Conduction and Radiation

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Lecture No. # 33

Conduction - Energy equation

In today's class we will go through a very important derivation for conduction, namely the derivation of the general energy equation for heat conduction problem. Then, you can look at simplifications of this general equation and arrive at energy equation for one-dimensional steady, two-dimensional steady, one-dimensional unsteady, two-dimensional unsteady.

This is basically applicable for three kinds of coordinate systems: Cartesian coordinate, cylindrical coordinate, spherical coordinate. Since it is easy to derive it for Cartesian coordinate, we will do it only for Cartesian coordinate and then, we will say it can be derived for both, the cylindrical and the spherical coordinate system and we will use, I will write out only the final forms of the equation.

So, the energy equation, the variable, the primary variable for which we write the equation is going to be, what is the dependant variable? Temperature; though the fundamental quantity of interest may be heat flux in most of the situation. In the last class we have seen, that it is easier to measure temperature and from temperature, temperature can be easily related to the heat flux via which equation? Fourier's law of heat conduction.

So, if you want to determine the heat transfer in conduction, if you are able to write down a governing equation, partial differential, ordinary differential, whatever equation in temperature, if you are able to solve it and get the temperature field, then at the boundary or wherever you want, you can apply k dt by dx and get the Q. You also saw, that the Q is a, Q is a vector, heat flux is a vector, which is orthogonal to the isotherm; temperature is a scalar, but heat flux is a vector. We saw isothermal, isotherms for the slab and I asked you to figure out, how the iso-flux lines will look like?

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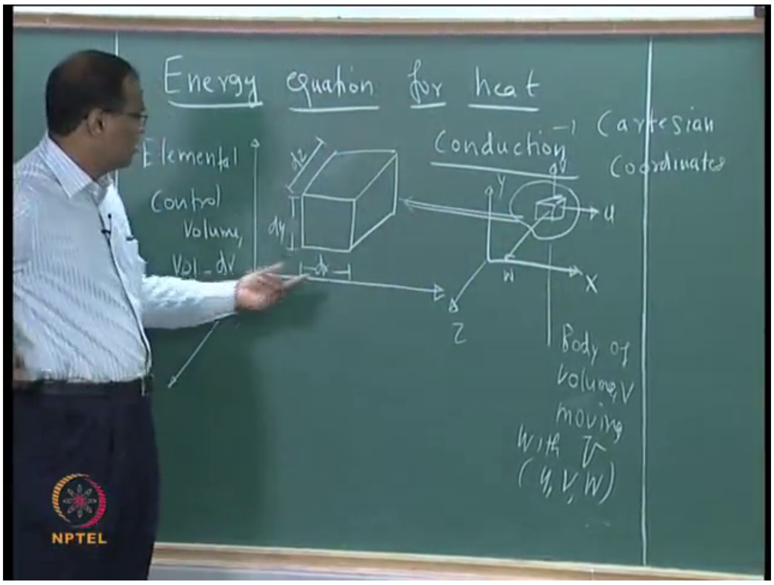

So, now, we are, going to, going to derive energy equation for heat conduction, this is basically in Cartesian coordinate. So, this is the X, Y, Z coordinate. So, Z is coming out of the blackboard; Z is coming out of the blackboard. I take an arbitrary volume, arbitrary shape, which has got a volume V and considering an elemental volume dv, elemental volume is a, cuboids with dimension dx, dx, dy and dz. This body is moving with a velocity, which is given by v; v has got 3 components: U, V, W. Sir, why do we want to have moving body in conduction? I told you welding rod problem, moving heat source problem or bullet or bomb, bullet or canon is fired, that resource is also moving or welding rod is moving.

So, we have to derive the heat conduction equation in its most general form and then, in many cases, we will say this velocity is 0 and reduce it for the stationary case because 99 percent of the time it will be the stationary case, but we should have a general equation to cover the cases, where the body is moving.

Now, we identified dx, dy, dz; now we have to identify the various terms, which contribute to the law of and apply the law of conservation of energy to this. So, we will start this.

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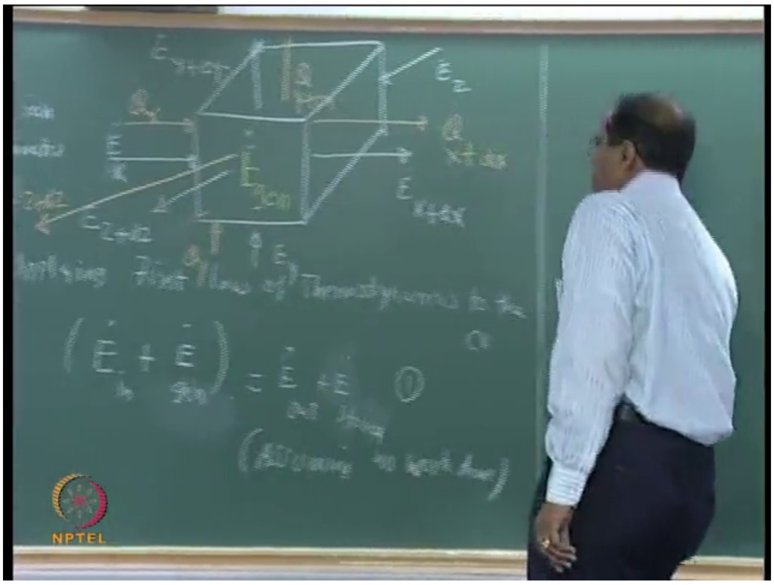

So, applying the 1st law thermodynamics, should the CV, the control volume, and putting E dot because this heat transfer we are talking about, rate per second, the transfer, that is what we are interested, what is joule per second. So, E dot in, what is, whatever the energy is coming in plus, these are correct.

Assuming, if there is no work done across the control volume, otherwise Q m is w is equal to delta E and considering only Q 1 delta E terms delta E is the change in the internal energy, which is given by E dot stored; E dot in is energy, which is coming in; E dot out is energy, which is going out; the energy which is coming in need not be just only because of the conduction, the body is moving, it can also be because of advection. So, there is an incoming energy.

If there is no heat generation and steady state prevails, the incoming heat, the incoming E dot in must be equal to E dot out, simple. But if E dot in is not equal to E dot out, then the body may be depleted of its internal energy or there may be accumulation of internal energy over time; that means, the temperature will increase. So, there will be E stored; E dot stored. Over and above all this, if there is a chemical reaction or nuclear reaction taking place in the control volume, that will also contribute to the increase in temperature, that is why E dot generator, generator and E dot stored will appear on opposite side of the equation.

You are getting the point? If it is continuously generating heat like Fukushima reactor and you are not able to, you are not able to dissipate heat, what will happen? E dot stored will increase. E dot stored means enthalpy or energy will keep on increasing, temperature will increase.

Now, the goal is to write down mathematical expression for each of these terms and convert it into a partial differential equation in temperature. Now, to generate its universally applicable, you are not applying any particular rate loss.

What, what, what?

Student: It depends on what you want to call E dot.

What, what, what, what?

Student: (( ))

I did not follow.

Student: (( ))

So, there are no problems on E in and E out. They are to be on 2 sides; everybody agrees that E, E generated and E stored must also be on opposite sides. Now, there is a sign convention, there is a problem, whether this must come, how you consider? So, what you are saying is E in is coming, E generate is also coming, let them, let both be on one side, fine. So, they are positive contributors. So, if they are balanced, it will be equal to E dot out, then it will be steady state, but if it is not happening, then it will be E dot stored, is that o.k.? I have no problem, I, even if I put the other way, towards the end, I will do some minus plus and I will change it; if conceptually this is better, you can take, no problem. Now, we will write expression for each of this, now E dot x… to 10:48

Now, substituting for these terms, substituting for these terms we have, yeah, please tell me to 11:34

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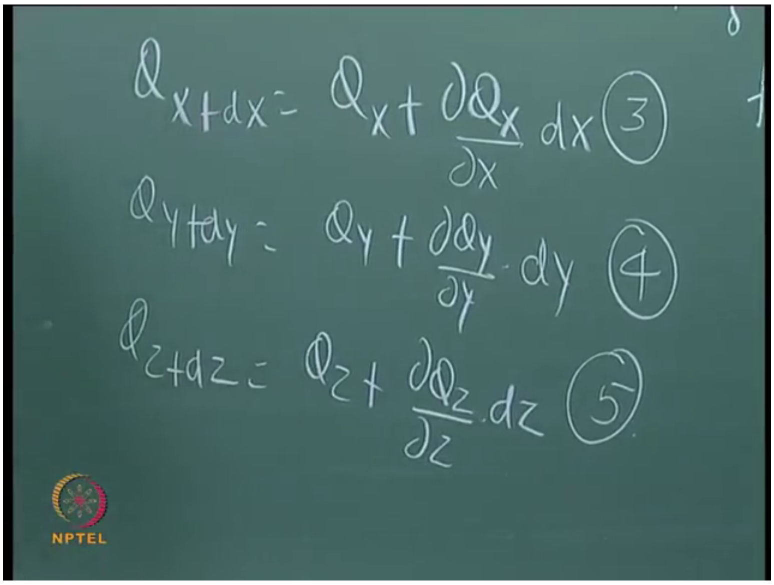

Correct, this is a, shall we write like this or if somebody writes, does it like this; you expand, plus the heat generation is there, E generation, I will call Q generation, Q, so that I know it is watts. Now, we start applying the particular rate loss. So, far I have not done anything else, just I am just doing book keeping, I am doing accountancy, I am balancing left side right side, that is all.

Now, I have to use my fundas. Q x, Q x plus dx, is that correct? 3. Q y plus dx y dy; Q z; E is different from Q; E is energy. E is energy, which is transpolar because of the motion. E terms will vanish. If the body is stationary, E terms will be the most important terms in the (( )) equation, I am formulating almost like (( )) equation, without discuss (( )) what is not like, I am not again using that. I am following an approach, which you will do for following, for deriving energy equation in convective heat transfer.

Now, this fellow we can remove, we will remove this fellow. I can straight away say, that the body is stationary and I, Vikram, I can straight away say, that the body is stationary and I can give you a very elementary equation, the body can move, source is moving, is an example. Do not think, that we are, I am writing the equation for welding, rod welding; rod is an example that is all.

The body whose temperature distribution you want, the body whose temperature distribution you want, may be the reentry vehicle, that body itself is moving. You will say, Sir, in that reentry vehicle, it is moving in one direction. Suppose it has got 3 components and is something like that, then we will take care of the general formula, is it ok? Finally, we will knock off all the Es, do not worry right now, what about this?

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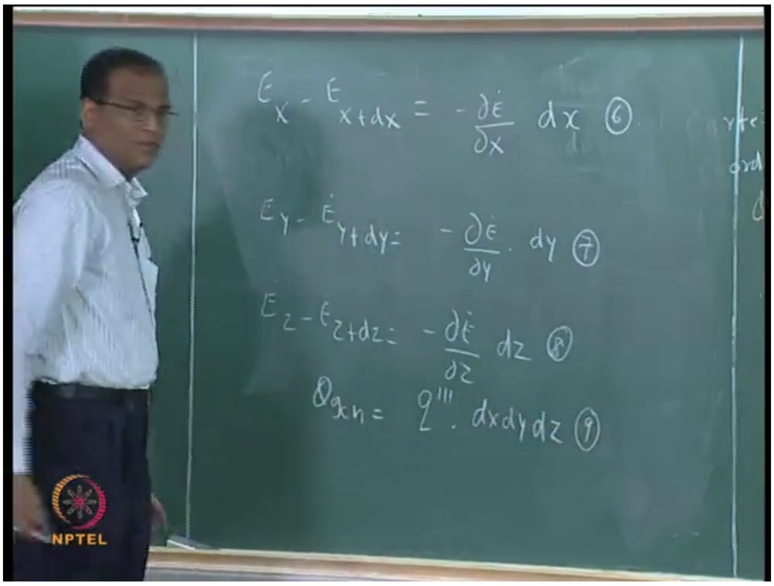

Student: To 15:57

Is it correct?

Student: to 16:31

So far, so good. Now, I will start doing something, is it fine up to this stage? So, we wrote down simplifying terms for each of these, let the Q dot generate remain, I will say that, I will also say that.

Student: to 17:20

E, E, capital E as units of joules or kilojoules. So, small e has units of kilojoules per kg, this also we can remove.

Student: to 18:08

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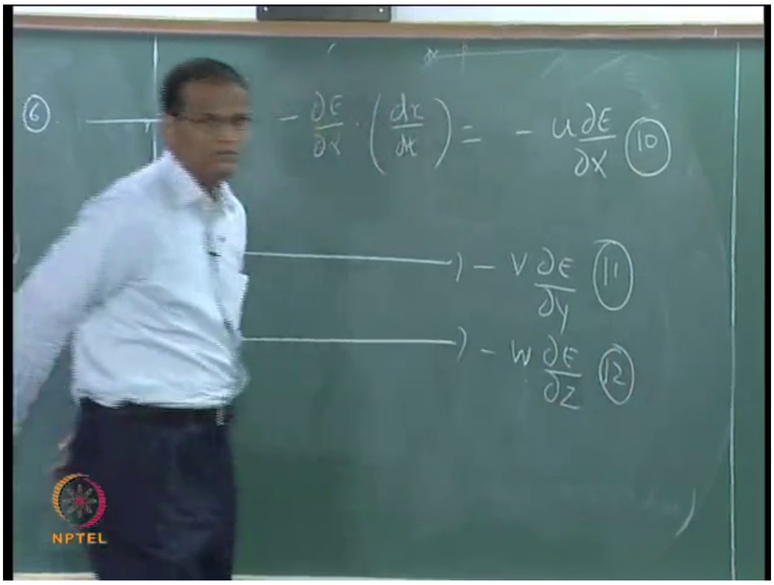

Is it correct? What is dx by dt? u, very good. E dot, I change it to E, I brought the time to the x. So, that is not the only thing we are capable of doing, we will do other things, fine. What is Q x doing here? Do not have better expression for Q x? minus k dou d by dou x; Q y is minus k dou d by dou y; Q z is k dou d by dou y.

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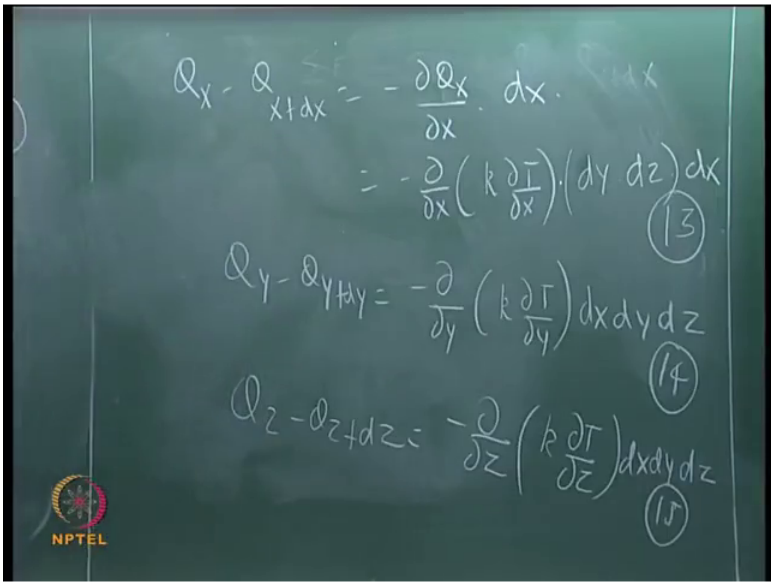

So therefore, yes, what will be the area? For people who have some doubt, this is a Q x, Q x is passing through this area, that area is dy, dz, I did one more thing, I removed the dy, dz out of the bracket, I am allowed to do that, correct? At this stage I do not want to pull the k out of the x, why? I can have a variable thermal conductivity formulation, correct? If I pull the k out, I am already making a statement, that k is independent of x. Many problems it may be like that, but let us keep it like this.

Now, Q x, what will be this, 13. Similarly, you can write an expression for Q z minus z plus dz, yeah. So, we have got expressions for each and every term in the energy equation, go back and substitute all the terms, simplify, every term will have a dx, dy, dz. Dividing by dx, dy, dz throughout is understanding, that dx, dy, dz cannot become 0. That is a whole beauty of a finite volume and control volume formulation, they are all valid, not at mathematical point, but at engineering points, where each point has an elemental volume associated with this.

Now, putting all this substituting, substituting into the, what is the equation number, E y E i minus E something, was that 2.

Student: to 24:04

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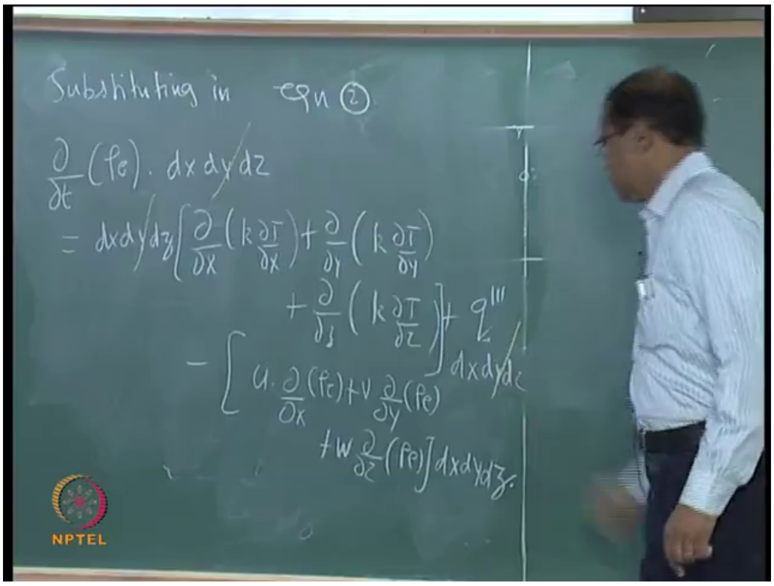

You are not allowed. Q x minus Q is minus dou Q and Q itself is minus, therefore this will be plus, you have to be very watchful, is it ok? You are taking minus dou Q x by dou x, fortunately the dou Q x itself has a minus, this will be…

Student: to 25:12

First, I need, I have those energy terms, what are those terms, minus, everything is minus; everything is minus.

Student: to 25:59

So, it is good for us, all the terms are dx, dy, dz that also reinforces our belief, that we have not made any mistake. Canceling dx, dy, dz throughout, dividing by…

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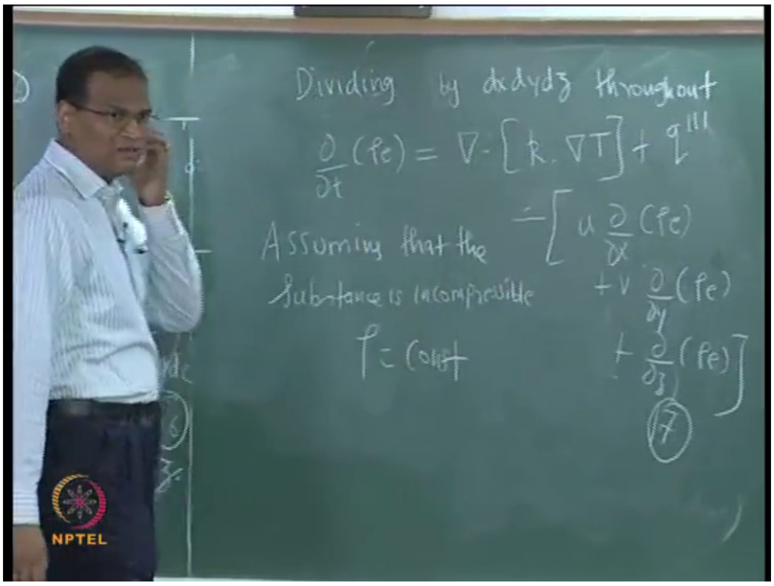

Student: to 27:07

Is that correct? If it is too, too difficult, we will keep it like this, now next step we will do. So, we will keep it like this, minus, yeah, we did not give any term. So, what was this number? 16, so this is 17. Any problem Vinay? Yeah, yeah, please feel free, no problem, now can we make some simplifying assumption.

Let us say that this substance is incompressible. So, the density is constant I mean it is not a great sin we are committing by doing the mostly we are dealing with solids we are not dealing with gases, gases anyway, convection will start. So, assuming that the substance is incompressible, so dou is constant, dou can be pulled out, what can you say about E? What?

Student: Incompressible.

It is all incompressible, whether it is time or x or y. Usually, the density changes, it is not, it is a necessary and sufficient condition from fluid mechanics.

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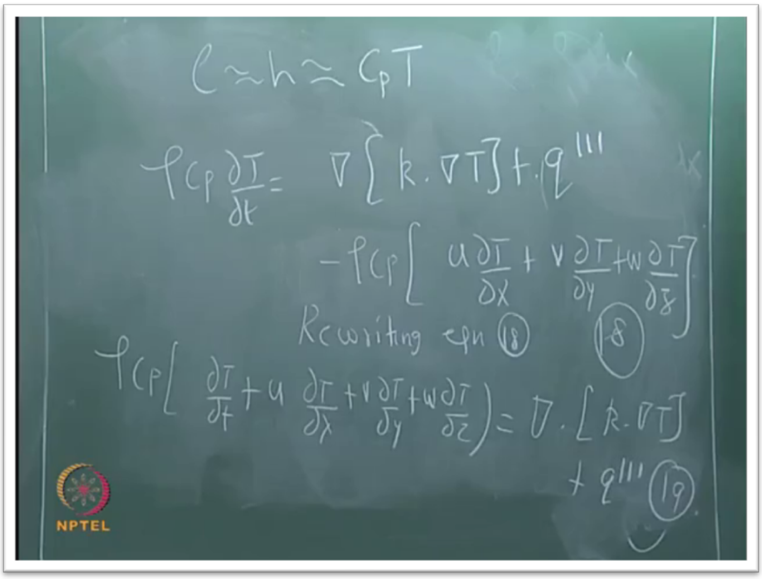

dou is constant, therefore, if density is constant, then I can put e (( )) h, I can write it as C P T, very good. It is C T, does not matter, specific heat. Now, I can say that or why C P also is constant, I can make it minus, now (( )) v dot, already. We will, first we will remove the rho, rho will come out, minus rho, C P will come out, rho C P will come out, u dou t by dou x plus v dou t by dou y, correct, that is it ok?

People who learn can make the heat transfer, you remember, (( )) will be the advection term in your energy equation, in convection it will be advection term (( )). So, we can take the advection term to the left hand side. So, we can have rewriting to 32:27. Now, I will apply the master stroke. If the body is stationary, u equal to v equal to w equal to 0. Once you have come to graduate level courses like this, the goal of learning any new subject is basically, you should not try to remember many things. Somebody says, what is energy equation, I forgot, what, give me half an hour I can derive it, just draw control volume, find out what are the, do a flux balance and then the key information, which you are putting is only 1, Q x equal to k a dou t by dou x, otherwise everything is in only thermodynamic knowledge. Then, your knowledge of vector calculation, you should be able to, you should be able to recreate or reacknowledge with just, basically you should know only a few things, from that you should be able to build.

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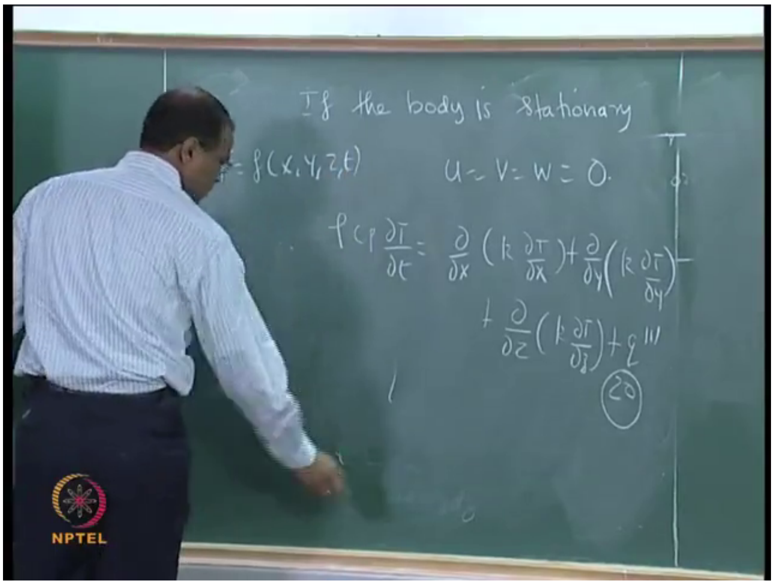

So, now, if the body is stationary…

Student: to 34:19

So, this is very familiar, this is the most widely used energy equation in conduction.

Student: Which one?

(( )), Outside, outside.

Yeah, delta. So, correct, finally it is a scalar, it is a vector, which is a vector finally, it is in between Q x, Q y, Q z, it is a vector.

Now, let us see. So, the equation 20 is a very formal mathematical statement of the law, law of conservation of energy wherein additional information has been injected, which has thermodynamics. You do not know the additional information you incorporate is, thermodynamics person is able to recognize this, thermodynamic person is able to recognize this, but this is, because you have studied heat transfer, that you are able to put Q x equal to k dt by dx. So, when information from particular laws, which are applicable to particular media or incorporated into the general equation of conservation of mass momentum energy, it leads to growth of new subjects. So, this is very formal mathematical statement of the law of conservation of energy.

Now, you can see that the net diffusion, the net heat, which is conducted in the x, y and z direction plus any, any heat, which is generated because of a nuclear reaction or chemical reaction, must be balanced in the event of this not getting balanced. Then, the enthalpy of the body will increase and it will follow rho c p d t by d tau. So, this is the most general statement, where T is a function of x, y, z and time. So, t is the function of... Now, we can look at simpler forms of this, if k is a constant, it….

Student: to 37:16

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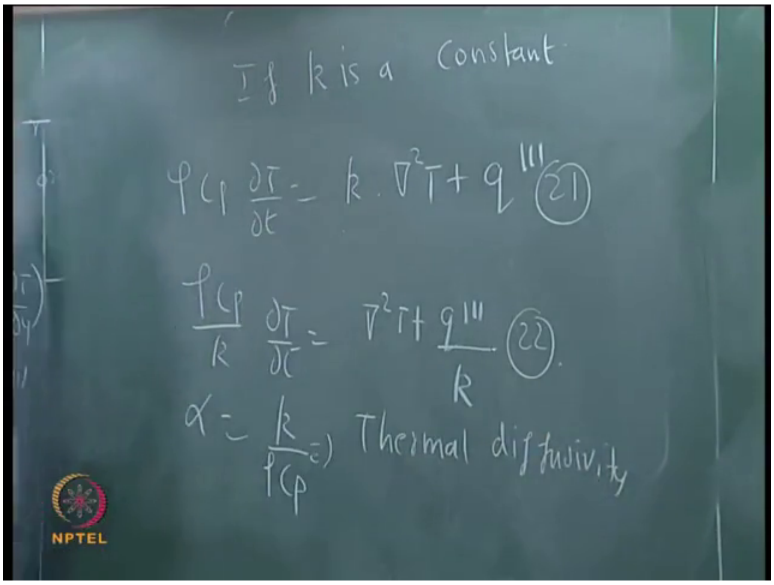

So, what is k by rho c p? k by rho c p is alpha; alpha. See, I always type check it like this, k is watts per meter, watts itself joule per second, therefore, when k is in the numerator, second will appear in the denominator; alpha, also as meter square per second, it should be always, unless this fellow is second cube, then we will get R j, but that is a different situation, generally it should be fine. So, this alpha is equal to thermal diffusivity.

Thermal diffusivity gives you an idea, how fast a thermal disturbance propagates in a medium. Milk is boiling in a kettle and you put a ladle or a spoon, you are trying to stir it and accidently you leave it. When you touch, within a few seconds you feel the heat if it is a stainless steel spoon, if it is a wooden, wooden spoon, it takes a long time. So, the issue there is not because, because thermal conductivity of steel is more than wood, anyway it is, but even for same thermal conductivity. If rho on c p is different, there the disturbance alpha, the disturbance thermal disturbance, how far it propagates or how quickly it propagates is decided by the property called thermal diffusivity alpha.

So, the thermal diffusivity alpha is a critical parameter only for unsteady heat transfer problem; for steady heat transfer problem, it will be the thermal conductivity, which will decide the flow of how much of heat is transferred, the speed at which a disturbance propagates or for the given time how far a thermal disturbance will propagate.

For example, for example, if you have an object like this, now this, suppose this water is at 30 degree centigrade. Now, I put, I keep ice on this left side, now this, the first layers of water, which are close to this ice will get, will feel the chillness. Now, this, say this thermal disturbance will exponentially decay with time, so the alpha will decide at a given time, how far into this water bottle will the effort of the ice propagate. If its alpha is higher it will propagate faster, if alpha is lower it will not propagate that much. In fact, later on we will see, that the penetration thickness.

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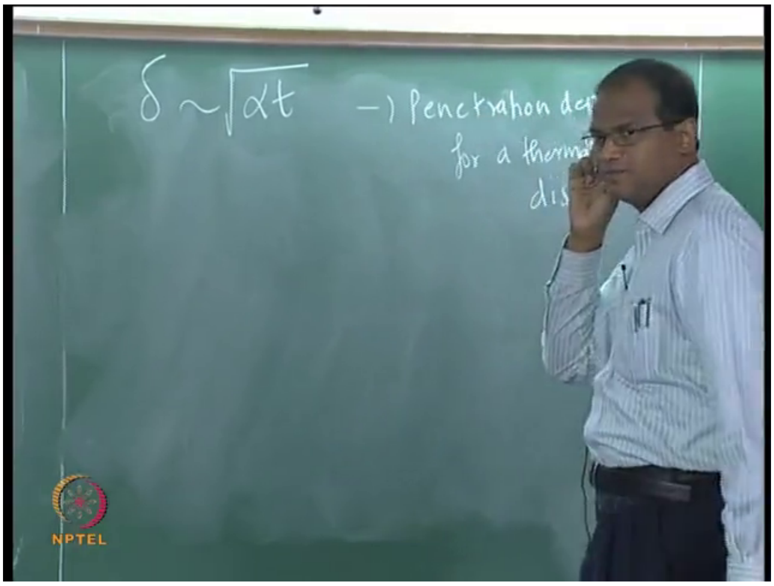

What will be the units of this? This is the penetration depth for a thermal disturbance.

Student: to 40:33

It (( )) of alpha into.

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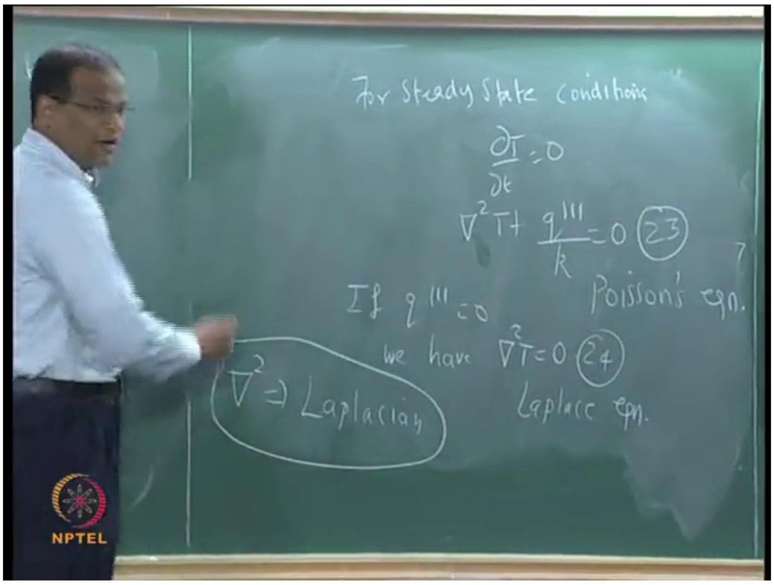

Now, if the net conduction plus, if the net conduction plus the heat generation is balanced, then there is no rate of change of enthalpy of the material for steady state conditions. So, there is no unsteady term. So, this is the unsteady term, which gives you a rate of change of temperature with respect to time. There is no unsteady term, therefore, we get this called the, this called the, called the Poisson's equation; this is called the Poisson's equation.

If Q triple prime is also 0, then it reduces to the Laplace equation. Del square is known as the Laplacian, Laplacian or Laplacian operator, it will always operate on a scalar; it will always operate on a scalar. You remember any other Laplacian kind of operator in fluid mechanics? Same function, del square (( )), an electric potential, electric potential will come in (( )), fundamental electrical engineering and all.

Now, can you solve the equation right away? The analytical method, method of separation of variable or numerical, whatever, what is required to close the problem? Mathematically boundary conditions and initial conditions are required.

Student: to 43:40

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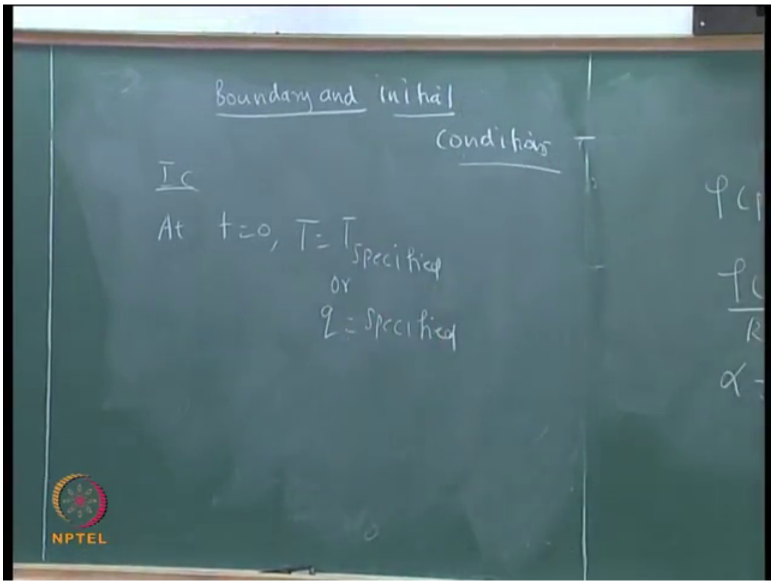

Initial conditions straight forward at T equal to 0 equal to t is equal to t specified or at t is equal to 0 is specified flux. It is only 1st order in time, it is parabolic in time from, at t, from T equal to 0 it will propagate, but however, there are hyperbolic systems in time vibrations of a (( )) membrane and all that, where it will lead to lot of problems, but this parabolic thing, time and it is elliptic in space, people are mathematically incline, can figure out why it is elliptic in space. We use the equation of characteristics and find out b square minus 4 S E less than 0, equal to 0, greater than 0, all that, fine.

Student: to 45:01

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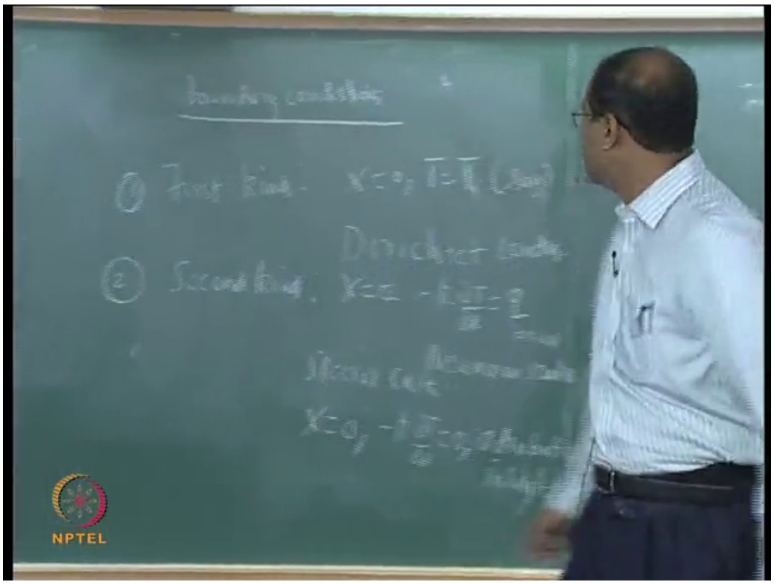

So, now, as far as boundary conditions are concerned, 1st kind is basically is a X equal to 0, T is equal to t 1, say, that is temperature is specified, this called a Dirichlet condition, you specify the variable directly. 2nd kind, you specify the information on the derivative of the variable, this is just an illustration, X equal to 0, X equal to somewhere, I can make it very general, X equals to X, this thing I am putting it in a manner, which is easy for you understand, say minus kQ boundary. So, this is, so this is called the Neumann condition.

What will be the special case of this at X equal to 0 minus dt by dx equal to 0? What we call this condition? Heat transfer, adiabatic, adiabatic or insulator; so, please remember, but you have to be very careful.

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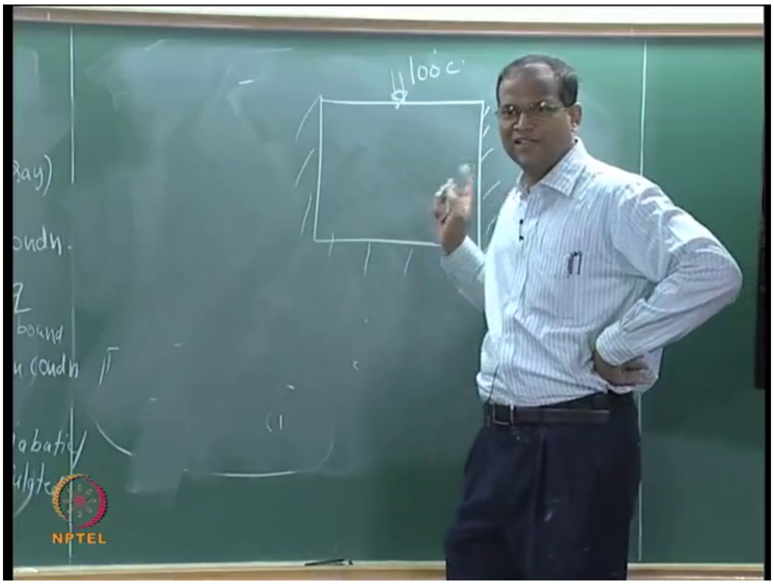

I have a slab like this, I say this 100 degree centigrade, I want a steady state solution; I want a steady state solution of the problem and I am keeping it insulated on all sides. You want to do your B. Tech project with me, on 1st day I gave you this problem, what will happen to you, heat is coming from the top, but there is no way for heat to… So, physically impossible boundary condition, you should not apply, you should know whether it is feasible first, anyway, quickly gives you a (( )), it is under the water, it is basically a, basically a difficult system to solve; are you getting the point?

So, if you are applying in, if some surfaces are insulated you have to watch out, if one or more surface are insulated you have to find out, whether there is a path for the heat to enter, whether there is a path for the heat to leave? Otherwise, steady state solutions will not come out of the problem, this is pure common sense.

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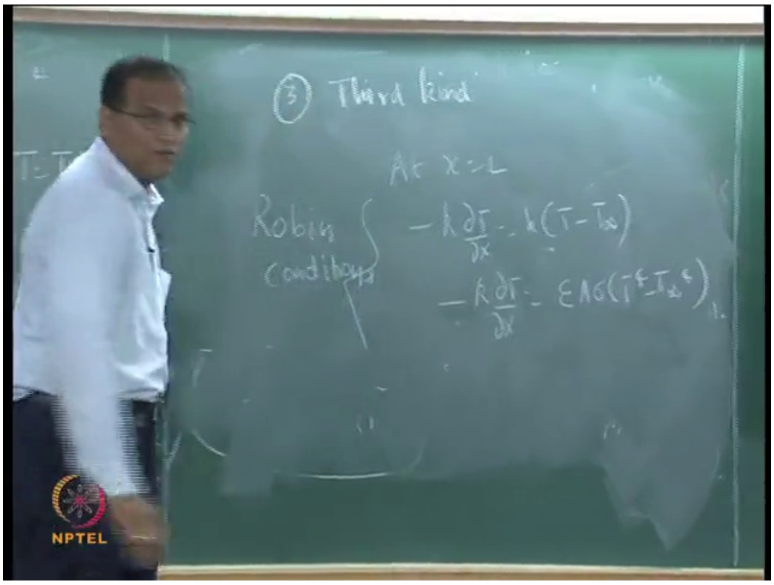

The 3rd will be the 3 kind, that is, at x equal to l, minus k dt by dx equal to x into T minus T infinity, what is this? There is a convection, which is taking place at the boundary or you can even say, minus k dt by dx equal to epsilon A sigma T to the power of 4 minus, that is, it is losing heat by radiation to the outside. This may be some fin system in a space craft, something space craft or something like that. So, this is called as the Robin condition that is 3rd kind.

So, if you complete, if you write down the governing equation pertinent to your problem, you first identify, whether it is temperature is a function of x, y, z and T or temperature is a function only of x and y or x and T write down the proper equation, see whether it is steady or unsteady. If it is steady, initial conditions are not required; if it is unsteady, initial condition is required. Write the initial condition and find out, if it is d square d by dx square alone, 2 conditions on x are required; if it is d square t by dx square plus d square t by dy square, 2 conditions on x and 2 conditions on y are required.

So, if you have two-dimensional unsteady problem, you will have the rho C P into dt by theta on the left hand side, right hand side you will have del square t, which involves dou square d dou x square dou square d dou y square. So, you require 5 conditions without which you cannot solve the problem, 1 on time and 4 on space. Once you have this, then you go to the problem and find out. Now, you take recourse to mathematics, can I analytically solve it, numerically should I solve it, finite element, finite difference, finite volume, fluent console star CD, I will write my own code, mat lab all these things will come. The first is, you have to sit down before going to the computer, sit down with paper and pencil, write down the governing equation, conceptually get a hang of a physical system, identify the various bonding initial conditions.

From tomorrow’s class, so we will have a busy start to conduction, we will look at simplified forms of this, where we can very nicely come out with an analytical solution, then slowly I will accelerate the difficulty level.