Spray Theory and Applications

Prof. Mahesh V. Panchagnula
Department of Applied Mechanics
Indian Institute of Technology, Madras

Lecture – 24
Linear stability analysis – Planar Liquid Sheet instability – 1

Hello, we will continue our discussion of cylindrical jet linear instability analysis from Yang which was the case of a cylindrical liquid thread or jet moving in another fluid.

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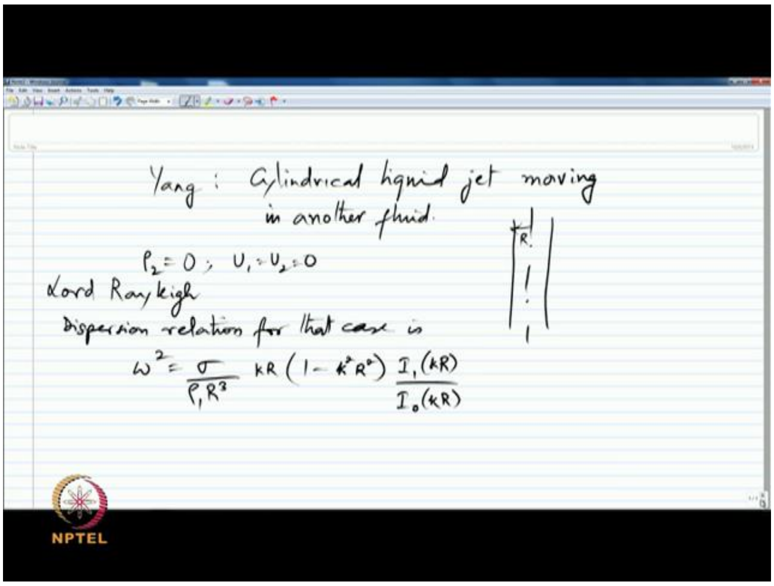

Now we will simplify that case to where rho 2 is 0 and both U 1 and U 2 is 0. So, essentially we are looking at a cylindrical column of liquid of radius R and infinite cylindrical column of liquid of radius R suspended in an ambient environment of density 0, that is essentially the case and this reduces to the original pioneering work of Rayleigh.

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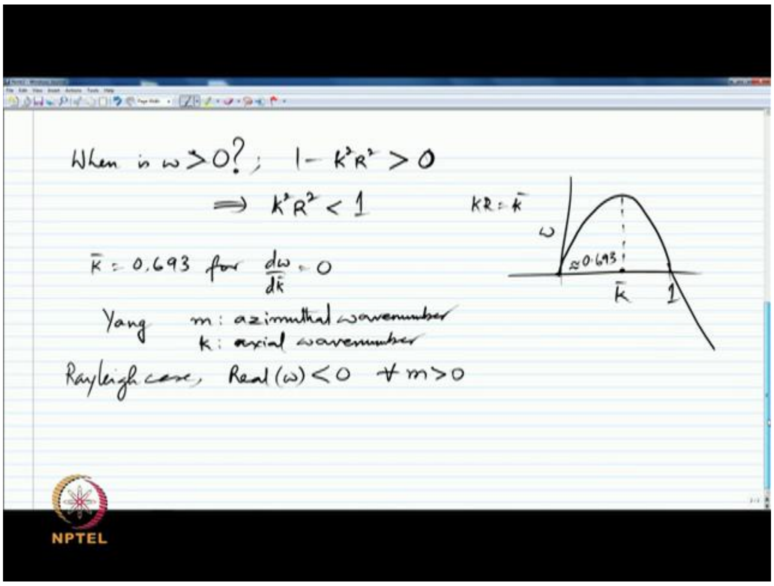

So, the dispersion relation for that case; so this is the dispersion relation from Lord Rayleigh, we are able to solve this closed form which is simply. Now I 0 and I 1 are Bessel functions are modified Bessel functions of the first kind and are positive for positive arguments, so I 0 x and I 1 x greater than 0 for all x greater than 0.

So, again we use the same condition that when is omega positive; the answer is in when 1 minus k squared, R squared is positive. So, this implies k squared R squared is a non dimensional wave numbers. So if I was to plot omega versus k, the number 1 on this k bar has a meaning; now if I was to plot the rest of this dispersion diagram again for all k bar greater than 1 omega is actually negative and this peak occurs at a value of 0.693. So, k bar equal to 0.693 for d omega d k bar equal to 0, k has units of 1 over length k bar is dimensionless.

So, this k bar being 0.693 tells us the wavelength associated with the disturbance that is likely to grow the fastest and again one of those remarkable coincidences of linear instability analysis when compared to experimental observations that this particular wavelength has been observed in very simple faucet breakup; liquid jet breakup experiments let us say in a from a bathroom faucet and the drops formed from a simple cylindrical jet breakup experiment; agree very well as a matter of fact with this prediction from linear instability theory. It was one of the early successes of linear instability analysis that has led to their widespread use in many different systems thereafter.

Now, let us quickly step back and understand the source of this instability. So, essentially if I have a cylindrical jet that is suspended in ambient air with no perturbation on the interface, it is likely to remain in that condition forever that is the meaning of the fact that; that particular state is a solution to our governing equations plus boundary conditions, but if perturbed and perturbed by a wave number of by an arbitrary wave number or let us say by if perturbed by a whole set of wave numbers between 0 and 1 then the wave associated with the wave number 0.693 is sub is likely to grow the fastest and the source of this growth essentially is surface tension energy.

So, a cylindrical jet that is infinitely long as greater interfacial energy in terms of the surface tension than liquid spheres that are discrete and arranged a spacing apart given by 2 pi by k bar in a non dimensional way. So, this is the finding from the linear instability calculation and this state overall as lower energy, lower interfacial energy then the initial state which is an infinite column of liquid.

Now, why would one prefer that the cylindrical column would only break up in an axis symmetric sense, there is really no reason to believe a priori that the cylinder would only breakup into axis symmetric drops or axis symmetric features, so if one were to take the full Yang analysis where you have m an azimuthal wave number as well as k which is our axial wave number then is it possible I may find an m star or a particular value of m for which the disturbance would grow the fastest and if I go back to the Rayleigh case for a moment, what we can show is that the real part of omega for all m is less than 0, that is all m greater than 0.

So m equal to 1, 2, 3 etcetera are all unconditionally stable that is even if you were to introduce the perturbation that has an azimuthal feature to it, they would be damped out they would be attenuated whereas, axial perturbations especially near this number 0.693 in wave number are likely to grow. Now if this is not the complete yang analysis, this is still the case where rho 2 is 0 and U 1, U 2 is 0 which is what we called a Rayleigh case.

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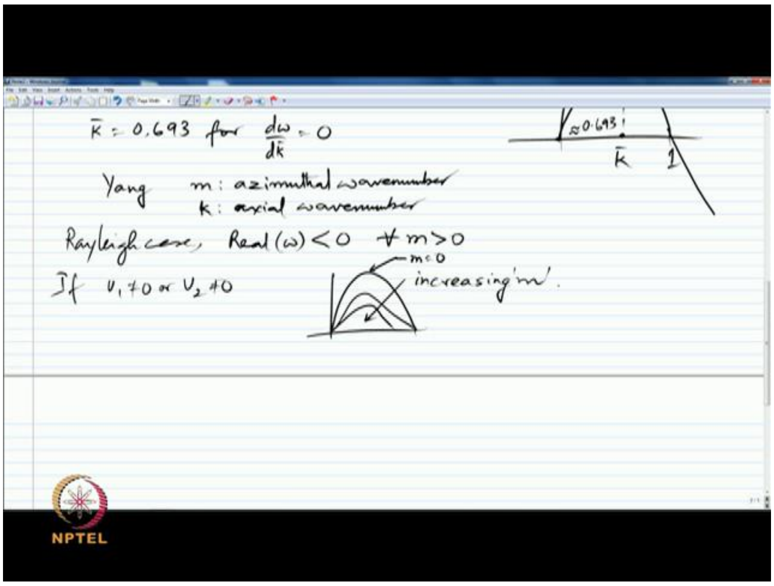

But if U 1 is not equal to 0 and U 2 is not equal to 0 or then this may not necessarily be true that all the azimuthal waves are unconditionally stable.

For this case the dispersion diagrams for different m, if the outer curve was for in the case of m equal to 0 all the other m values 1, 2, 3 etcetera would all have growth rates less than the growth rates for m equal to 0 and this basically means that even if the flow field had access to disturbances that had an azimuthal nature to them all the azimuthal disturbances would be damped out and only a axial disturbances would show up because the growth rate associated with a particular value of case k star and in when U 1 U 2 are not 0 this k star is not equal to 0.693.

Whatever, be the value of that k star; but m star equal to 0 meaning the most unstable azimuthal wave number would remain 0 would dominate the flow field. This is the story even when 1 extends to the full three dimensional to, imposing fully three dimensional disturbances with an azimuthal and an axial wave number associated with the disturbances.

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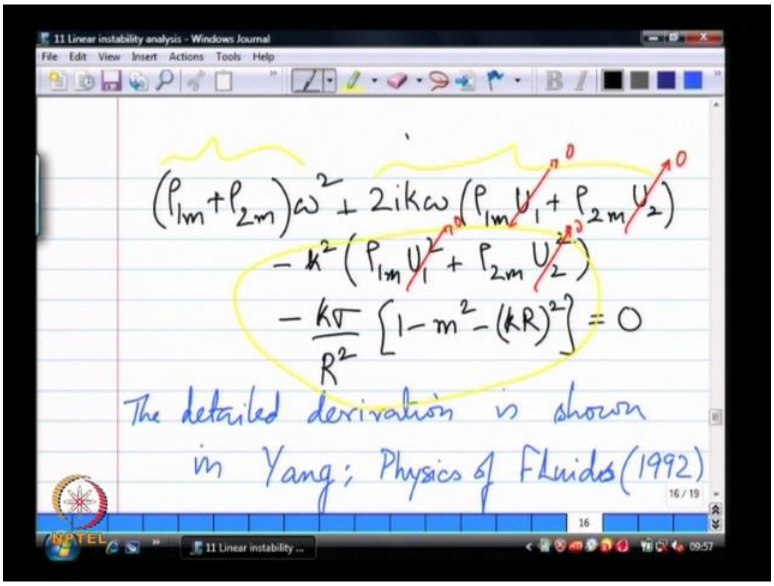

So, if I go back, so that is the physical meaning of m and k. So, if I go back to this full three dimensional equation of a quiescent cylinder of liquid of radius R in a fluid of density rho 1 with the inside fluid being rho 2, this is essentially the surface tension part of the instability that is the only part that matters the density is do not matter if you look in this red equation here remember we have we simplified the previous more general equation by setting rho 2 equal to 0 and this rho 1 m has a rho has there is a rho 1 inside here.

So, essentially the fluid density and the surface tension of the fluid are the only parameters that matter and I can find the growth rate if you do this calculation what you will find is that of all the different modes of instability possible.

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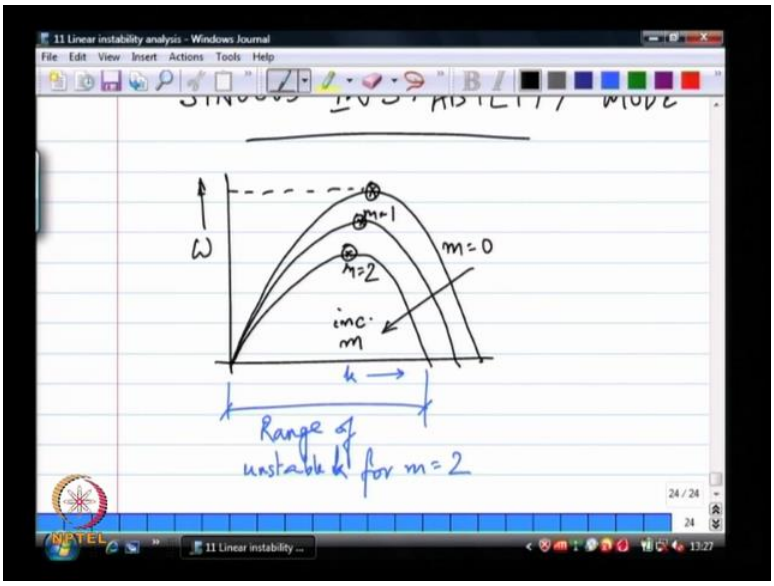

Now, this is pure surface tension instability say if this is the instability for; m equal to 0. m equal to 1, 2 etcetera would all be subsets of this. So, not only is the range of unstable wave numbers, so what I am marking here is the range of unstable wave numbers for m equal to 2 and you can see that the range of unstable wave numbers is decreasing with increasing m, not only that the actual growth rate of the m equal to 0 mode which is this part is the highest of all m greater than 0 that is if a jet were to break up purely under the action of surface tension, it would only breakup in an axis symmetric fashion.

So, I would break up in that fashion and give rise to drops whereas, if you go back to mode general equation you might find that there are certain values of U 1 and U 2 for which m equal to 1 mode which is the sinuous mode would become the dominant mode, but beyond that it would not be possible in these for a cylindrical jet geometry to give rise to the higher circumferential modes. In other words m equal the growth rate associated with either m equal to 0 or m equal 1 would always give you the highest growth rate of all the values of m that is in general quiet unless you pull up some very esoteric situation involving even like swirl.

So, this is as far as this is basically cylindrical jet, linear instability analysis of instability jet in full three dimensions that is what we have looked at thus far. What we will do today is move on to a planar sheet and look at how a planar sheet breaks up. The first example that we looked at was just a single interface of liquid, so like air over water. So, what we now want to look at is a liquid sheet of water that is coming out into air let us say alright.

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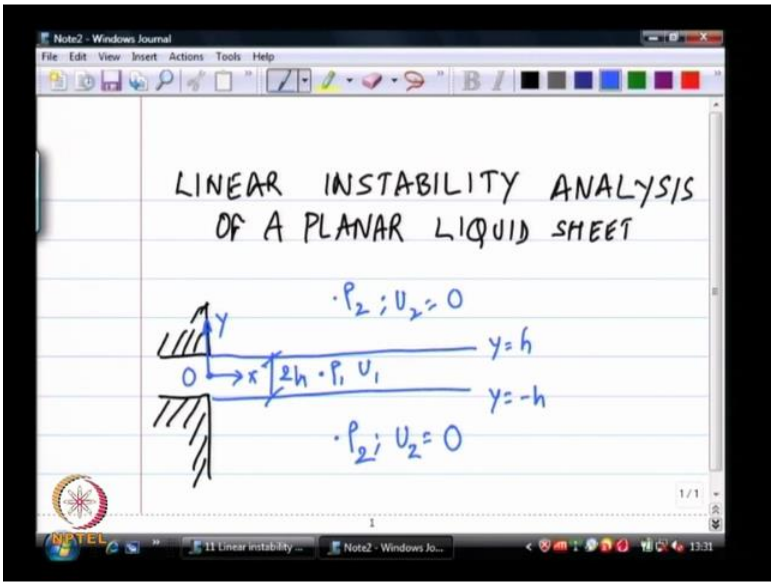

I am going to simply this geometry slightly because this kind of an analysis could go out of hand very quickly if you are not carefully. I mean go out of hand for the sake of a class room or for the sake of hand calculations, the algebra just gets too some combustion beyond a point. So, we will look at a cylindrical sheet of thickness 2 h with the origin placed in the middle of the liquid sheet. So, we know that the top meniscus is given by the equation y equal to h, the bottom meniscus by the equation y equal to minus h we are going to ignore gravity so it is just simply liquid sheet coming out of a slot and the z direction is the sheet is infinite in the z direction.

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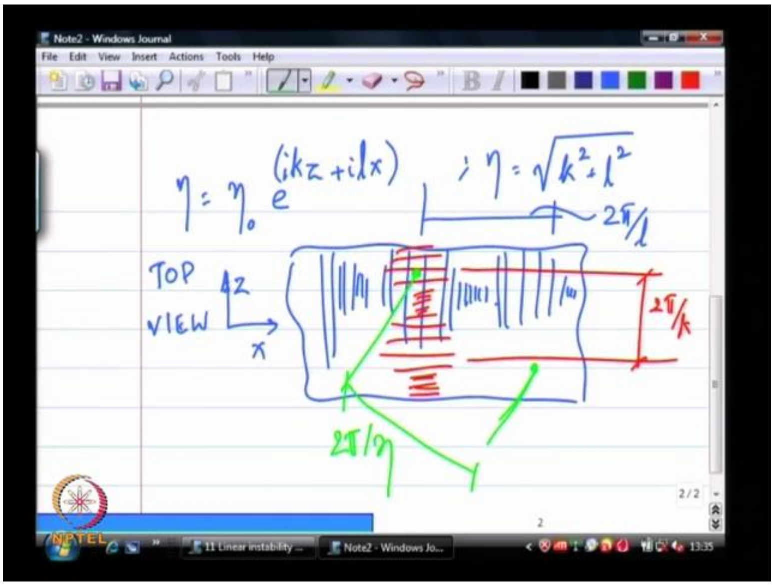

Now I will write the normal mode form first just show you a couple of things and then we will move on. Say if I want to get fancy and I want to include sinusoidal perturbations in both the z and y direction and somehow believe I want to look at sort of waves coming in and out of the; so if this is the liquid sheet, there are waves into the camera as well as waves in this direction. The wave number in this direction is given by l, the wave number in the direction towards the camera which is my z axis is given by k, but the liquid has a velocity U 1 that is along the positive direction or positive x direction wait, no, this is x naught. So, the wave number in the x direction is l and the wave number in the z direction is k and the velocity of fluid is U 1 in the x direction.

If you look at this kind of a configuration, you essentially have the velocity in one direction, but you want to look for instabilities in that grow, in the z direction as well as in the x direction. Another way to look at this is the compound wave number given by both l and k. So, if I now define another eta which is k squared plus l squared, so if I take a piece of this liquid sheet look at the top view; I might create waves that are in that direction and let us say this wave number is given by this is the x direction and if I make this my z direction, this is given by 2 pi over l; I am going to change pens here.

The spacing on the lines is intended to show the sort of the slope, the higher the spacing the higher the slope. So, it is sort of like a topological map basically, now in this direction this is intended to give you something like 2 pi over k. The 2 pi over l, the super position of these 2 waves would essentially look like there is a high spot here and another high spot here found by the intersection of these and this put together is my new wave length eta or new wave number 2 pi over eta simple Pythagoras theorem application at the very least.

So, having two wave numbers in two orthogonal directions is really no more complicated than having one wave number in one other in one sort of a rotated direction and 0 wave number in the other. So, this combination of these two waves is going to look like there is a wave there are waves of the order of 2 pi over eta, wavelength 2 pi over eta in the direction given by tan inverse l over k essentially or tan inverse k over l; however, the velocity direction is still x. So, it is like I have a velocity direction x, but I have chosen to super pose waves in some other arbitrary direction that is the only result of having two wave numbers in directions that are along the flow direction and perpendicular to the flow direction.

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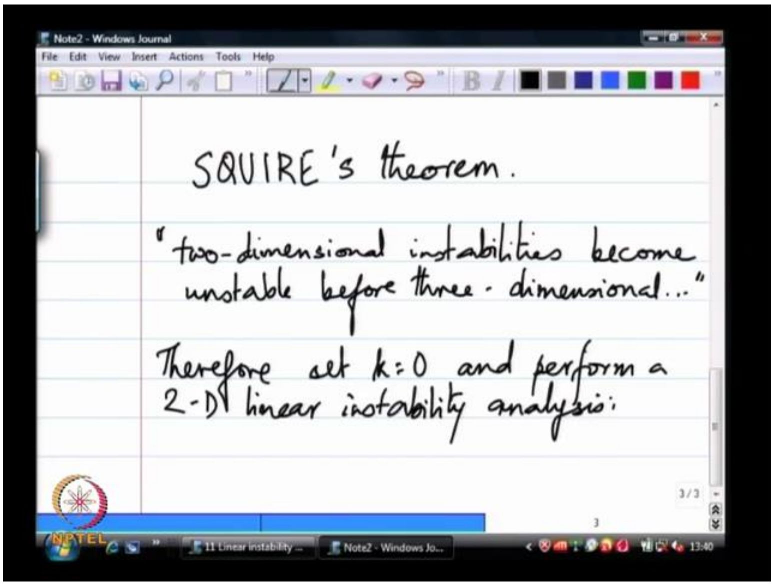

Now, what we learned from an old theorem due to a man name Squire, I am going to paraphrase this, but it is going to be reasonably accurate, so you can write this down. So, essentially 2 dimensional instabilities, so what we showed is; that it is if I want to look at the situation which is the most unstable. I only I am interested in the situation that is the most unstable what he says is the k equal to 0 mode gives rise to growth rates where in at some value of l whose magnitude the growth rates magnitude is higher than that for any other nonzero value of k. So, it is sufficient for me to consider the k equal to 0 mode because that is k equal to 0 or l being finite mode is where we only consider in stability along the flow direction and that is what he referred to as 2 dimensional modes.

So, if I was to see in order for me to show you both k and l, notice how I had to draw like a plan view alongside the elevation view. So, it is inherently a three dimensional instability, so there is a 2 dimensional instability. It may be l equal to 0 or k equal to 0, but both of them are pure 2 dimensional instabilities, there is a 2 dimensional instability that becomes unstable before a 3-D instability becomes unstable meaning becomes unstable means shows a growth rate greater than 0. What that also means is that the wave number set or wave number pair with the maximum growth rate is going to have is going to be 2 dimensional in nature meaning either k or l being equal to 0 and that would happen before both k and l being not equal to 0 shows the maximum growth rate.

So, for a given flow condition for a given rho 1 rho 2, U 1 and surface tension sigma; it is sufficient if I said either k equal to 0 or l equal to 0, it does not matter which; that is as for as squire is concerned, but what we know is that we really need to be setting k equal to 0 because it is this instability in the flow direction that is going to give us the maximum growth rate.

So, we can go forward with this analysis by setting k equal to 0 and perform a 2-D linear instability analysis. So, if somebody ever asks you to perform a full three dimensional linear instability analysis, first look for a reason why the 2-D would not give you the highest growth rate. So, if the flow direction is unique then the only linear instability mode I need to consider is the one in the flow direction.

Now I will give you, let us take one more extension sort of a 1 more thought experiment. Let us say U 1 is the magnitude of the velocity in the x direction and I am going to add another v 1 as the magnitude of the velocity in the z direction. So, this sheet of liquid coming out of this like a thin slit is moving in the x direction with the velocity U 1 and in the z direction with the v 1, but the slit itself is infinite in the z direction. So if you think about it for a moment, it is just like the sheet is coming out of an or phase of a slit and moving in some other direction with the velocity square root of U 1 squared plus v 1 squared and not having any other velocity in any other direction.

Like if I say I have I am moving at 3 meters per second in this way, 3 meters per second this way, it is exactly like saying I am moving in this direction at 5 meters per second when I say those two statements are exactly equivalent. So, when I move when I say moving at 5 meters per seconds, I have no velocity in this direction at all. So, this sheet has a velocity root of U 1 squared plus v 1 squared in some are arbitrary direction with respect to my current axis, but if I rotate the axis to be aligned with that flow direction. I can now consider a 2 dimensional instability in that direction let be done with it. It is only in situations that this thought experiment fails, that you should look at look at even doing a three dimensional linear instability calculation.

If I cannot rotate my view point, rotate my axis to be aligned along a unique velocity vector direction for the liquid sheet, then I need to be looking at a three dimensional linear instability calculation and towards end of next class we will allude to a few such situations where you do really need to look at a three dimensional linear instability calculation, but otherwise for the most part, a 2-D linear instability calculation is sufficient not because it is easy to do, but because there are theoretical reasons to believe that growth rate would actually be higher than anything you will find from three dimensions alright.

So, with that understanding now let us go forward and do this calculation; I just redraw this picture one more time.

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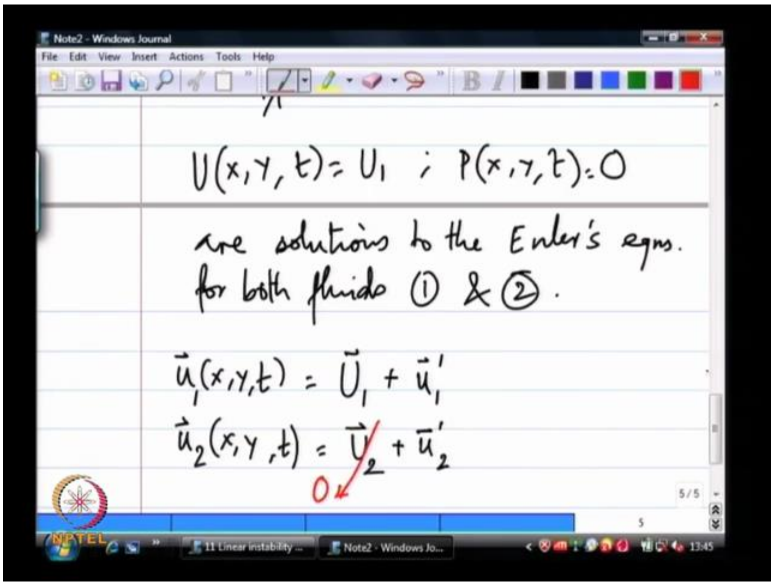

Now I am not going to go through all of the calculation but I am just going to write down the result because we are now familiar with it, u of x comma y comma t equal to U 1; p of x comma y comma t equal to 0 are solutions to the Euler's equation for both fluids 1 and 2. Now if I take a perturbation quantity, so if I take my perturbation velocity field to be equal to U 1 plus u 1 prime; I am using the vector notation.

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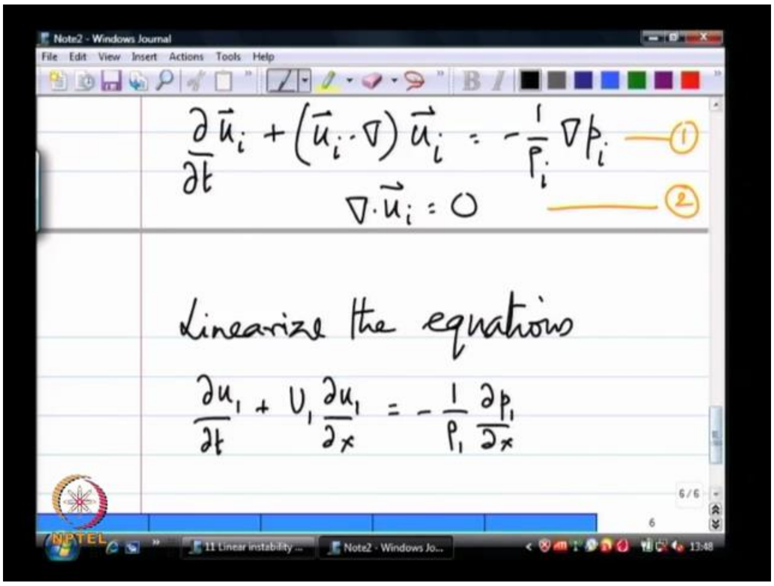

We have chosen U 2 to be equal to 0, how about we just simply write this in terms of i notation like we did in the past it will save us some space.

Now, if I make the substitution of these of the perturbation quantities. So, these are the perturbed fields, so if I linearize the equations of course, I have the continuity equations that says del dot U equal to 0 go to write that term. So, let us start numbering these; I am only starting with the perturbation quantities as my starting point on the equations because there is no mean velocity in the y direction.

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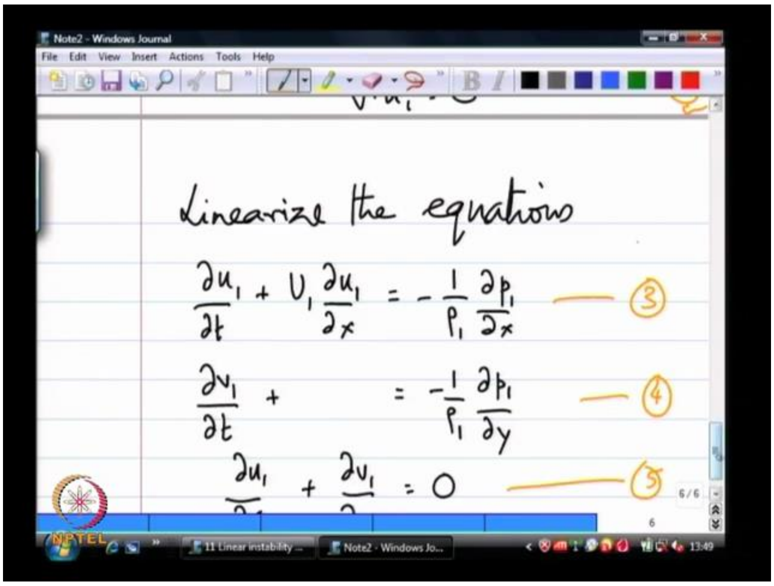

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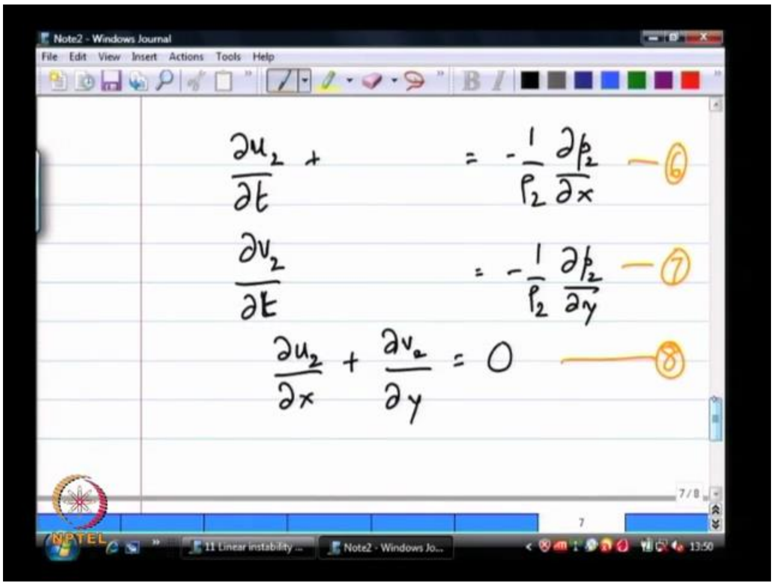

So these are the equations as far as fluid 1 is concerned, if I rewrite for that fluid 2; it is easy to show, so let me number these. If I take the derivative of 6 with respect to x, the derivative of 7 with respect to y and add them up using 8, what I get.

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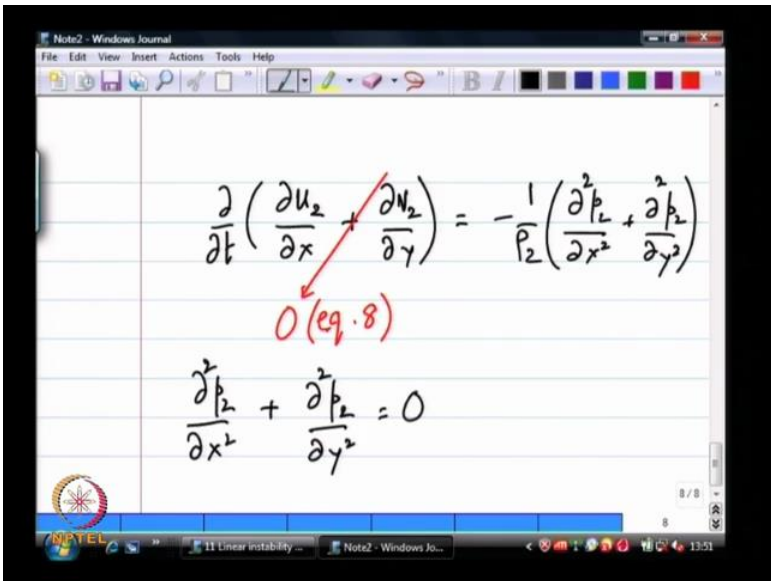

So using 8 that is what you get, we know how to solve this equation that is our good old Laplace equation and I can solve this by separation of variables; I am not going to do that here I have already done it once.

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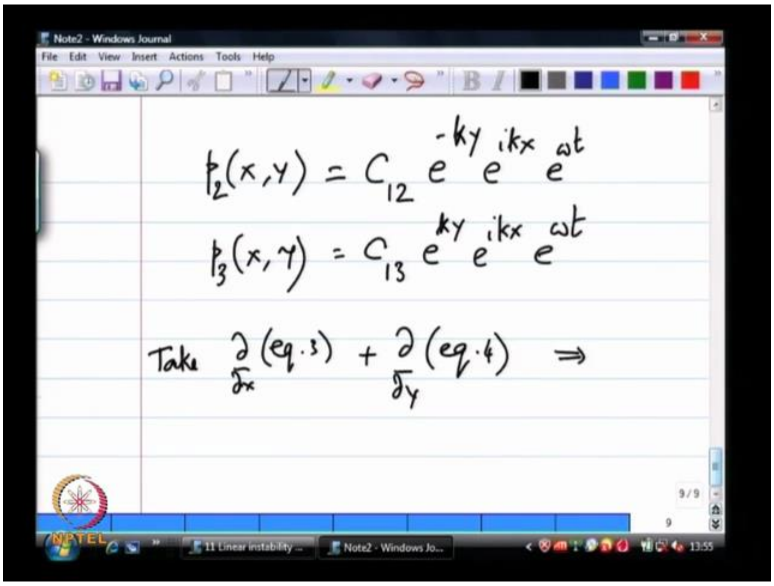

What you will end up seeing p 2 is the fluid outside the liquid, it is like the air. Now if I go back I really do need to make the distinction between 2 and 3 because what I called boundedness condition for large y would be different for this fluid 2 which is above and this fluid 3 which is down below because in fluid 2 what I want is that as y increases; y tends to plus infinity the pressure field has to vanish. In this fluid 3 which is down below the liquid sheet, I insist that as y intends to minus infinity the pressure field has to vanish.

So, it is a little, so I do need to write this out as you know for fluids 1, 2 and 3 but let us just say this is our real 2 fluid and what I end up getting from that is; I will call this C 12 e power minus ky because now times e power minus ikx, this is the solution to the pressure field in 2 times of course, this C 12 is a constant and of course, you are going to have e power omega t, p 3 which is a fluid below. I can write as e power ky, e power ikx, e power omega t. I can do the exact same thing for fluid 1 which is here; now these are the equations that govern my fluid 1. I can take the derivative of 3 with respect to x and I can take the derivative of 4 with respect to y and add them up and what I end up getting.

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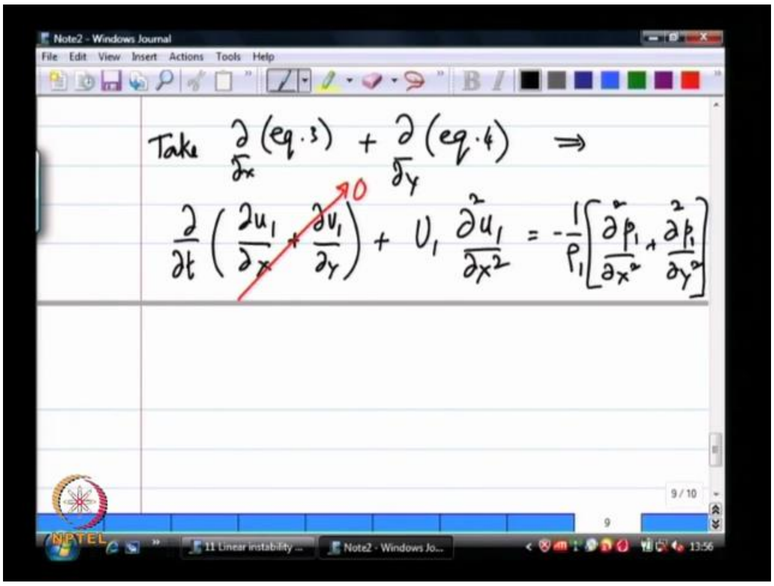

I can eliminate u 1 and v 1 in favor of p 1 from these, so essentially this is these are three equations u 1, v 1, p 1 in terms of three variables. I can eliminate, now what I have done is I have eliminated v 1, I still have u 1 here.

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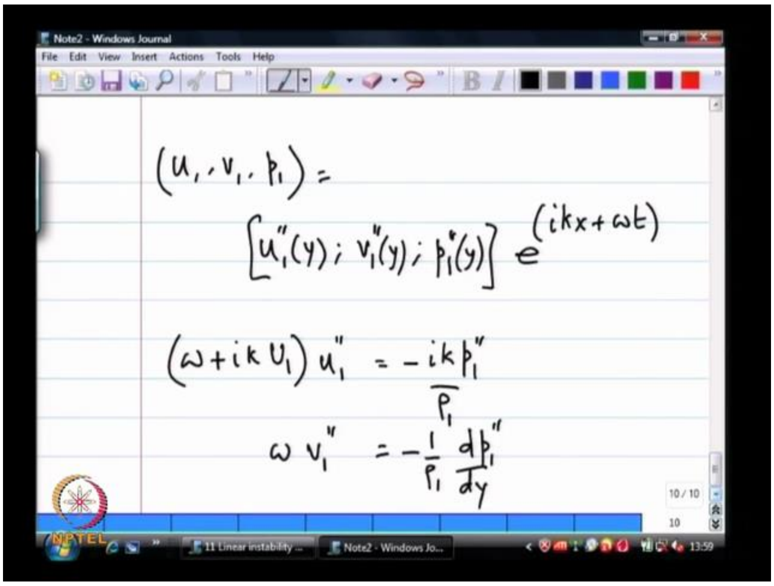

If I make the assumption that u 1, v 1, p 1 are all equal to sum u 1 double prime of y, v 1 double prime of y and p 1 double prime of y times e power ikx plus omega t; if i make this, so in other words I can remember how we showed that the normal mode is not really an assumption.

We can eliminate u 1 and v 1 and in terms of p 1 and get an equation that will yield these normal modes or for simplicity, we can assume normal modes in the quantities substitute back and get equations in terms of this u 1; in terms of the double prime quantities. When we do that, what we find is omega plus ik U 1 times u 1 double prime is minus ik p 1 double prime over rho 1, omega v 1 double prime is minus 1 over rho 1 d p 1 double prime d y and the last one is ik U 1 double prime plus d v 1 double prime dy equal to 0.

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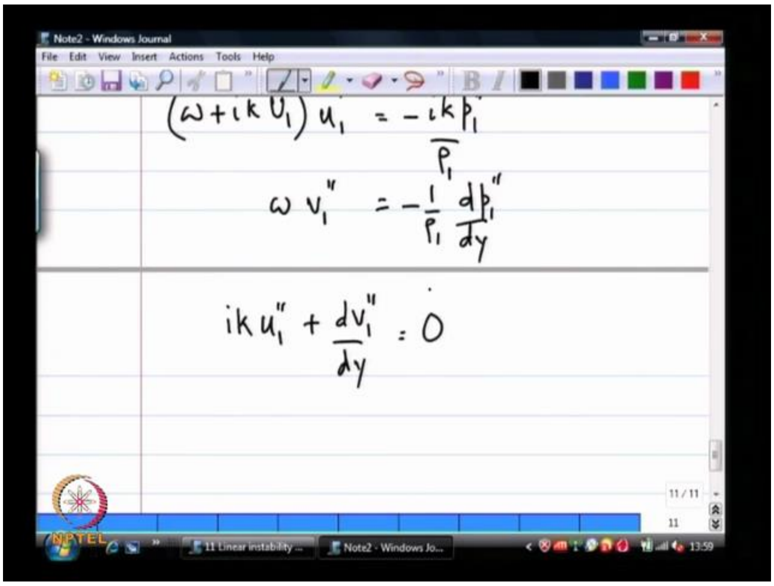

So where we had partial differential equations, we have now converted them to ordinary differential equations in y and in fact, equation I will number this 9, 10 and 11. 9 is simply an algebraic equation, where 10 and 11 have differential quantities in them. We will start from here and complete this analysis in the next class.