**Spray Theory and Applications**
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# Lecture – 25
# Linear stability analysis – Planar Liquid Sheet instability -2

Welcome back. We were looking at the linear instability analysis of a plain liquid sheet. This is essentially if you imagine a liquid sheet that is flat, but has a finite thickness which is about we called it 2 h exiting into and otherwise quiescent atmosphere. We look at the density of the liquid sheet was rho 1 and it was moving with the velocity U 1. The density of the air outside, if you want to call it the fluid outside was rho 2 on the top rho 3 on the bottom and U 2 and U 3. But essentially rho 2 and rho, 3 we will set equal to each other. Say it is exactly the case of a plain liquid sheet exiting into atmosphere.

And one point we have to understand in all this linear instability analysis is that we are looking at the stability of an infinite liquid sheet. There is no substring as a nozzle from which the liquid sheet is exiting. Now you can do an instability analysis of what is called a semi infinite liquid sheet where you show the presence of a nozzle and then say you cannot have the possibility of disturbances upstream of that and you can only have the possibility of disturbances downstream of the nozzle.

But, we are not going to focus our attention. We are only going to focus our attention today on the linear instability analysis of a plain liquid sheet; exiting that of an infinite plain liquid sheet.

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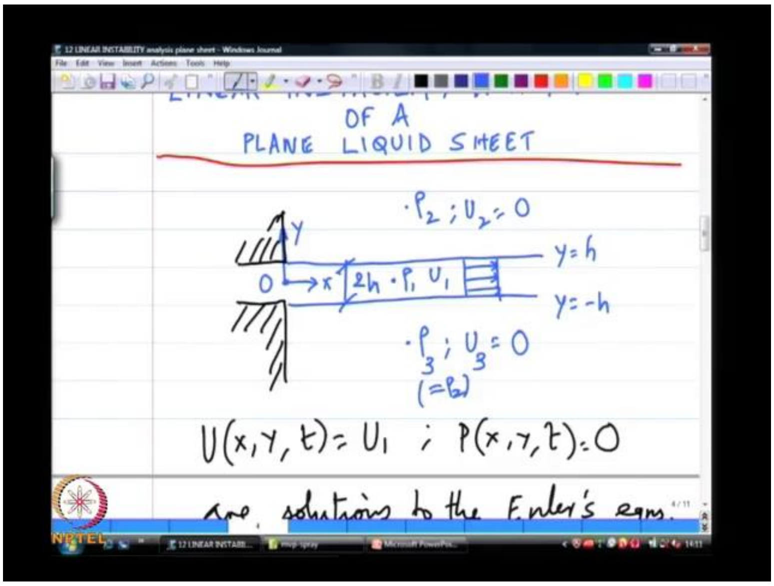


Let us look at the schematic that we have, like I said the sheet is of thickness 2 h, the density of the fluid is rho 1 and it is moving with the velocity U 1, uniform in the entire sheet and rho 3 we are going to set equal to rho 2.

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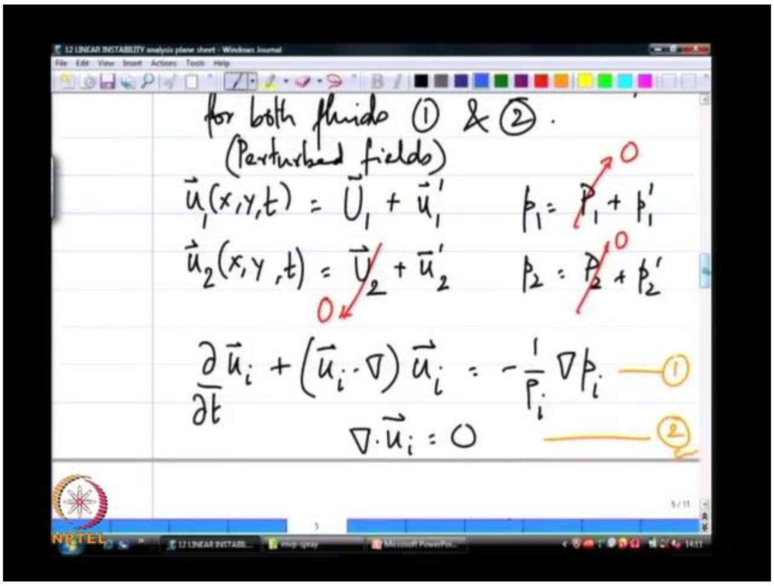


We went through the process of writing the Euler's equations for both the fluids, and then going through the perturbation process for the 2 fluids, and we came as far as calculating the pressure field in the fluid above and the fluid below.

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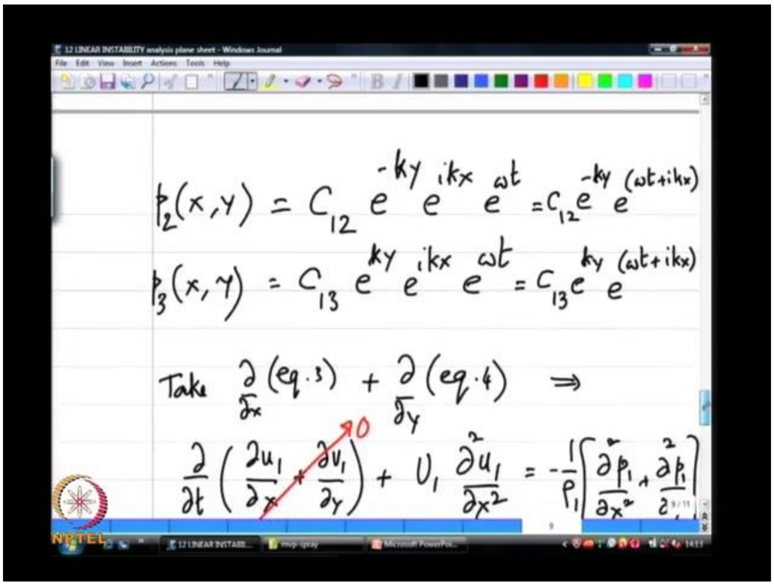


The pressure field in the fluid above p 2 is given by this expression C 12 e power minus ky e power ikx times e power omega t. I can write it slightly more compactly in the form that we are otherwise use to which is likewise p 3, and then this is essentially the fluid above and the fluid below p 2 and p 3. As far as the liquid sheet itself is concerned we went through the process of eliminating one of the variables.

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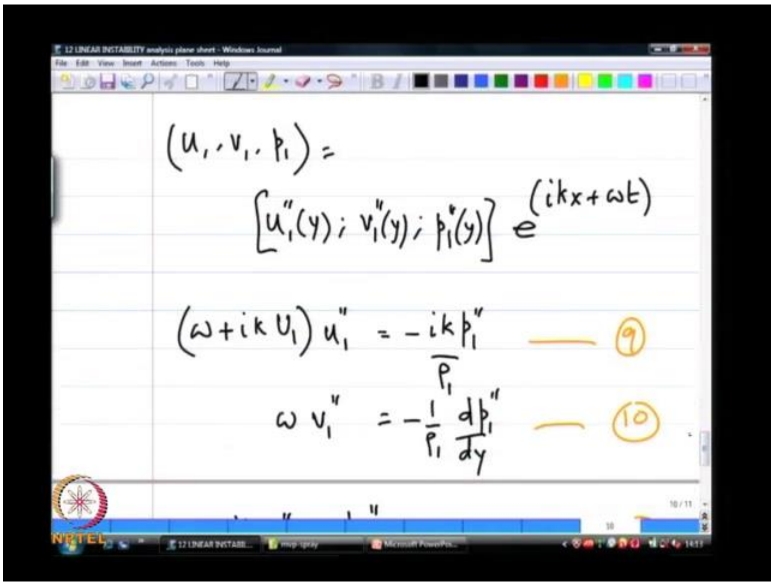


And using the normal mode assumption, we got these equations 9 10 and 11. I can multiply equation 9 by ik and take one over omega d dy of equation 10.

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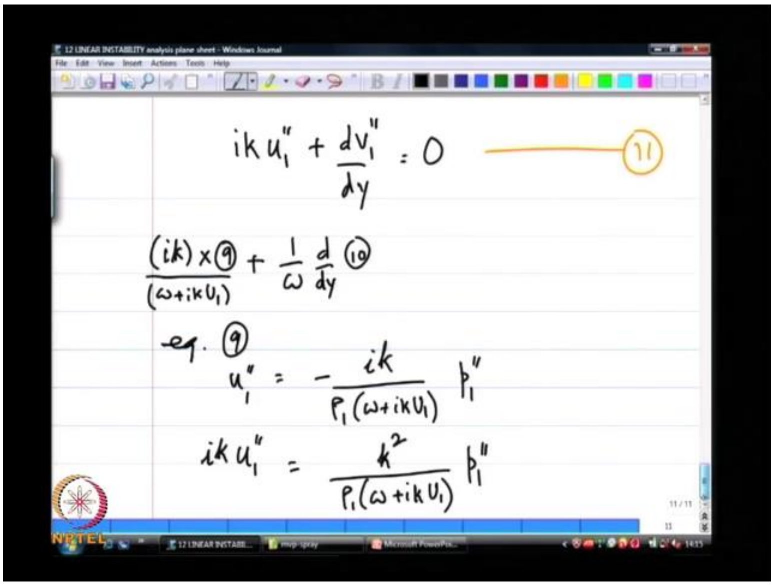


You will see why I am doing this; it is essentially to get to the form that is in equation 11; if I divide this by omega ik U 1 and multiply by ik.

Let me do that equation 9 first, becomes U 1 double prime equals minus 1 over rho 1 plus omega ik U 1 p 1 double prime. So, ik U 1 double prime, you have ik times ik that is minus k squared, but with the other negative sign is this becomes k squared over rho 1, I call this equation 12.

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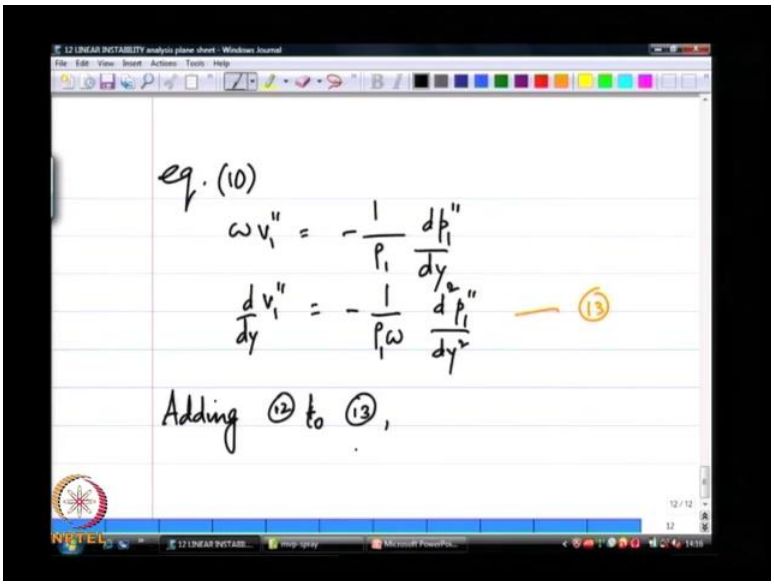

Now, equation 10 is omega v one prime, if I take derivative of this with respective to y, I call this equation 13.

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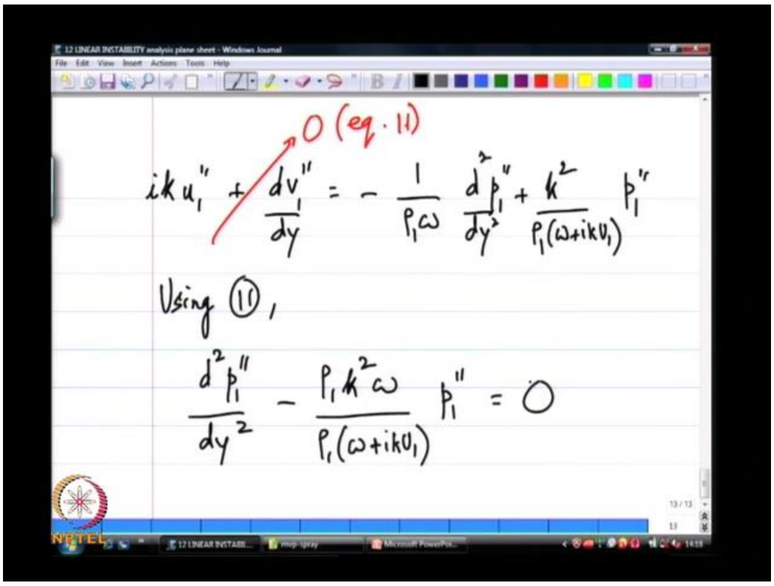


Now using 11, we will find that this right hand side is 0 because of equation 11. What we have, I will write this in a slightly different way, but still recognizable.

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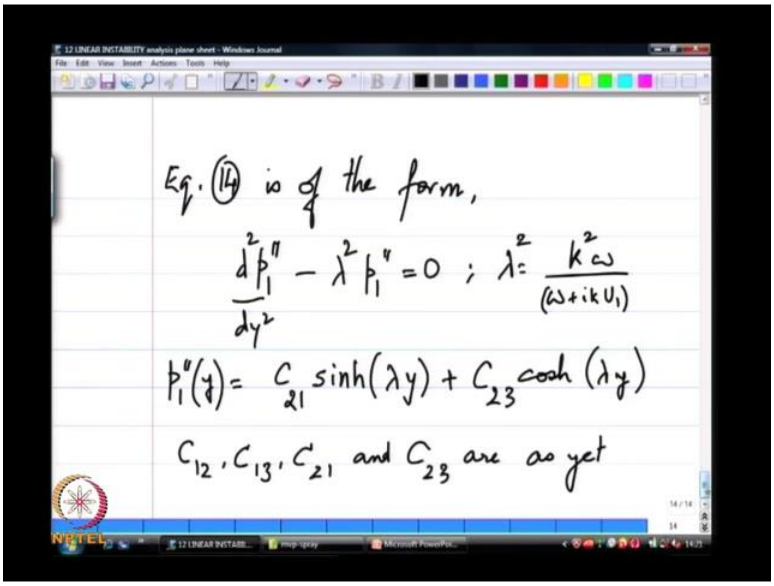


Essentially I have now an equation for p 1 double prime. This is of the form equation 14 where lambda squared equals k squared omega over omega plus ik U 1. The rho 1 cancels out. The solution to this; I call this 21 and 23 just for consistency of nomenclature. I have this C 21 and C 23 are as yet; undetermined, and likewise you have these 2 constants in the pressures C 12 and C 13.

I have C 12, C 13, C 21, and C 23. Now, I am going to go back to the schematic that we had the plain liquid sheet.

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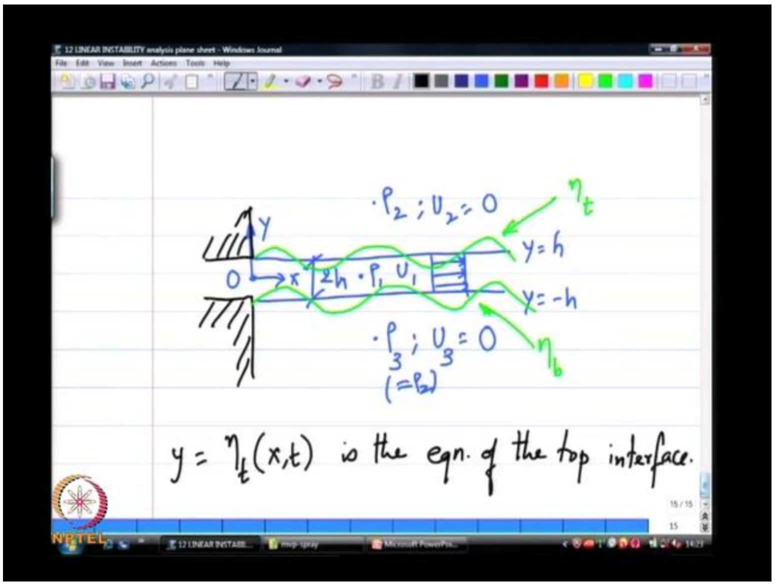


I will just copy this whole thing; this is my unperturbed configuration. If I look at the perturbations on this, so I can have 2 kinds of perturbations on this and this is the fundamental difference between a plain liquid sheet and all of the analysis we had before. This problem has 2 interfaces as suppose to one that is always be in the case.

What do we do with the 2 interfaces? I will call the above the interface on top as eta top and the one on the bottom as eta bottom. If I write the form of eta t and eta b, so essentially y equal to eta t of x comma t is the equation determining of the top interface and y equals eta b x comma t.

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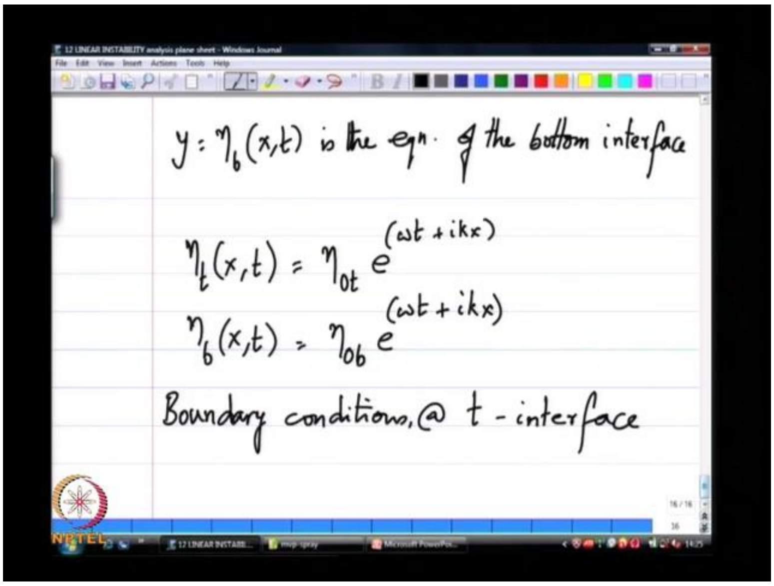


Now, because of the linearity in the problem this clearly has a functional form that is similar to the functional form that already has occurred in the pressures. I will call this eta 0t e power omega t plus ikx. This is coming from the normal mode assumption.

Now that we know the interface shape, we can write down the boundary conditions. The boundary conditions at the t interface which is the top interface.

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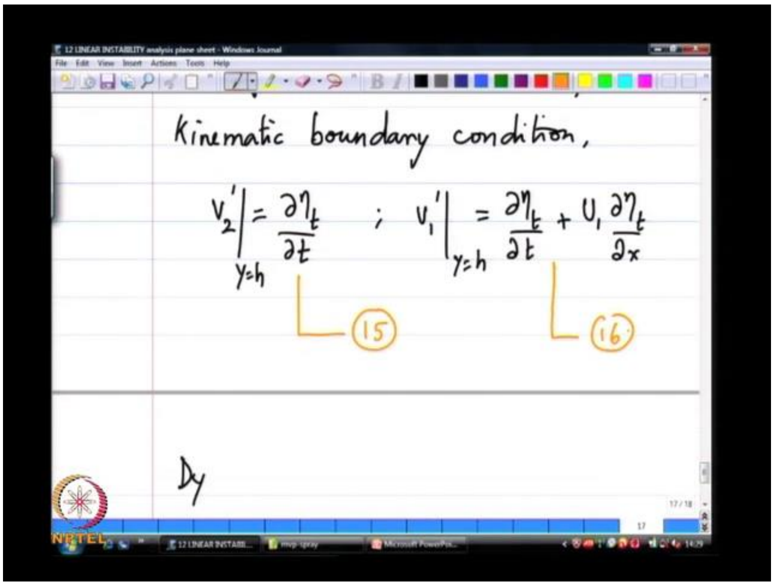


The first one like we have always written is the kinematic boundary condition. I will write it from the direction of fluid 2 and from the direction of fluid 1. From the direction of fluid 2, what I know is that v 2 prime, which is the y direction velocity in fluid 2 has to equal, the partial derivative of eta t with respect to t. This evaluated at y equal to h has to equal this. From the fluid ones point of view v one prime at y equal to h has to equal the partial derivative of eta t with respect to t plus U 1 partial derivative of eta t with respect to x. And all these derivatives evaluated at the unperturbed free surface.

We have discussed this at least twice during the course of the previous lectures, but I reiterate one more point of you sort to say that v 1 prime and v 2 prime in these two. In this equation are the Eulerian velocities at the interface where as if the right hand side of each of these 2 equations is the same velocity at the same point written from a Lagrangian point of view. If I was a material particle on the interface and I am governed by the equation eta t of x comma t which means the rate of motion that I am subjected to is fixed therefore, the material velocity at the interface say either in fluid 1 or fluid 2 has to equal the corresponding velocity coming from the Eulerian description that is the physical essence of the kinematic boundary condition.

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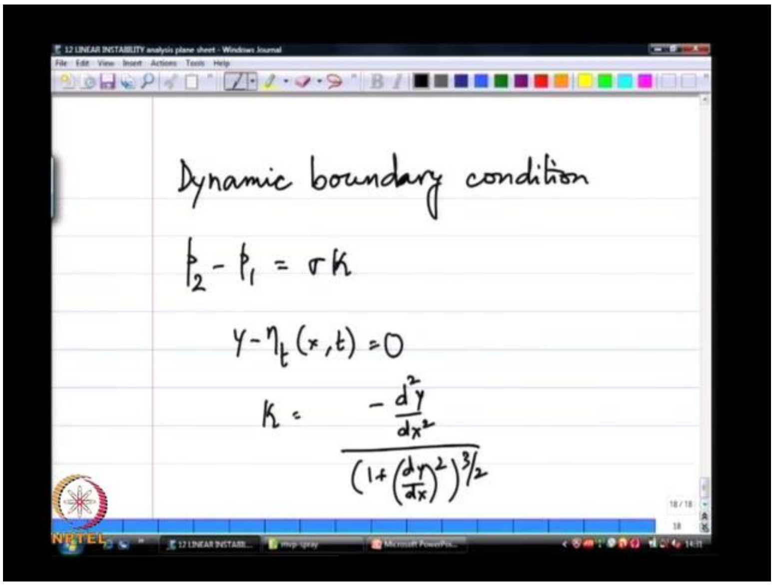


Now, if I write down the dynamic boundary conditions, I will call these, I have 2 kinematic boundary conditions at the top interface. How many dynamic boundary conditions do I expect to have? It is only going to be one because what that, what the dynamic boundary condition amounts to is essentially a force balance on a tiny element. If I take an element of fluid just like that and write down the force balance on that element of fluid, that element of fluid is chosen to span both fluids, but of an infinitesimal thickness. It is own mass is nearly 0, which means the pressures acting on both sides and the net result and forces should all balanced out. If I say force equals mass time acceleration the mass of the element itself is small, then you are essentially looking at something that is where the forces are all equals since static equilibrium that is what we call the dynamic boundary condition.

With that, if I write down the pressure in fluid 2 minus the pressure in fluid 1 equals sigma times kappa, we have already gone through this process of identifying what kappa would be. If I know y minus eta t of x comma t equal to 0 is the equation of the surface, kappa is given by this minus d square y dx squared divided by 1 plus dy dx the squared raise to the power 3 half.

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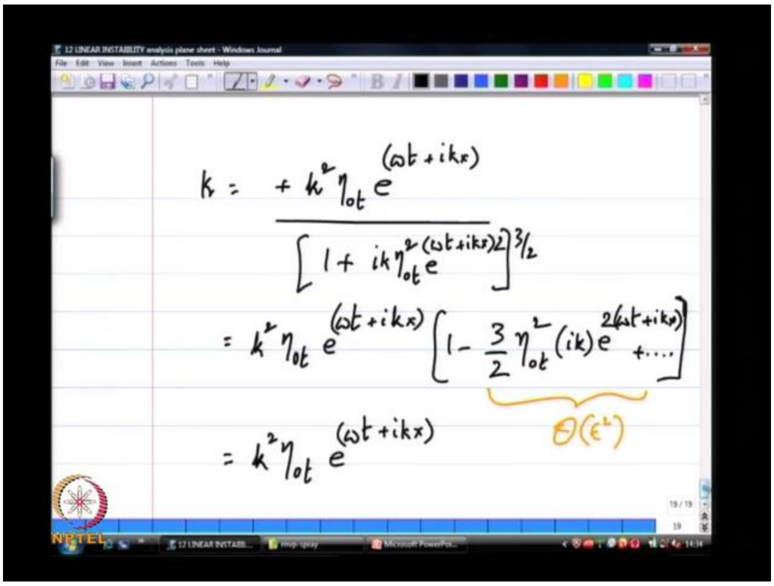


If I simplify that knowing what the form of eta is, this is eta 0t e power of omega t plus ikx. This if I take derivative twice, one derivative gives me ik times. This is another derivative gives me ik times ik which is minus k squared. This becomes plus k squared eta 0t dy dx is essentially ik using the Taylor series expansion that we had.

I can clearly see that the minus 3 half this term here is order epsilon squared. This simply becomes, this is the formula for the curvature.

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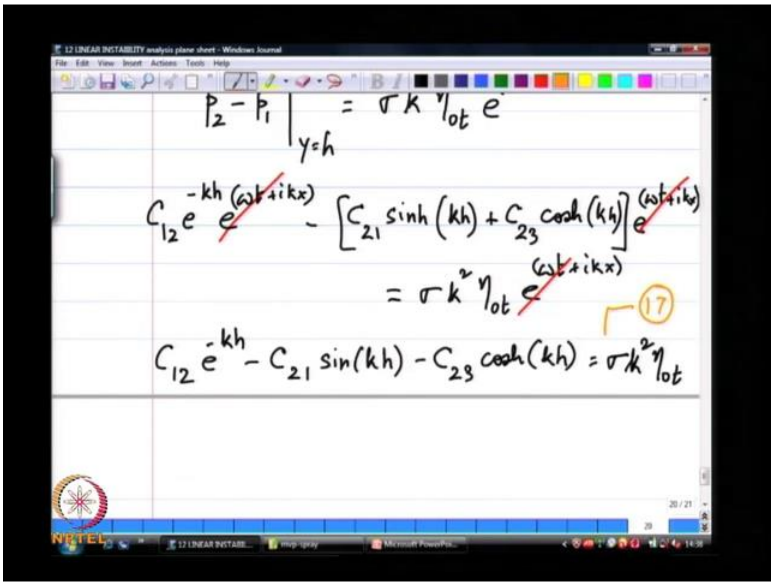


If I substitute that in the dynamic boundary condition, I have p 2 minus p 1 evaluated at y equal to h is equal to sigma times k squared eta 0t e power, we will actually do this just to sort of show, what it looks like p 2 is known really speaking these are the prime quantities p 2 prime and p 1 prime because the mean pressure condition was where everything was at the same pressure.

I can go get the forms for p 2 and p 1 p 2 is given by this C 12 e power minus ky times e power omega t plus ikx. This is C 12 e power minus k h times e power omega t plus ikx. This is p 2 prime at y equal to h p 1 prime at y equal to h has these 2 C 21 and C 23, but this is p 1 double prime. We just have to multiply this by e power omega t plus ikx. This is simply the right hand; the left hand side of this equation. The right hand side says sigma k squared eta 0t e power omega t plus ikx.

From here I have one equation that C 21 e power minus k h minus C 21 sin k h sorry C 12 minus C 23 cosine k h equals sigma k squared eta 0t likewise. I have this is my equation 17 which is, I have 15 16 and 17 as the 3 boundary conditions at the t interface.

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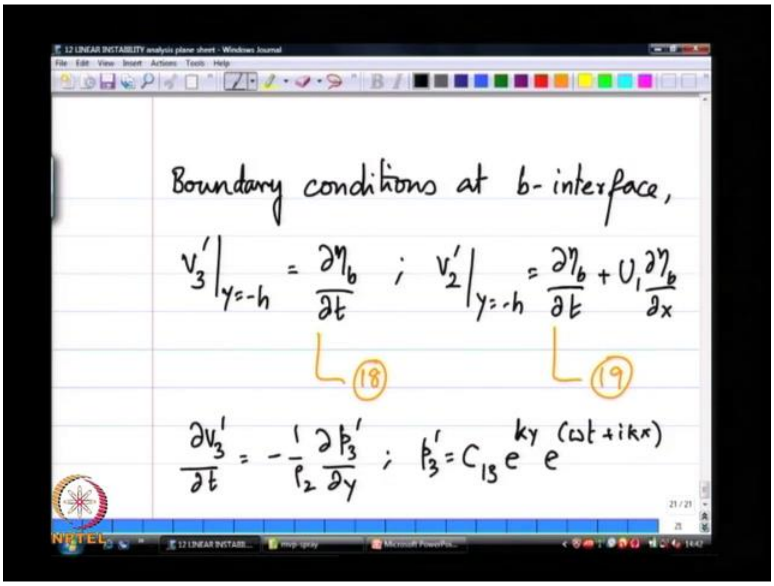


Likewise, if I write the boundary conditions at the bottom interface, I will have the following first of all v 3, v 3 prime at y equal to minus h is equal to d eta b dt and v 2 prime at y equal to minus h equals d eta b dt plus U 1 d eta b dx these are the kinematic boundary conditions at 18 19.

Just as we did the dynamic boundary condition part with the top interface, I am going to implement this in some detail with the bottom interface. If I go back, I know what p 3 double prime looks like, p 3 x comma y comma t is given by this whole thing, If I evaluate, from this p 3, I can go to equation 7 or as the equation, that is look like equation 7 essentially because the fluid on top and bottom 1 and 2 are essentially the same the equation for the v velocity in fluid 3 will look exactly like equation 7 with the subscript 2 replaced by 3.

I know dv 3 prime dt equals minus 1 over rho 2 rho 2 because rho 2 and rho 3 are the same and knowing p 3 prime is C 13 e power ky e power omega t plus ikx.

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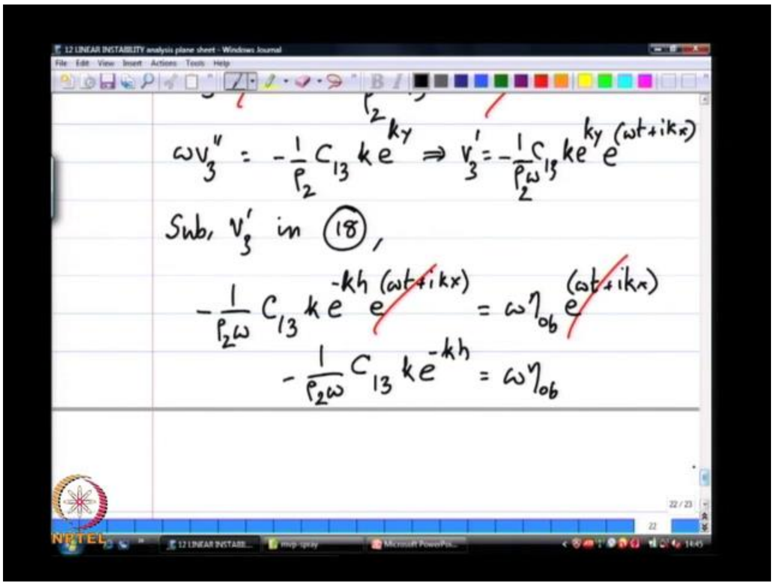


What we find is that knowing v 3 prime is also of the normal mode form. From here, I find omega v 3 double prime is minus 1 over c 2 times C 13 k times e power ky. This is the functional form, this is equation it gives you the functional form for v 3 double prime.

If I use, if I substitute that in the boundary condition 18, what we have at for 18 is v 3 prime evaluated at y equal to minus h equals d eta b dt , the partial derivative of eta b with respect to t. If I evaluate v 3 prime at y equal to h, I have minus 1 over rho 2 omega C 13 k e power plus k minus k h because y equals minus h e power omega t plus ikx equals omega eta 0b e power omega t plus ikx note is how the exponentials cancel out in pretty much all these equations what we are left with is.

So, this is 20, but comes from 18, I can likewise go through and do the same thing for 19, I am just showing you, how you can take the solution that we have for the pressure and find the y direction velocity and use the kinematic boundary condition to come up with the expression relating the unknown constants and eta 0s, eta 0t and eta 0b.

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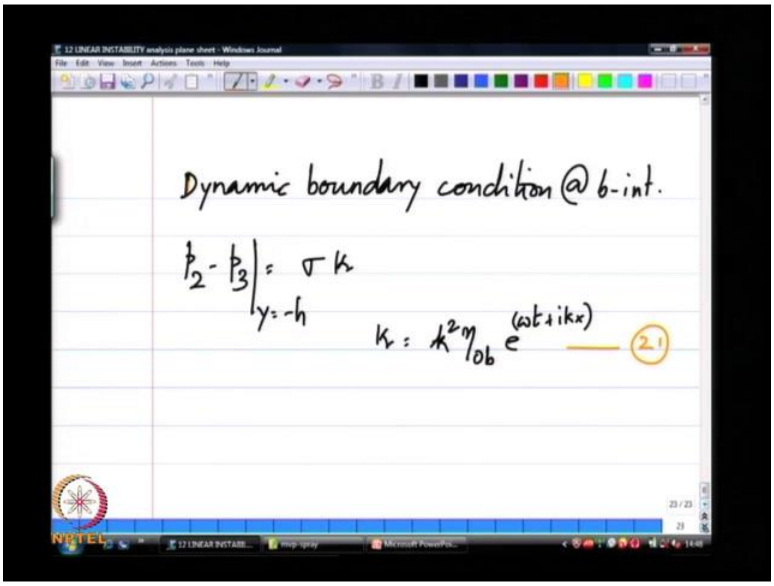


Now the last part is the dynamic boundary condition at the b interface and what we have here is p 2 minus p 3 equals sigma kappa evaluated at y equal to minus h. You can find kappa to be the same keys k squared eta 0b e power omega t plus ikx up to order epsilon.

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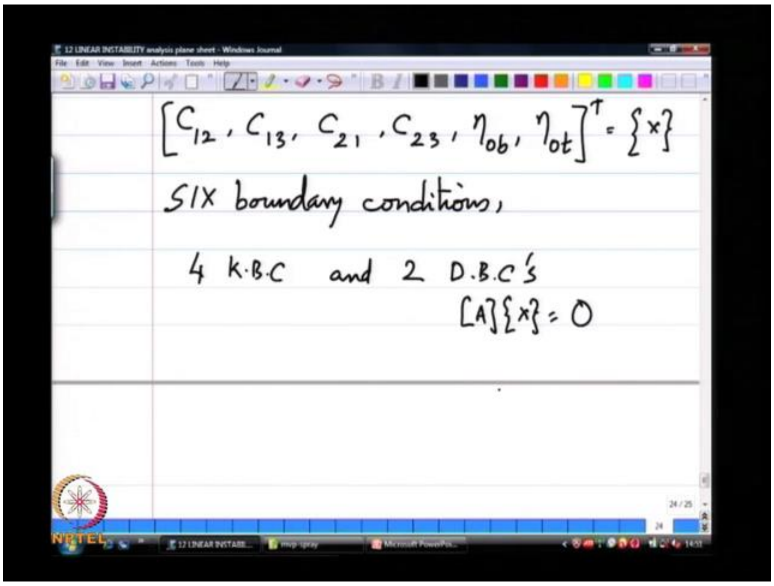


Let us quickly take stock, what are the unknown constants that we are yet to find? We have C 12, C 13, C 21, C 23 eta 0b and eta 0t we have 6 boundary conditions, basically 4 kinematic boundary conditions and 2 dynamic boundary conditions. In total, when you go through this process though it looks like a nice consistence Eigen value problem.

Now, minds you that all these boundary conditions yield only homogeneous equations if we look at equation 20 as an example, C 13 is my unknown constant eta 0 v is one of the unknowns in the list that we had just here under likewise, everyone of our equations would be an like for example, here C 12, C 21, C 23, and eta 0t. Every one of these equations would yield you a homogenous equation, every one of the boundary conditions would yield a homogenous equation in these 6 unknowns that are shown here.

But the problem is that you will find that when you go through this process, the rank of the matrix is less than even 5. Essentially you have a 6 by 6 matrix. If I write this out in the form of i can, where x vector is basically that I can, write these 6 equations in this form, what we will find is that we, in order for me to set the determinant to 0 i do need one more condition the because some of the equations give you the same, they are essentially the same equation.

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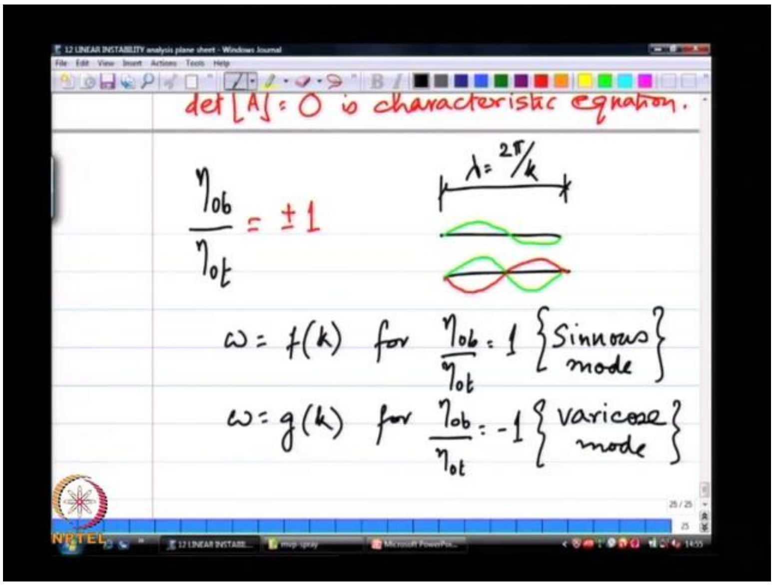


You will not be able to eliminate all these equations to get you to the characteristic equation. The condition that is used is basically assigning a value to this ratio. Now let us quickly look at what this ratio means. If I take 2 menisci and if one of this menisci's has a perturbation of that form, I have 2 possibilities for the bottom 1, it is in phase or exactly out of phase.

This number can either be plus or minus 1. Depending on whether you choose this to be equal to plus one or whether you choose that equal to minus 1, you get 2 different characteristic equations arising from setting this determinant equal to 0. These are of this form, but determinant of a equal to 0 is the characteristic equation. Whether you choose this ratio to be equal to plus 1 or whether you choose it equal to minus 1, gives you 2 separate characteristic equations that is essentially if you go back, what is the characteristic equation characteristic equation is sum f of k. So, you have one equation for eta 0b over eta 0t equal to one which are called sinuous perturbations sinuous mode would be a better way to say and you get another equation this is call the varicose mode.

For a given wavelength or for a given wave number for a given wave number, I can get one growth rate, if the disturbance was in the sinuous mode and I can get another growth rate, if the disturbance was in the varicose mode. For a given flow situation, I need to know what disturbance that I am looking at. I can plot my dispersion diagram which is omega versus k. I want to ultimately identify the value of k for which omega is a maximum and for me to e get to that condition; I need to know which mode, it is that I am looking at.

Typically, what you could do is that you could plot a dispersion diagram for both the sinuous mode and the varicose mode and choose the one k that has the maximum growth rate of all the modes possible of both the sinuous and the varicose modes. In fact, this is kind of nice if you go back and look at what this actually physically means the, if I assume the sheet is of the, if I assume the sheets to be composed of parallel menisci, initially in the unperturbed condition the sinuous mode says that both of them are in phase which is like saying that the sheet is going to. If I look at the top meniscus in the bottom meniscus as being the top and bottom surfaces of my palm essentially they are both in phase sort of like that.

This is like a flag flapping in a breeze for example, would be the case of a Para of a sinuous mode except for a rigid object you cannot have a varicose mode, but since you have a you are dealing with the liquid sheet the liquid sheet could go in to a mode where it thickens and thin at one point and thins at another point completely symmetric about the half way axis about the x equal to about the y equal to 0 position. There are situations where we see that a sheet thickens and thins within the varicose mode there are situations where the sheet is more of a flapping nature.

Depending on the flow condition you may get one kind of break up or the other now if you go back and look at the analysis that we have that we have performed there is nothing in this analysis that says that the liquid sheet that you have a liquid sheet exiting into air. They are just fluids of some densities I could have a gaseous sheet. Like a sheet of gas, that is let us say entering an otherwise quiescent liquid now you can imagine, how if I did that the varicose mode would essentially mean bubbles forming the menisci sort of being anti symmetric to each other or menisci being mirror images about the y equal to 0 axis would among to sort of like a bubble becoming trapped.

If you did this analysis, you would find that the situation where rho 2 is less than rho 1 and rho 1 is less than rho 2 and rho 3 would give you the varicose mode showing the dominant behavior whereas, if you had a liquid sheet exiting into air generally speaking you would find a sinuous mode dominantly.

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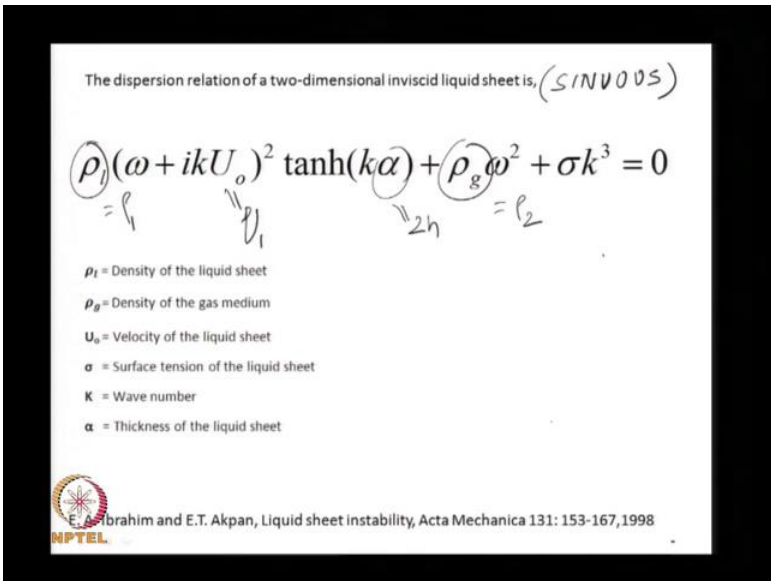


Let us look at some results. If I completed this analysis, what I will get is a dispersion relation that looks like this. With this alpha being equal to 2 h in our analysis this is from this paper by Ibrahim and Akpan from Acta Mechanica. This U 0 in their nomenclature is our U 2 our sorry our U 1. The rho liquid is what we called rho 1 and rho g is what we call rho 2.

So, this is essentially a quadratic in omega. So that I can solve just like the previous situation, I can write omega is minus 10 hyperbolic or essentially I can write the dispersion relation in closed form where I will get 2 roots for a given value of k. Now, this is for a sinuous mode. Somewhere along you will see if you go back to this reference they would have assume that eta 0t or eta 0b over eta 0t is equal to 1.

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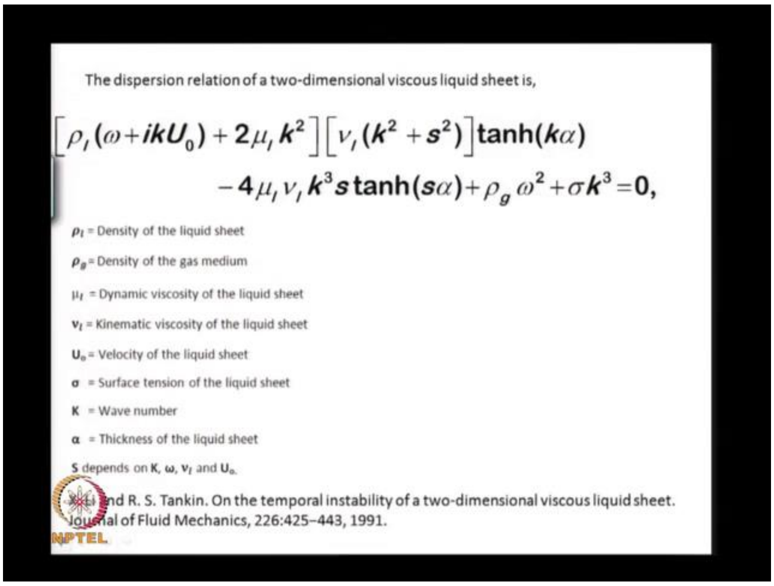


Now, if I look at the results from; if I look at an extension of this now, all over thus far we have only looked at in viscid analysis. At the beginning of the next class we will start off with including the effect of viscosity; liquid viscosity and see what we find.