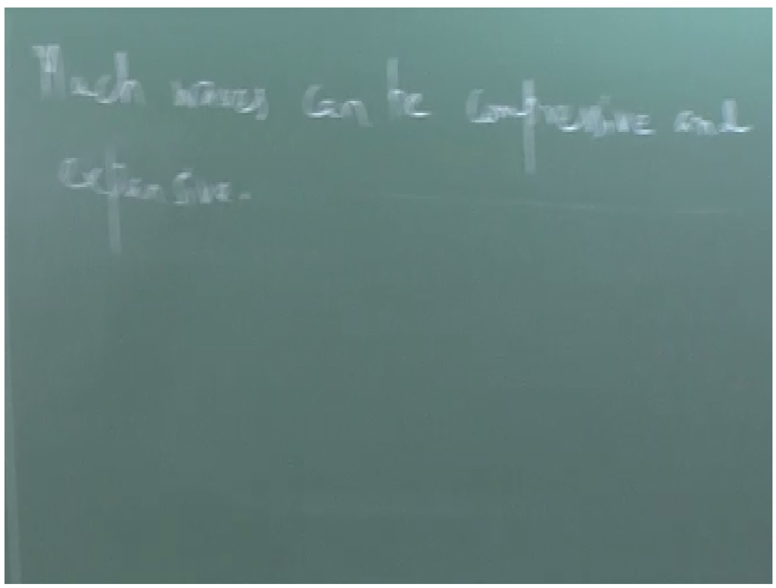

So Mach waves can be compressive and expansive. In the earlier situation that we drew when we looked at a point disturbance which was traveling with supersonic speed that was traveling with a constant supersonic speed, but if it starts accelerating then we are going to degenerate different Mach cones. So if it starts accelerating then the Mach waves that it would generate would look like this.

The same point disturbance if it starts accelerating then we will get different Mach cones like this. The same point disturbance if it starts decelerating then we will get Mach waves which look like this which is why are saying that Mach waves can be both compressive and expansive. **“Professor - student conversation starts"** sir in concave corner the flow downstream of both the shock and the isentropic compression flows parallel to surface and slip line should be parallel to surface.

Slip line usually will be parallel to the surface, but that can be situations where you actually instead of getting 1 slip line you can also get may be one more slip line. There can be more discontinuity. You can also get instead on situations slip lines like this. So there is a discontinuity across each one of this and then it comes out like this.

So the exact orientation of the slip line depends upon many other quantities so that is not something that we can predict from the theory that we are discussing here from the frame work that we are starting about. **“Professor - student conversation ends"**.

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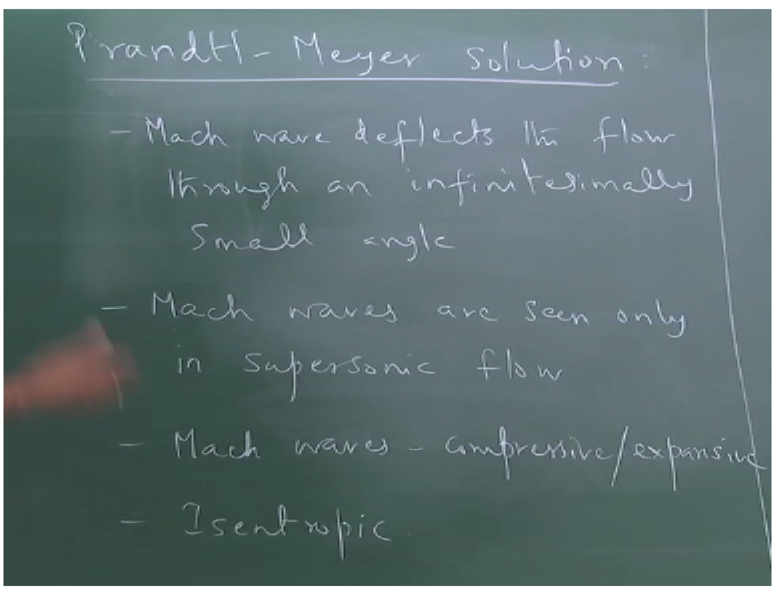

So what we are going to do next is discuss solutions to this problem which were first proposed by Prandtl and Meyer so what we are going to look at the solution that we are going to derive is that each Mach wave so the important aspects are that Mach wave deflects the flow through an infinitesimally small angle and Mach waves are seen only in supersonic flows and as we just wrote down Mach waves can be compressive or expansive and fourth point the process is isentropic.

So what are going to do next, or what this Prandtl Meyer solution does is it relates the flow deflection angle to the Mach number that satisfies these constraints that is what we are going to do next. Remember for the oblique shock we said that there were 3 parameters. Mach number, flow deflection angle, and wave angle, but the advantage here is that the wave angle we already know the relationship between the wave angle and the Mach number.

Because mu is nothing but sine inverse 1/M which means that we need to relate only the Mach number and the flow deflection angle that is what we are going to do next that is what the Prandtl Meyer solution does the additional condition that the process should be an isentropic process. Okay let us see. What I am going to do next is you have to follow this very carefully.

I am going to take one such wave from a compressive solution and 1 such wave from an expansive solution and draw the velocity triangles before and after. So you are going to use the same orientation and draw this. Let us drew this very carefully.

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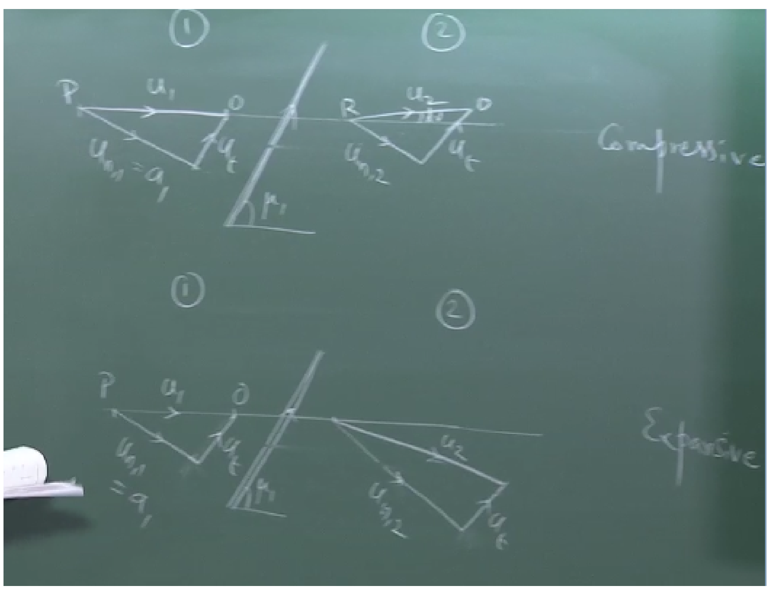

So let me call this is this is the direction, let me call this region 1 and region 2. Let us call this u1, this is normal to the wave and what did we say earlier about the normal component, the normal component is equal to the speed of sound. So this is the normal component and this is = a1. This is the tangential component ut and this angle is the wave angle mu1 correct. So as I said the observer sees 2 components one u1 and another one normal to the wave that is what we have sketched here.

Let us mark this point I am going to mark this point as O and I am going to mark this point as P. This is not the same triangle as earlier. This is the different triangle. O and P are different from our earlier triangle. Now this is a compressive wave so after passing through the wave the normal component of the velocity decreases just like in an oblique shock so the normal component decreases so I am going to show that like this.

So this is the normal component, this was the normal component before, this is the normal component now, so this has become smaller so I am going to call this un2. The tangential component remains the same, same as earlier and so u2, the new velocity vector looks like this and the flow was actually like this the velocity vector was like this before the wave and now after passing through the velocity vector as deflected and this angle we will denote this angle.

This is the flow deflection angle and we will denote that angle as dnu that is the flow deflection angle and we are denoting it explicitly with the differential to indicate that each Mach wave deflects the flow through an infinitesimally small angle. Remember the solution must satisfy these constraints so the deflection angle has to be infinitesimally small and we will use this fact also later on.

For now, we have shown it as dnu to indicate that it is infinitesimally small later on we will actually make use of the fact in our mathematical derivation also. So this is the compressive wave solution and let me label the vertices like this. So let me call this o and let me call this apex R. **“Professor - student conversation starts"** Any questions, sir what exactly is the difference between this and oblique shock? I am going to show that in a minute.

The difference is this is dnu and I will make use of that. There are 2 differences. Number 1 this is equal to speed of sound which was not true before and number 2 this angle is very small however we are using that fact we will use that fact then we will go through the mathematical derivation.

For example, you can probably guess that sine of dnu is what it is going to be if dnu is very small dnu itself that I will use later on. Mn1 will be 1 in this case. Mn1 is 1 in this case correct.

This normal Mach number is 1 that is correct because you are saying this is an -- you know this is like an acoustic wave Mn1 should be 1 for an acoustic wave it is consistent that is correct and if Mn1 is 1 what would be the strength of shock wave? 0 infinitesimally weak that is exactly what this wave is that is very perfect. **“Professor - student conversation ends"**. Now let us look at the expansive solution.

So I am going to take one of this wave and then draw this diagram like this same diagram on the left hand side, but different on the right hand side. So this is region 1, this is region 2. So this is un1 which is = a1, this is ut, this is u1. This angle is mu1 and as before we will label this vertex as o and label this vertex as P. These are supposed to be the same okay. This is the best I can draw. I will show you the actual picture after we finish this.

Now after passing through the wave this is an expansive solution so let me write it here. This is an expansive solution so after passing through the wave un1 increases so this is un2 now, this is ut, and this is u2. So now as a result of un2 increasing the flow has been deflected away from the wave through an angle dnu. So in the earlier case the flow was deflected towards the wave through an angle dnu, it was a compressive solution.

Now the flow is deflected away from the wave through an angle dnu, and let us label these vertices same as before 0 and R. any questions so far. Alright now what we are going to do is combine the velocity triangle. So what I am going to do now is I am going to take this velocity triangle and notice that this is the same as this so I am going to take this side and make it coincide with this side so I am going to combine these 2 into a single triangle for this solution. I am going to do the same thing for this solution also and draw a composite triangle let us do that.
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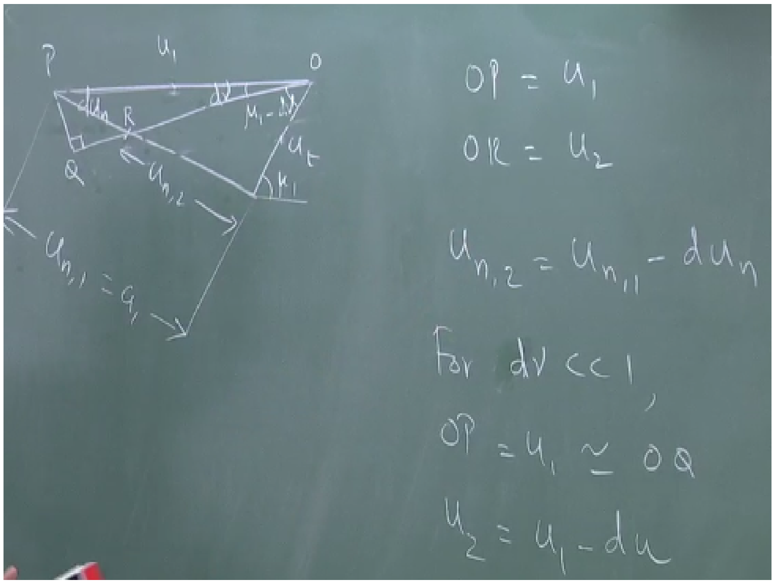

I am enlarging this so that things are clear not drawing to the same size as the other one. I am enlarging it so the things will become very clear so this is 0, this is P, this is un1, the arrows are not required I think it should be alright without the arrows. So un1 = a1. This is the triangle before the flow and the triangle afterwards looks like this. This is un2. This angle is mu1.

This angle is dnu correct notice that I have taken the second triangle pasted it here and this angle is going to be mu1 - dnu. and okay so we have OR so this is point R, this is vertex R, and so OR is equal to let me write this explicitly here OP = u1, OR = u2. Now let me do the following. This is 90 degrees let me do this geometric construction. I have extended OR up to Q and I have dropped a perpendicular from here to that.

Now notice that if you look at this triangle, this is un2 this whole thing is un1, so this PR is nothing but un1 - un2. S0, I can actually write this as dun because we expect the change to be very small. So what we are saying is un2 = un1 - dun that is what we have done here. Similarly, if you look at this, this is u2, OR is u2. Now if dnu is a small angle here is where we are going to make use of the fact that dnu is small.

If dnu is a small angle, then OP is approximately = OQ. For small dnu much much < 1, OP = u1 is approximately = OQ which means that since OR is u2 or Q is going to be du, OQ = u1 so this whole thing is u1, this is u2 so this is going to be the change in velocity du so I can write u2 = u1

- du for this solution. So now we are making use of the fact that this is small unlike the oblique shock solution. I am going to do the same thing for the expansive solution also combine the 2 triangles, let us do that.

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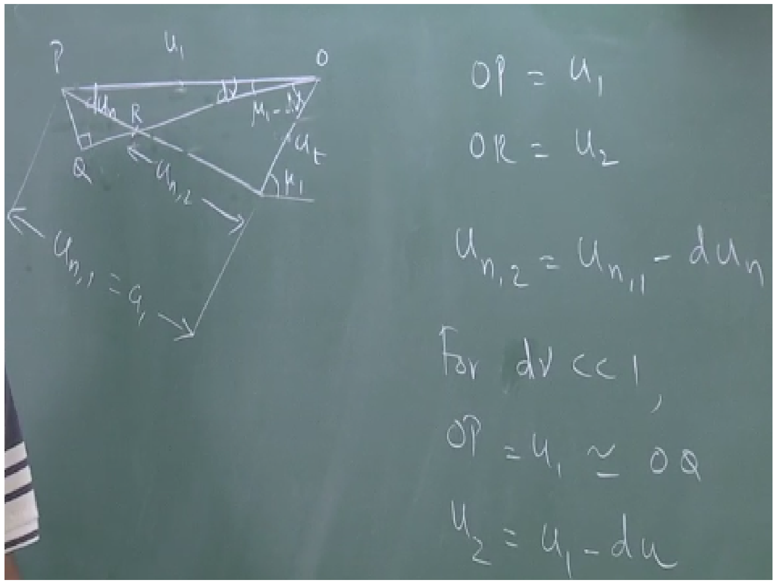

We will do this slightly like this, so this is supposed to be on the same line. So this is our OR, this is u2 and what I am going to do is do a same construction as before call this point Q, let me call this point, let me right this over here so I call that point Q now this whole thing remember this whole thing was u2, this was my this is my flow deflection angle dnu, this angle is mu1. Now I used the same trick as I did before.

I noticed that un2 is going to be un1 + dun. un2 = un1 + dun and so let me write that down dun now if then OP = u1 which is approximately = to OQ. So this whole thing is u1 and if dnu is very small then OQ is also going to be u1. So u2 is going to u1 + du. Now we can see that I have brought both these solutions into the same framework except for so here the angle is mu1 + dnu, this angle here that angle is mu1 - dnu, here this is un2 is un1 - dun.

Here it is + dun, so the + - I have brought both this into a common into a same framework. Let us look these triangles done in the neat way.

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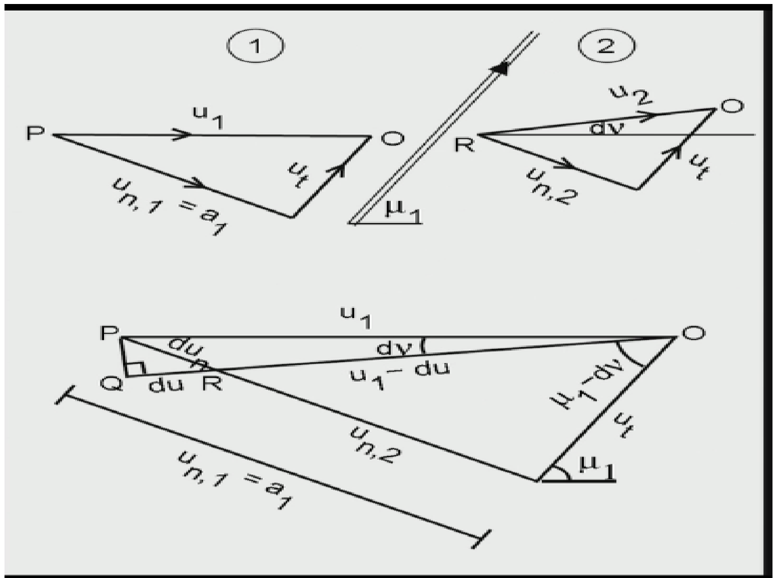

Here it is so you can see the same construction 1 and 2. This is the velocity vector before un1 = a1 ut. This wave angle is the Mach angle mu1 and after passing through un2 decreases is < un1 so the velocity vector shifts that way because ut is the same. This angle is dnu and I have combined these 2 triangles by merging these 2 sides like this and you see the same construction so u2 is u1 – du.

And we have made use of the fact that OP and OQ are the same because dnu is small, but on the same thing here combine these 2 triangles, but the flow is deflected away from the wave combined these 2 and use the fact that OP = OQ = u1 because dnu is small. Alright now are ready to pursue the solution to this equation. Let us get started.

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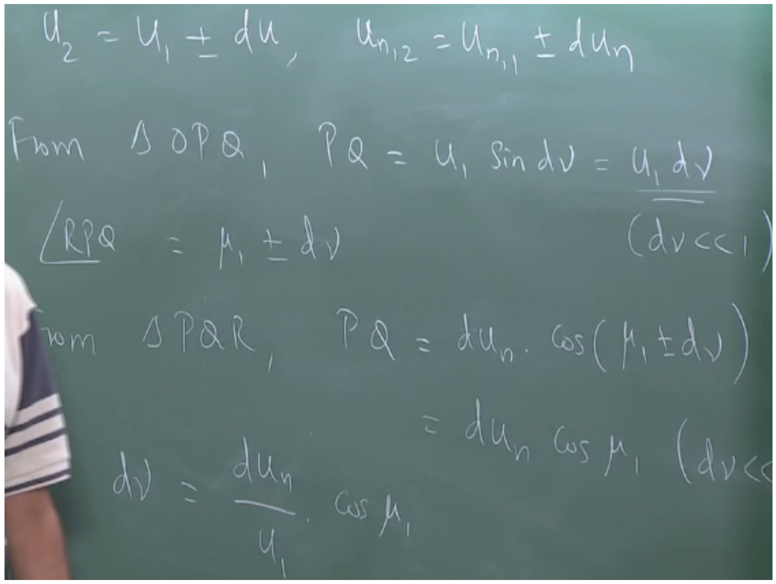

We will not be able to complete this today, but let us get started. We can write in the general case for the Prandtl Meyer solution or for the compression or expansive solution u2 = u1 + - du and un2 = un1 + - dun. So from triangle OPQ, PQ = u1 sinus dnu. OPQ in both the cases. PQ = u1 sine dnu and for small values of dnu I can approximate this as u1 * dnu. Now, angle RPQ = mu1 + - dnu.

So that from triangle PQR we get PQ = dun * cosine mu1 + - dnu and for small values of dnu this can be approximated as du and cosine mu1 itself for small value of dnu. So we are making use of the fact that dnu is very small in this case and we are making use of the factor dnu is very small here also. So we have 2 expressions for PQ and if you equate these 2 expressions for PQ we have 1 expression for PQ here.

Another one there and if you equate these 2 expressions we get dnu = dun divided by u1 * cosine mu1. Remember, we are trying to relate the flow deflection angle to the Mach number. The wave angle itself is already known in terms of the Mach number so that is where we are going with this. So we will continue with this expression and try to this is a function only of Mach number. So I need to make sure that I rewrite this also in terms of Mach number and then we will go from there. This we will pick up in the next class.