Spray Theory and Applications
Prof. Mahesh V. Panchagnula
Department of Applied Mechanics
Indian Institute of Technology, Madras

Lecture - 23
Linear stability analysis- Cylindrical jet instability -2

Good morning, we are going to start of from where we left at the last class, we were looking at a cylindrical jet.

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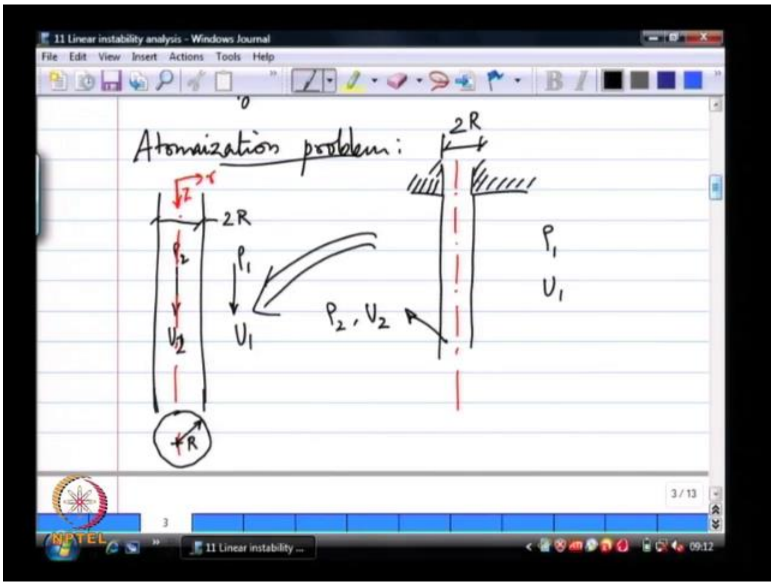

We were looking at applying linear instability analysis to a cylindrical jet, and the jet itself is moving with the velocity U 2 and the fluid outside in general could be moving with the velocity U 1, and we wanted to see how this extends to you know the how the stability analysis that we discussed extends to this case; by the way the this.

So, once we start of we ended by writing the by actually not just writing, but deriving this normal mode assumption. So, the fact that the Eigen modes of this problem or orthogonal in that space comes out of the for the realization that we are dealing with a strum lionville problem, which is basically it is a Laplace the pressure field in each phase is governed by a strum lionville problem, and because of that we went through this process and showed that the modes have to be of the form e power im theta and e power ikz, which automatically mean that we are going to be orthogonal with respect to the waiting function that that comes from the strum lionville problem, which in this case is I think one.

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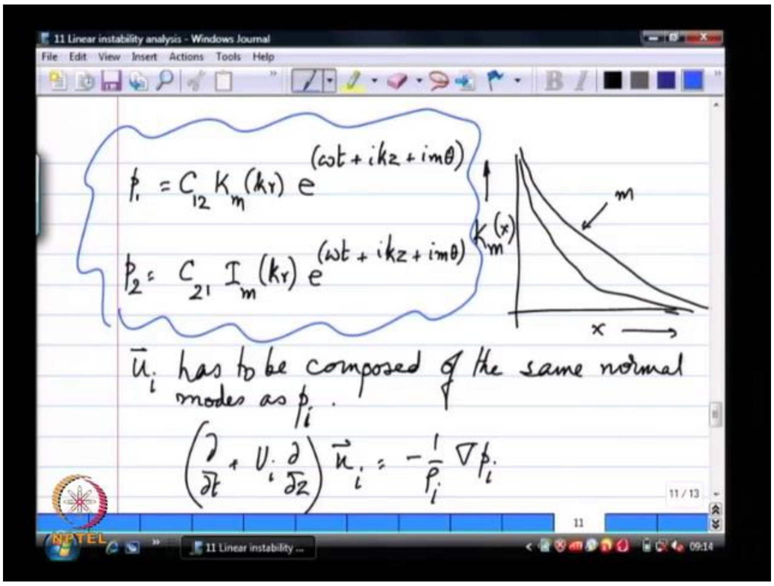

So, given that so these are the this is the form of the eigen function that we came up with for the pressure field p 1 and p 2 and from this we can go back to the differential equation, that he that we have and solve for the velocity field U i. We can also from this actually find that any sort of a surfaces perturbation the cylindrical interface perturbation also has to be of the same Eigen mode form.

Now, so we now need to formulate a set of boundary conditions to solve to complete this Eigen value problem. The first set of boundary conditions that we wrote down, or what we called the kinematic boundary conditions. Now I want to make sure we I am going to repeat some of the material from the earlier linear instability analysis just to make sure we have a complete grasp of the physical meaning of these boundary conditions, the kinematic boundary condition, see if I now take I am going to write these cylindrical polar coordinates in this fashion.

So, this is our jet that is moving with the linear velocity U 2 and the outside fluid is moving with a velocity U 1. If I take this is eta as a function of z and time eta meaning this is coming from the normal mode. In fact, z theta and time so if I take a material point on this free surface I am going to use a different color for that, if I take a material point that is on the liquid jet surface, this material point inside the liquid surface has a radial velocity U R which is given by the perturbation field U i dotted with e R. So, the magnitude of this radial velocity U R is given by U i dotted with this radial unit vector in the radial direction e R this U R is the Eulerian velocity inside the jet that is essentially what we find, that also since the material point happens to be on the interface the rate of change of eta should equal that material velocity or the eulerian velocity at the interface. So, what do we mean by that this interface point can exhibit a radial velocity due to two reasons; one eta itself under goes an increase with respect to time at a given location.

So, we have to go back to our fundamental understanding of a partial derivative. So, at an R and theta or at a z and theta location on the jet interface d eta d t or partial derivative of eta with respect to time gives me the rate of change of eta with respect to time. So, that in some sense is like a radial velocity at that point the second part corresponding to U i partial derivative of eta with respect to z accounts for the fact that even if there is no growth of eta with respect to time, but eta has a sinusoidal motion and that sinusoidal form is being advected down strain.

So, if I take a sinusoidal form and if this sinusoidal form some time later is at this location the same material point, which was there is now here. So, the material point essentially has move down that much distance in that period of time that this waves are shown the two snapshots that represent these two waves. So, even if they amplitude of this sinusoidal wave did not change the fact that the material point moves with the interface is basically, what is kinematic boundary condition implies in physics terms.

Now, the rate of movement, so this wave itself has moved a distance U i times delta t. So, if I took the material point as I have shown here, just inside the interface inside the jet then that wave is being advected with the velocity U 1, which is inside the U 2 which is the jet fluid velocity, I can draw this exact same pictures with the orange dot and the pink dot just above the meniscus in the other fluid, when I do that the material velocity in the axial direction of that point is U 1.

So, as viewed from fluid 1 the waves are being advected with the velocity U 1 as viewed from fluid 2 the wave is being advected with the velocity fluid 2, consequently the radial velocity in fluid 1 would be different from the radial velocity in fluid 2. So, we are used to understanding this from a flat meniscus saying, if I have a flat meniscus and if let us say v 2 is the radial velocity in the in this part of the fluid that has to exactly equal v 1, which is the radial velocity in the other part of the fluid, but if I am dealing with a curve meniscus especially the curvature is moving.

So, if the curve meniscus is still stationary then the radial velocity has to exactly be continuous, it is sort of analogous to our heat flux boundary condition at an interface between two materials, it does not even if the curve meniscus or the interface between the two fluids is curved it does not matter. But if this curvage if this curve meniscus is being advected then the advection velocity is different in the fluid 1, because of a inverses the fluid 2 now this is essentially coming from our idea that I am going to throughout all the terms that are not order one in this equation, in writing this equation I am going to throw out all the terms that are not order one, and I am only going to retain the terms that are order one, because of that the interface is assumed flat or unperturbed for all practical purposes. So, if I have this interface and it in the unperturbed condition the fluid the material particle inside the jet is moving with a velocity U 1, the material particle outside the jet is moving with a velocity U 2.

So, this U 1 and U 2 being different at the interface is accounted for in this kinematic boundary condition. So, we are going to have to write something like this time and again every time we do a linear instability analysis of any kind of a problem. So, we need to understand where this comes about, what is the physical significant of this and just to make sure that we are able to make sense of it, the second set of boundary conditions are what we call the dynamic boundary conditions.

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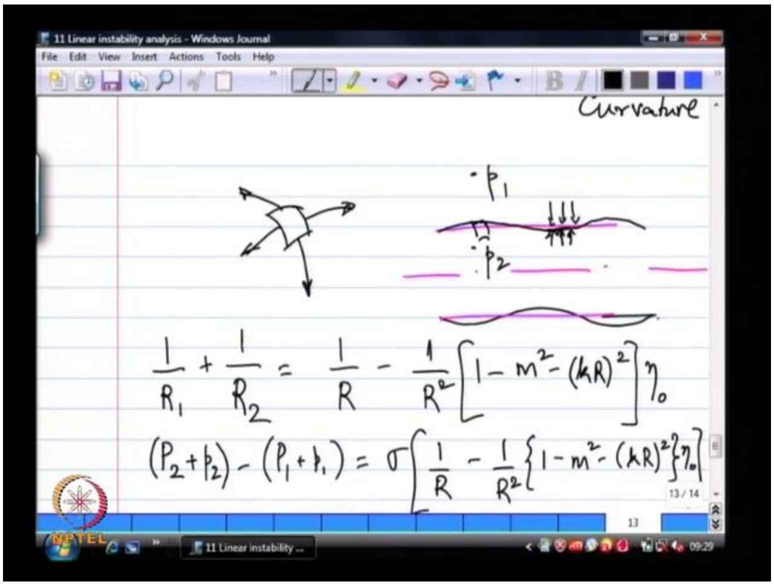

So, if I now again make go back to the cylindrical jet this is the unperturbed form, the perturb form has this sinusoidal wave now the pressure inside here, is P 2 and the pressure here is P 1, we are only talking of the perturbations from the unperturbed condition. So, these lower case P 1 and P 2 are the perturbed pressures additional pressures due to additional pressure fields, due to the perturbation capital P 1 and capital P 2 are were our mean pressures before the perturbations.

So, if I now look at this unperturbed or the perturbation pressure fields the perturbation pressure fields give us require that the forces at this interface remain in balance. So, if I take a short patch of this meniscus. So, write here there is surface tension sources acting in all four directions now, because that interface is curved in both those directions it has it curvature in both those directions. So, because of that we have to define what is called principal radii of curvature. Which may not necessarily be in the R and theta directions per say if this, if for a very general free surface, but if I am dealing with a free surface that was originally cylindrical, and I am only going to retain terms that are order epsilon or order eta in the perturbation quantities then those quantities are order eta.

So, in that condition the radii of curvature are in the coordinate directions. So, you have a radii radius of curvature in the R direction, and you have another curvature in the theta direction and that essentially comes to as you will recognize m is the circumferential wave number and k is the axial wave number, R is the unperturbed jet radius as shown in this figure. So, if I write the complete equation P 2 minus P 1, if I write the total pressure in fluid 2 minus the total pressure in fluid 1 has to equal sigma times this 1 over R 1 plus 1 over R 2.

But we also know from matching the mean pressures the unperturbed pressures this is coming from the mean pressures being matched. So, if I use that, what I end up getting is this, now I should be clear that this is P 2 minus P 1 evaluated at the unperturbed jet radius.

So, if you go back we did find P 2 and P 1 as a function of r t z and theta, all we are saying is that add this meniscus the pressure field just inside, and the pressure field just outside, on an infinitesimal strip of this on a infinitesimal piece of the interface balance out while including surface tension force that is the physical meaning of this dynamic boundary condition. So, since the mass of that liquid or mass of both fluids put together right at that meniscus is. So, small that when I say f equal to m a or some of all forces on that infinitesimal piece of material is equal to m a that m is so small that the sum of forces has to be equal to 0, where essentially taking a thin slice right at the interface where fluid 2 is pushing from the inside fluid 1 is pushing from the outside.

So, I can the take P 2 and P 1 forms that saw in this here, evaluate them for the condition that this lower case R equal to capital R. So, that gives us the pressure inside fluid 2 and fluid 1 at the coordinate R equal to capital R.

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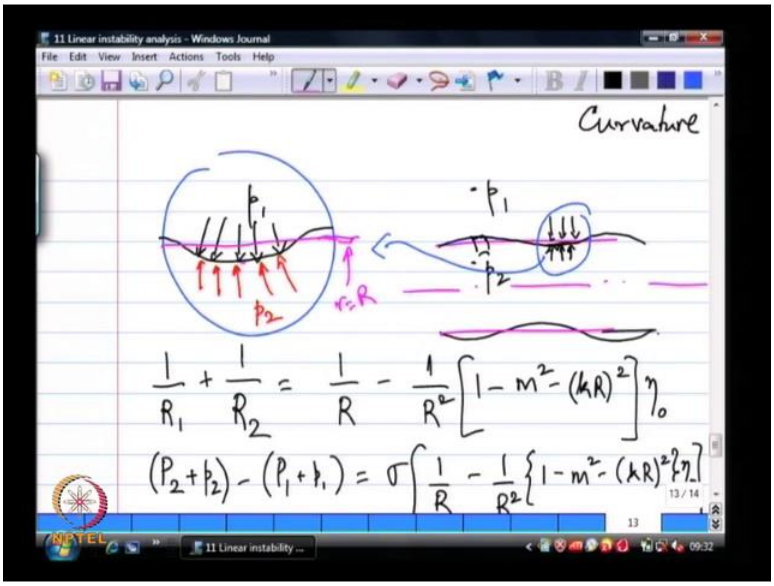

Now, I want to also make sure we understand one more concept here, when I take this and draw a free body diagram of this, and I will erase this part I have just drawn this is the pressure field p 1 and this is another color for the pressure field p 2 and the unperturbed interface is here. Now if I really want to understand, if I really have to write this force balance if I have p 2 as a function of R theta z and time which I do, and if I have p 1 as a function of R theta z and time which I also do.

I have to evaluate this at the interface location at the perturbed interface location not at R equal to R. So, this coordinate this unperturbed location is my R equal to R, but. So, this is let us say; I will write this at the interface for.

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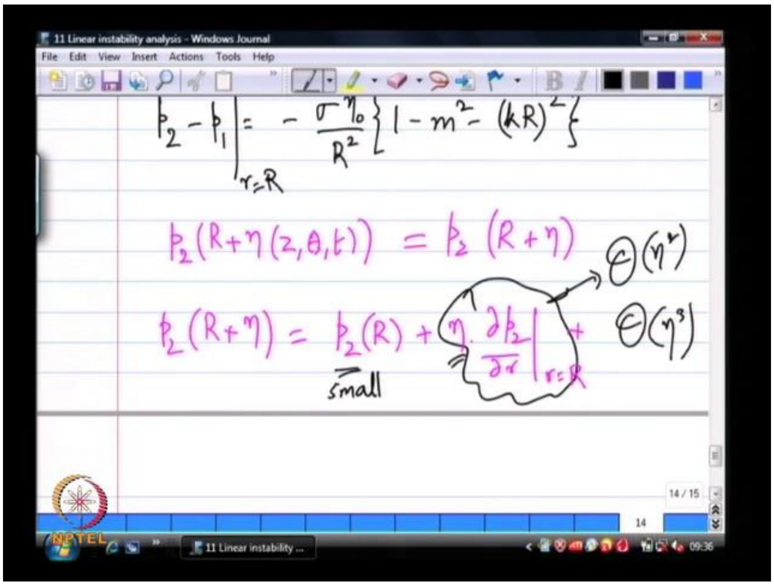

Now, and I will show you how this comes about, if I take p 2 as an example, which is inside the jet p 2 at the interface which is R the interface location is given by R plus eta which is a function of z and time, if the radial location of the interface is capital R plus this perturbation eta which is a function of z and time eta given by or in fact, z theta and time right.

we are all doing a full three dimensional analysis here, I will write this more compactly as R plus eta you know that eta is the interface location, I can write p 2 is a p 2 at R plus eta to be equal to p 2 at R plus e 2 eta times d p 2 d R evaluated at r equal to R plus order eta squared terms, simple Taylor series now p 2 itself is a perturbation quantity p 2 itself is order eta or order epsilon quantity. So this itself, so this term is order eta squared already.

So, technically if I was to write this next term it would be order eta cubed, because p 2 is a small quantity eta is another small quantity d p 2 d R the derivative of p 2 with respect to R is also a small quantity. So, this already is a second order effect, second order correction to p 2. So, to the order into which we are maintaining the fidelity of r equations which is order epsilon the pressure field at R plus eta is equal to the pressure field at R especially since we are only dealing with a perturbation pressure field. So, this perturbation pressure field, I can now take even though I do rigorously, if I have to do this to higher orders I do need to retain this terms up to a higher order, but to the order into which we are retaining these equation to we are writing these equations we can write this we can evaluate these perturbation pressures at the unperturbed interface and still get away with it.

Now, this process of doing this Taylor series expansion and then checking the terms that are required to be included in the dynamical pressure match condition is very important. Because, they are situations and one simple situation is where you have a swirling jet if the cylindrical jet had a mean swirl velocity. Which we are considering two cases where the fluids are simply co flowing if the inner fluid had a swirl velocity to it this order epsilon correction order epsilon terms itself would require the second part, where the perturbation pressure would now had you would have terms coming from here which are order epsilon. So, with I do not want to go in to the details of it we will probably take that of as a home work problem later on, but we now have this pressure matched condition I call this equation 6, this is 7 or rather I want to make this 7 8.

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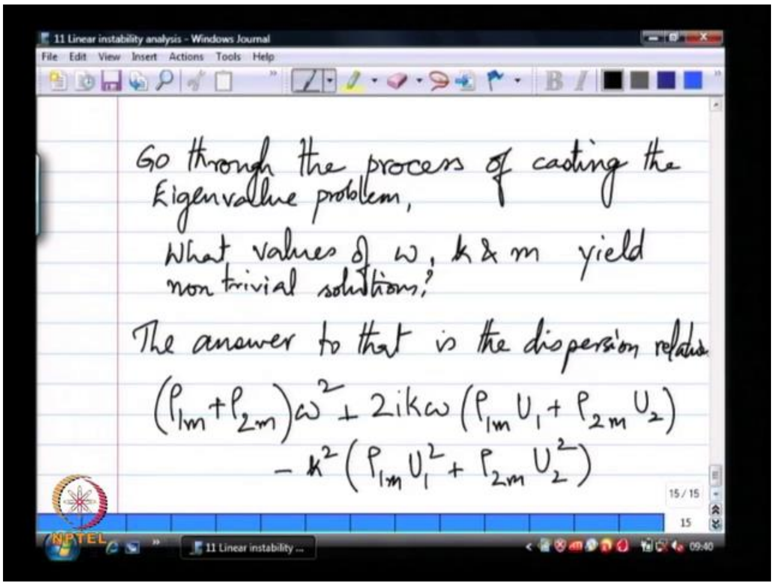

So, if I take all of these equations and go through the process of casting the eigen value problem, that is we asked a question what values of omega k and m yield non trivial solutions the answer to that is the dispersion relation that you get.

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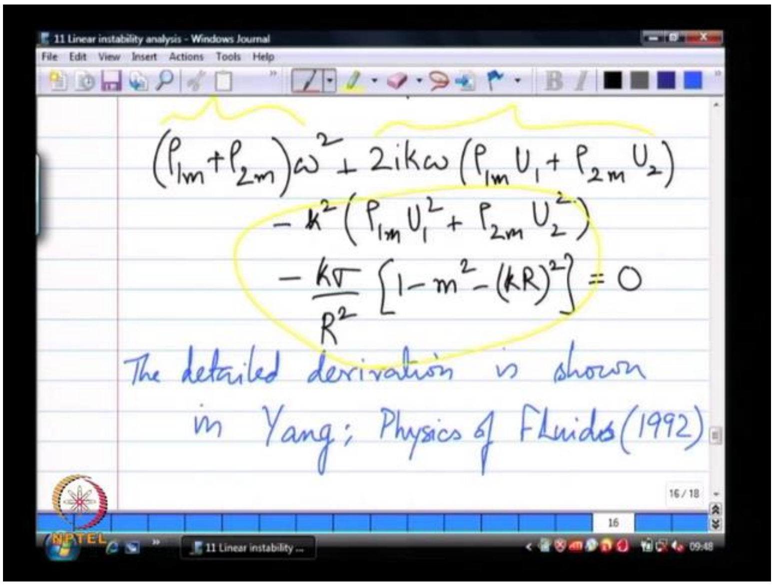

Which in this case, would be of the form the detail derivation of this is shown in this paper by yang, from the general physics of fluids, 1992? So, let us look at this for a moment and try to extract some physics out of it the equation as you will see is quadratic. So, we have it is we have one term that is order omega square that is of the form omega squared omega plus constant term here so in general.

If I want to complete this write down the terminology that is being adopted here, rho 1 m is equal to k times I m k R divided by I m prime k R times rho 1 rho 2 m is k times minus k times k m k R divided by k m prime k R times rho 2 rho 1 and rho 2 are the material densities k m is the Bessel function of the second kind, modified Bessel function of the second kind prime k m prime and I m prime denote differentiation with respect to the argument of the function. So, if I go back look at this there is for a given k and m this is a quadratic equation in omega which means in general it has two complex routes.

The one complex root that gives us the higher real part, the one complex root that has the higher real part is the one that is going to dictate growth of that particular imposed wave. So, the wave itself in this case is characterized by two wave numbers an axial wave number k and an under circumferential wave number m. So, these circumferential and axial wave numbers have a characteristic growth rate, which is given by the real part of the omega real part of the omega and the one of the two that has the higher growth rate is the one that is going to be significant.

So, in general if I take any particular any instance and plot this omega as a function of k and in I will plot this in some dimensionless sense I plotted with respect to k R, which is saying my length which is like non-dimensionalizing my length unit the radius of the jet to one. In general you will find this to be you will have the characteristic value omega as a function of k R take on a graph that looks like this.

So, this end here is a k cut off beyond which, if I extend this graph beyond the k cut off both the routes of that quadratic equation for m equal to 0 and k value greater than this k sub c, both the routes of that quadratic equation have real parts negative meaning that any perturbation that has a circumferential wave number m equal to 0 and an axial wave number greater than k sub c. If you could generate that kind of a perturbation on the meniscus is going to completely die down exponentially die down in time, and any perturbation with k less than k sub c and m equal to 0 is going to grow in time with a growth rate given by that particular value.

So, for a given k r this is the growth rate that corresponds to that imposed disturbance now as you can see each of these disturbances is going to grow exponentially. So, if I had a source of perturbations that expand all k r going from 0 up to infinity, all k r that are k is greater than this k sub c are going to be stable. So, this is my neutral stability boundary now all values between 0 and this case, of sub c are actually unstable meaning the real part is greater than 0, but there is one particular value which achieve which takes on an even greater significance going to the fact that its growth rate is the highest among all of the wave numbers that are possibly unstable.

In because each of these is growing exponentially in time very quickly you saw you see that this particular value of k sub m is going to dominate in amplitude over all other wave numbers that are imposed. So, this is a characteristic of most linear systems and weekly non-linear systems that we see like for example, we looked at the Kevin helmus instability on a lake, and this we said explains why all the repels on a lake are nearly of the same wave length. Now I want to take one more instance, we will look at this equation and simplified by setting both the U 1 and U 2 velocities is to be 0 ok.

So, I am just now looking at a cylindrical column of liquid, that is somehow stationary in air with no gravity now what simply going through the process of scratching out the terms that do not apply. Now this is actually a beautiful result due to lord relay which describes the instability of a cylindrical liquid column in vacuum because, we said way rho 2 equal to 0, due to surface tension alone because U 1 and U 2 are both equal to 0. What you found was that, if I plot this omega as a function of k R, like I said you will get something like that and this k R at which you get the maximum growth rate k is very nearly equal to 1, which if I translate into physics into real into the physical system. If k equal to 1; that means, it is wave length lambda is equal to 2 pi.

So, what he showed was that if I took a jet of diameter r, this cylindrical column would break up into drops at equal intervals in length of the cylindrical jet equal to 2 pi R. So, this if I took the cylindrical column of liquid and sort of sheared of pieces that are 2 pi r long r diameter and let each blog, each of the cylindrical blobs go back to it is lowest energy shape which is a spear that would give us the size of the drop that would form, and he showed experimentally that all of this mathematical analysis, and the experimental observation of the actual size of a drop form a dripping fosset from a just passed dripping fosset, where you have a cylindrical column of liquid coming out a slow moving cylindrical column of liquid coming out of a bathroom fosset the drops forms from that are very close to this prediction.

Which is quite remarkable because if you go back, the whole analysis is based on the fact that I have an unperturbed cylindrical column that I introduced at small infinitesimal perturbation and look at the growth rate of that infinitesimal perturbation in the neighborhood in time of when the perturbation was introduced to start with? So, we are only looking at the case where I have introduced a perturbation and we are just watching it grow. But to then take the results that come out of that calculation and compare to an experiment where clearly the meniscus is well passed the infinitesimal perturbation states we are at a case of I am able to visibly see the sinusoidal perturbations on the meniscus to show that the linear instability analysis also holds all the way to the stage, where the perturbations can grow and grow to a much larger amplitude than infinitesimal. Then being infinitesimal in nature all the way to where the amplitudes are comparable to the jet radius itself, even in that instance this kind of an analysis the results from this kind of an analysis still hold good.

There is no theoretical reason to believe that they should hold good, but in the fact that they do compare well with experiment is an indirect testament of the fact that the non-linearity present in these equations is weak in comparison to the linear effect themselves.

So, we have been save by the grace of navier here in the sense that for the case of jet break up problems we are linear instability analysis has performed remarkably well in compare in pre in being able to predict drop sizes, being able to predict the atomization characteristics and at the. In fact, quantitatively, but at the very least qualitatively where for example, if I look at this exact same situation and see what effects surface tension may have on this you will find that you that qualitatively the predictions of this model hold good you know when you compare the results to experiment.

Likewise if you go back to the dispersion relation now in the most general quadratic form that we had if I choose to retain U 1 and U 2 right here, I can investigate the individual effects of U 1 and U 2. Like for example, if I increase U 1 what would that do to the drop size, if I increase U 2 what were due to the final drop size obtain those kinds of qualitative analysis are the trends that you that this analysis would predict are very close to what experiments have shown.

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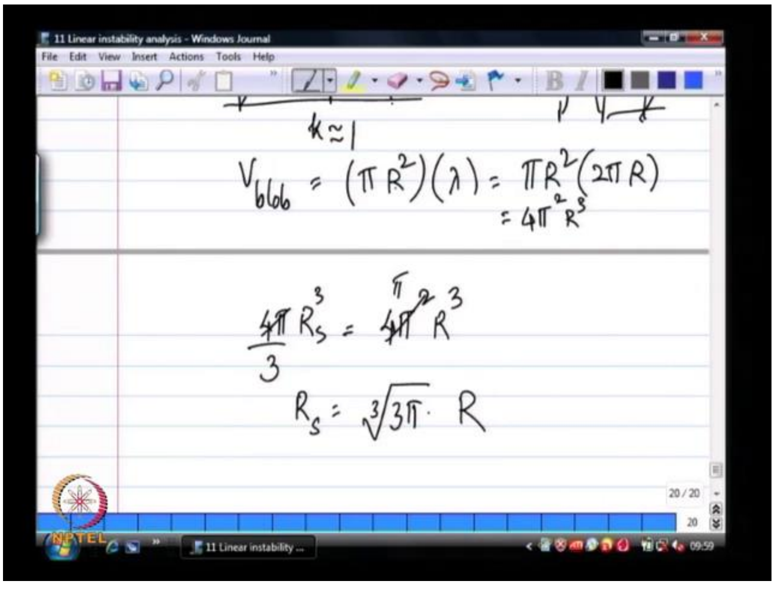

So, this case, where we took a linear stability result and we are able to predict drop size from knowing that the volume of the blob formed is pi R square times lambda, which in this case happens to be pi R square times 2 pi R. Now if I take this so, this gives me 4 pi R squared R cubed. So, if I take an equivalent spherical drop radius. So, if I take this column of liquid and allow it to (Refer Time: 44:11) less into a spherical blob. So, the drop formed from a cylindrical jet of radius R is going to be cube root of 3 pi times the radius of the cylindrical jet itself. So, you can take at the very least you can say that the size of the drop formed are going to linearly scale with the cylindrical jet for the case of rely type or surface tension driven break up process.

We will stop here we will continue this discussion in the next class.