**Spray Theory and Applications**
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**Lecture - 22**
**Linear stability analysis- Cylindrical jet instability- 1**

Good Morning. We will continue our discussion of linear instability analysis, and we will start by quickly recapping what we did in the last time.

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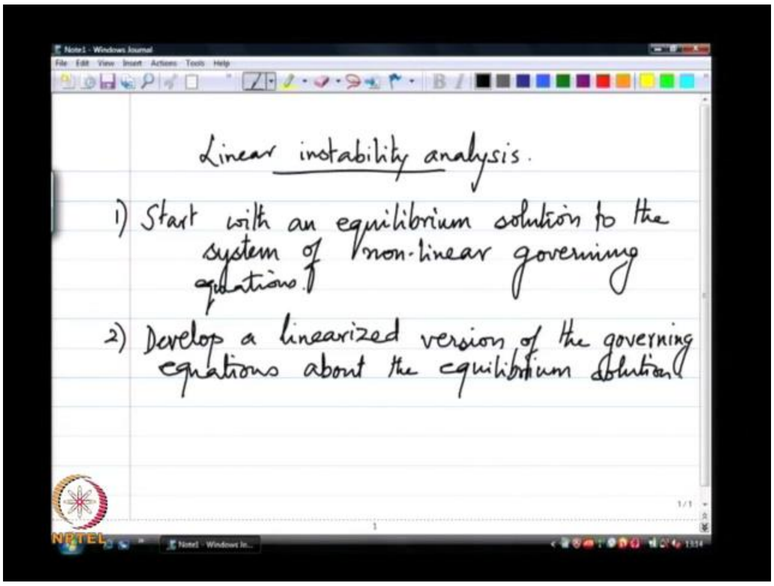


We wrote down a series of steps that we go through; to start with an equilibrium solution, and then, we develop a linearized version of the governing equations about the equilibrium solution.

So what this does is it helps us understand, the behavior of a small perturbation around the equilibrium solution.

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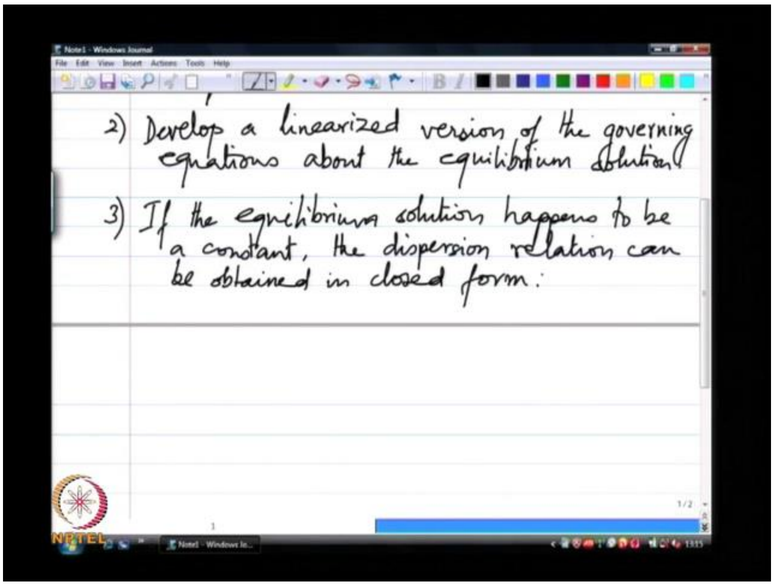


Now, if the equilibrium solution happens to be a constant, like the case that we discussed the dispersion relation can be obtained in closed form. If not one would have to solve for these eigenvalues, essentially the dispersion relation gives us the eigenvalues at any point in time. If the equilibrium solution is not a constant one would have to solve for those eigenvalues numerically. Say for example, you look at if the; I can apply the same exact methodology to look at the stability of the parabolic velocity profile in a round pipe.

When I take u equal to some 1 over r square over capital R square, which is like my parabolic velocity profile. It is not a constant in space essentially the mean the equilibrium solution is not a constant, therefore what we end up with is a forth order ordinary differential equation that and an eigenvalue problem, involving this forth order ordinary differential equation to which governs the growth or decay of the perturbation quantities. The forth order equation is very often called Orr sommerfeld equation.

So, essentially the Orr sommerfeld equation is a process is obtained through exactly the same process, except in that case you would have to solve for the eigenvalue with the largest real part, which is going to determine the growth of a given perturbation numerically. Whereas, in this case we are able to obtain it in closed form meaning: analytically, because the mean flow is simply a constant.

Now, I do not want to show that this is not something that is, that came out of the vim or fancy of a mathematician's mind, this process of obtaining the most unstable wave number.

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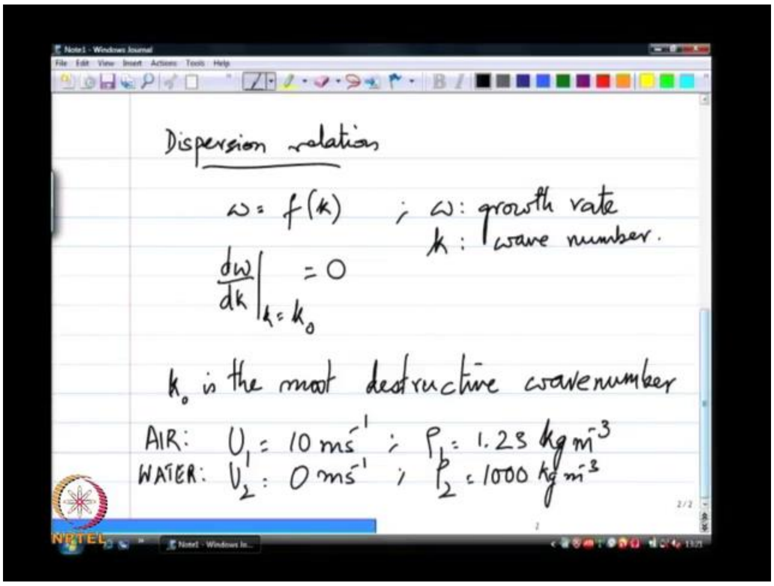


So, essentially if we go back to this dispersion relation. It is some omega is a function of k omega is the growth rate, also in mathematical terms you would call that the eigenvalue, k is the wave number. This growth rate as a function of the wave number is some algebraic equation, in our case it turned out to be a polynomial, but in general it could be any algebraic equation explicit in some instances even implicit. But it still it is a closed form solution, that if I give you a k you can give me an omega, and it could be not 1 omega or 2 omegas, but many omegas, if omega happens to be a transcendental equation. If this equation happens to be a transcendental equation, but the 1 eigenvalue that I am interested in is the one with the largest real part, for that k it is that 1 eigenvalue that is going to govern the growth of that disturbance.

Now, so from this we can find at one particular value, I call this k 0; k 0 is the, what we would often call the most destructive wave number, which means that for a given disturbance of this wave number k 0 the growth rate omega the corresponding growth rate omega 0 is the maximum of all the possible growth rates for all the other wave numbers. Which means, because of the functional form e power omega t this particular wave number is going to cause, is going to appear to grow the fastest in relation to all other wave numbers.

So, and this has been and has been validated experimentally in many situations you know like the example that we just solved, you can show for yourself that when the water, you can go back to the dispersion relation that we solved. So, if you replace what we called fluid number 1 with air, and the fluid number 2 with water.

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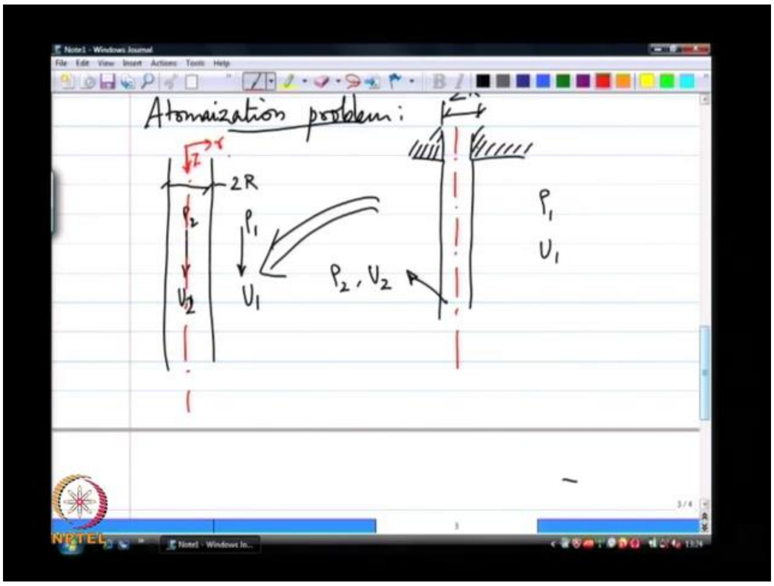


And also said sigma to be equal to the surface tension of water, you will find that the lambda 0 corresponding to k 0 is on the order of about 10 centimeters, 7 to 10 centimeters.

So, that is when air is blowing over a lake, that is otherwise at rest and air is blowing at the velocity of 10 meters per second, the repulse that we observe this theory predicts will be about 7 centimeters in wave length. This kind of a prediction can; obviously, be very easily validated in experiments and it has been validated; now what we want to do is look at how this theory can be applied to sprays and atomization which is what we are after now. So, if I take the first instance of an atomization problem, if I take a cylindrical jet ensuing out of a nozzle of radius r, I am going to assume, so this is now the case of an axisymmetric flow.

I am going to assume rho 1 and U 1 are the density and velocity of the fluid outside, rho 2 and U 2 are the density and velocity of the fluid of the liquid itself. So, imagine just a water faucet, faucet through which a jet of diameter 2 r is exciding into let us say air the air is at some velocity U 1 and the fluid is at some other velocity U 2. Now what we do actually end up considering is not this problem, but the case of an infinite cylindrical jet. So, if I replace this with an idealized problem that is infinite in the zee direction, I can go through the exact same process.

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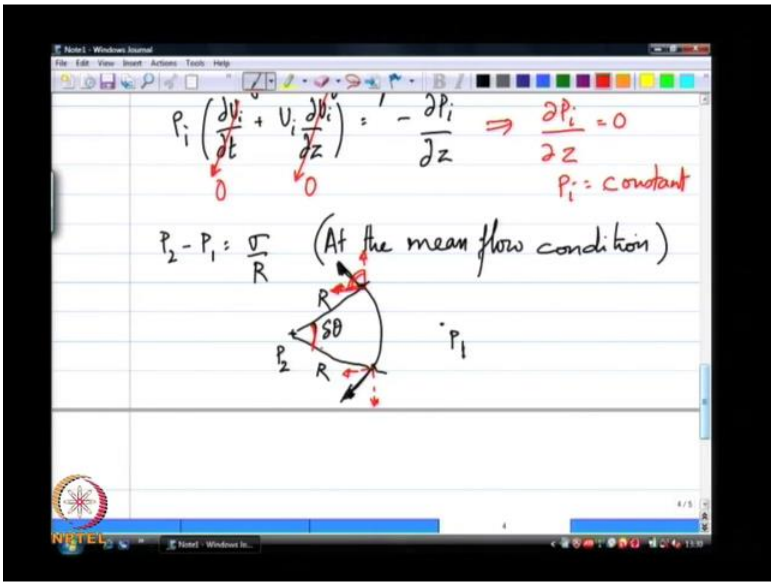


That is step one; write the governing equations, I leave the subscription notation i, like an our old notation, we said capital quantities are the mean flow quantities and in this case U i happens to be a constant, U 1 and U 2 both are constant in their domain of definition.

So, if I now simplify this equation, U d U i dx is 0 because, it is not a function of x and d U dt is also 0 which implies just like we found in the other case, du dx well in this case I should write not x, but z because, what we can show from this. If I write the same equation in the r direction, what we find is that P i is a constant except if you now, if you remember in the flat interface problem we found that because, the interface curvature was 0 P i P 1 is also equal to P 2. So, the pressure in both the fluids put together was a constant, we will see here that because you have this interface having a curvature.

So, if I this happens to be a round jet in cross section and the radius of that round jet is this capital R. So, what we find is that P 2 minus P 1 is equal to the sigma over capital R. So, at the mean flow condition the capital P 2, which is the pressure inside the liquid is constant inside the liquid P 1 is the pressure in the fluid outside that is in itself constant, but P 2 minus P 1 has to be equal to this sigma over R, because of the surface tension pressure that occurs at the interface. Let us just make sure we understand this understand the physics of how this comes about, if I take a piece of that some delta theta in diameter in angle, if I take a piece of the jet and if P 1 happens to be the pressure P 2 happens to be the pressure here, and P 1 happens to be the pressure outside there is a surface tension force that acts at the cut section.

So, if I cut the fluid at some point here and here, there is a surface tension force that acts in either direction and this surface tension force as a component in this direction, and a component in this direction. What you will find is that this P 2 minus P 1 has to be sufficient to balance the normal component of these forces, the horizontal the vertical component as shown in this figure will cancel out, and from that you get this curvature dependence. So, if I take this angle the sin and the cosine essentially, if I say that the normal component of this force which is this force is a function of this delta theta. And therefore, you start to get these curvature effects, so if I take the perturbation quantities.

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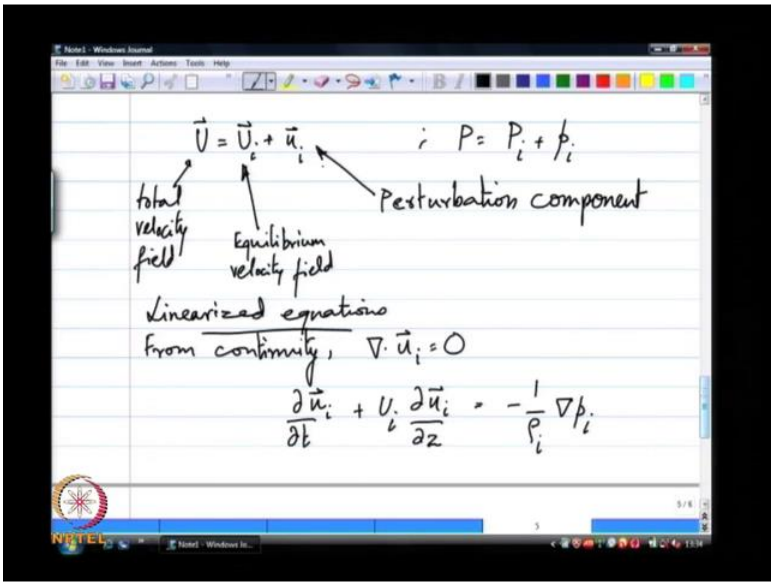


If I write u as the u bar i plus U i vector, U is the total velocity field, U i is the mean velocity field, mean or I would like to call it the equilibrium velocity field and this is my perturbation component. If I go through the same process that I went through before, and linearized these equations, I will keep these in the vector notation just too sort of make things a little easy for us, from continuity equation. What we find is that del dot U i is equal to 0 this says that the perturbation velocity field is also divergence free meaning the perturbation velocity field has to instantaneously obey this compressibility condition.

Essentially, del dot u being 0 is coming from the fact that the fluid is in compressible and. So, if are these are now the linearized equations that I am writing. I do not want to go through the same process of showing you how I substitute the mean plus perturbation into the full governing equations, take out all the order epsilon squared terms keep only the order epsilon terms, and that is when you get these linearized equations.

So, I am skipping those two steps, in the interest of time to just show you the equations that you get from this linearization, first equation is del dot u equal to 0 and the second equation. So, if I take the total pressure to be equal to this P i plus little p i just like we wrote in the previous case.

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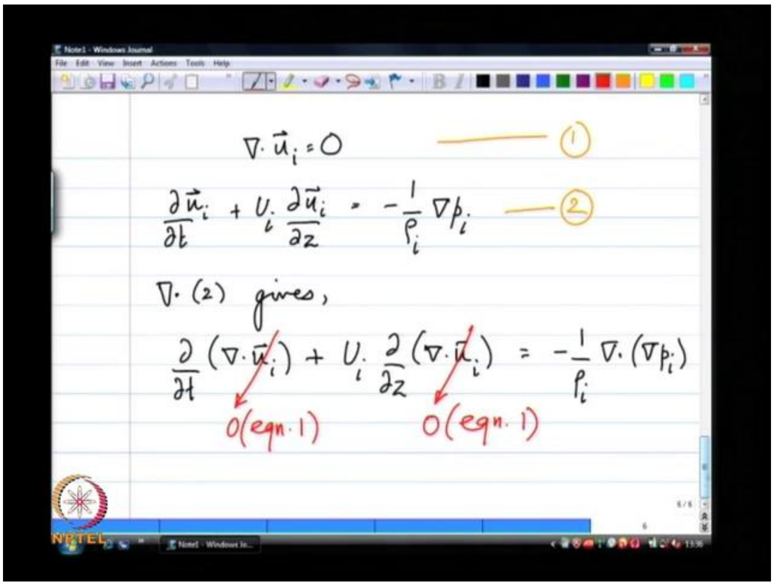


So, if I now take, I will mark these now as equations 1 and 2. If I take divergence of equation 2, and if I use equation 1 what we end up getting, is this equation del dot grad of p i the divergence of gradient of the pressure in each of the fluids is 0, which can also be written as del squared p i equal to 0.

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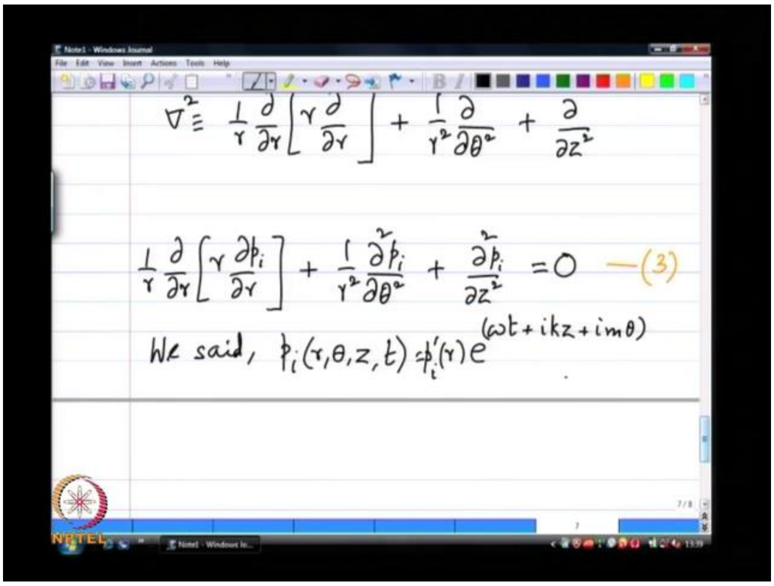

Now, del squared is a linear operator. So, I will just sort of write this out in x in open form, this is our equation that we get from just combining the system of linear equations that we have. Now I want to point out one thing here, the that will reinforce some of the concepts that we discussed in the earlier linear instability analysis, at this stage we introduce what we call the normal mode expansion, correct we said we will write p i as e power omega t plus i k z plus i m theta.

In previous case we did not have this i m theta, in this case I am, I have two spatial variables over which the perturbation can vary, in the z direction as well as theta direction, we will see what this means in just a moment, but essentially I am assuming that my perturbation has two spatial variable forms, two spatial variables involved and one time variable. If I do that, I have this and of course, I have the pre multiplier this pre p prime i of r is what we said would be the eigen function in the r direction. Now this is what we call the normal mode expansion, what I want to show you is that really speaking you do not even need to think of that as an assumption, the normal mode expansion is not an assumption that we make I show you how, if I look at equation 3.

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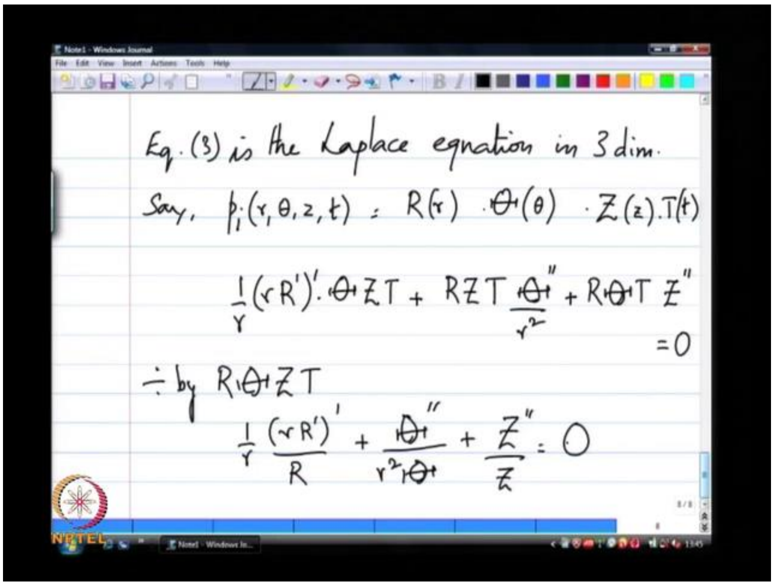


So, we want to answer where does this come from, we in the previous analysis I showed this to you as though it was an assumption, but I want to show you that it is really not an assumption, that there is fairly simple procedure to obtain it. So, if I look at equation 3 is the Laplace equation in three dimensions. So, this is in if I replace this p i with capital T for temperature, that is standard heat conduction equation in three in cylindrical polar coordinates, steady state heat conduction equation in cylindrical polar coordinates, how would you solve that equation? You have to first identify which are the homogeneous directions and which is the direction in which you have may have some sort of an in homogeneity.

In this particular instance z is a coordinate in the vertical direction. So, the z coordinate basically is infinite, the theta coordinate is a periodic coordinate. So, clearly those are the two directions in which the solu the coordinate as homogeneous boundary condition, homogeneous or periodic, really I should be say in periodic boundary conditions right. So, if I simply do a separation of variable solution on equation 3. So, if I say, p i of r theta z and time is equal to r of r comma t times theta of I am going to write this as capital theta of theta comma t times capital Z of z comma t. In fact, we do not even need the t in every one of this; I will show you in a moment that even that is not required, time sum capital T of t, if I take this assumption, which is essentially coming from separation of variables.

And I introduced that into the governing equation, what I will end up seeing here is. So, if I divide by capital R, I will write this first term also in this notation of in this prime notation indicating differentiation with respect to it is own argument. Now if I the standard procedure of solving you know three dimensional heat conduction equation, if I divide by R theta Z and T, what you end up seeing is this. Now if I non dimensionalize the r variable, using capital R then essentially what I have is that each of these terms is only of this term is only a function of Z this term is only a function of theta.

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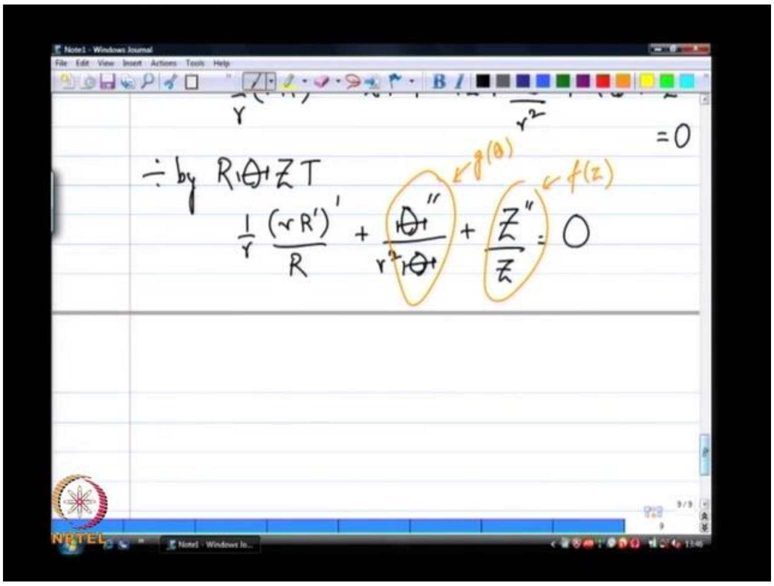


And if the sum of three functions, that are each of a particular independent variable have to all add up to 0 the only way they can add up to 0 is if we each of them as a constant.

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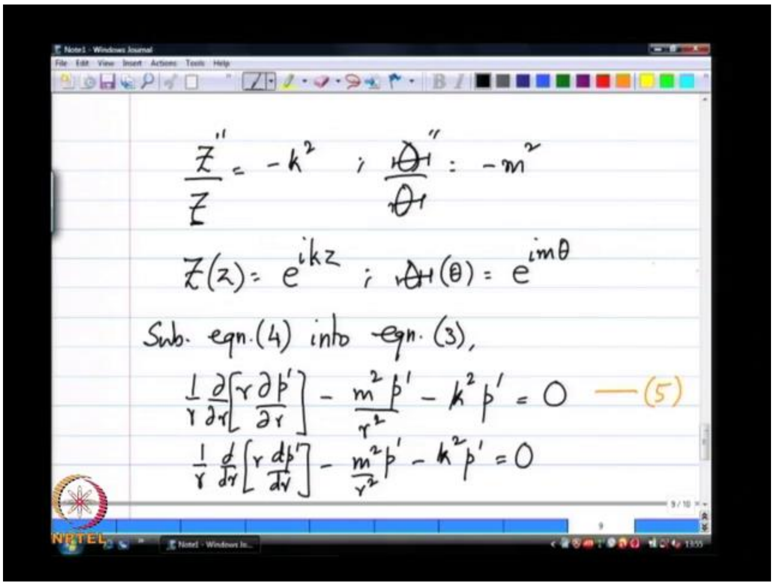


The standard solution by separation of variables, so from there you get the equation that Z del equals minus k squared, if I say theta double prime over theta equals. So, minus m square and I get do this by separating out the r part. So, if this part is equal to some minus k squared, then the solution for this. So, essentially if I start out so the assumption of periodicity in the z direction and theta direction comes from taking the system of equations that we have which is an equation 1 and 2 eliminating some of the unknowns in favor of the others.

So, like for example, in this case I eliminated U i vector, which is u v w velocities in favor of p. So, I end up getting one homogeneous equation in terms of the pressure and after I have that, which have written out in form as equation 3 identifying which are the periodic directions. Once I identify which are the periodic directions, I can then solve this as though I am solving this problem by separation of variables, and this is and what I end up with is what we call the normal mode assumption in the past in the last class. So, essentially in a mathematically this is a sturm liouville problem that yields orthogonal eigen functions in k in the z direction, and orthogonal eigen functions in the theta direction and essentially this normal mode assumption is just coming from the fact that after eliminating these equations you get a Sturm-Liouville problem.

So, now let us just, so it comes from the fact that we are dealing with a strum lionville problem. This is the answer in the previous case it was a strum lionville problem in two dimensions x and y, if we did this in the previous case we would have simply gotten del square p i equal to 0, and there del square p i would have simply been the partial derivative of p i with respect to x twice, partial secondary derivative plus del square p del y squared equal to 0. So, you just have been del square p del x square plus del squared p del y square equal to 0, you identify that x is the periodic direction in that problem y happens to be the depth direction.

So, it is not the periodic direction and so, you get sin function e power k x in the x direction, and corresponding to each e power i k x you get either e power minus k y or plus k y depending on which ever the two fluids, you are in the standard solution to Laplace equation in two dimensions. So now, let us take this forward if I take this normal mode assumption of the form four and substitute into three, what do I get? Now p prime is only a function of r full building, this p prime we said is a function of r and. So, essentially this equation will call this equation five really should be written as an ordinary differential equation. So, just to be precise is an ordinary differential equation in p prime.

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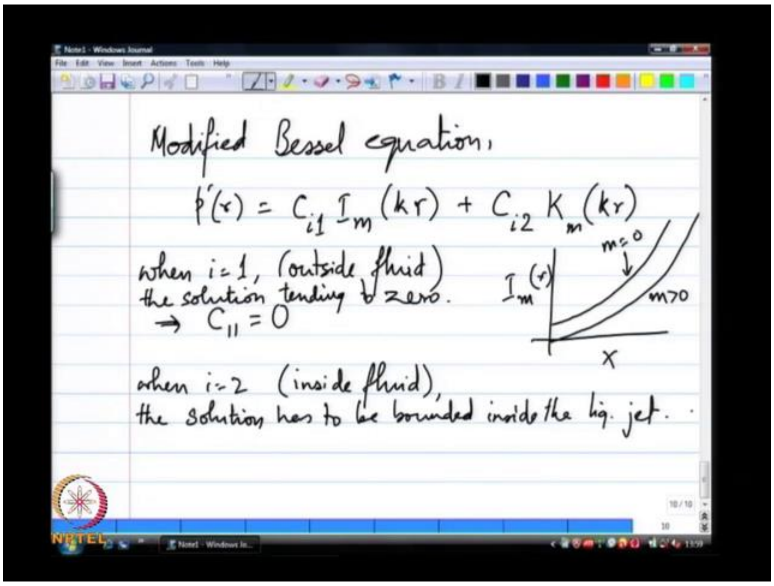


As it turns out this is called the Modified Bessel Equation, and the solutions of this are the general solution.

This C i1 and C i2 are actually four constants we have seen 11 for i equal to 1 and i equal to 2, C 11, 12, 21, and 22. Now when i equal to 1 what we do want to know is that the solution is bounded inside i equal to 1. So, just to be clear i equal to 1 is the outside fluid, the graph of i m k r for i m x as a function of x, looks something like this for different values of m this happens to be m equal to 0 and this is qualitatively how any m greater than 0 looks. So, as r becomes large k r becomes large and i m of k r increases indefinitely as r becomes large. So, all we know is that C 11 has to be equal to 0, it is coming from the solution not just being bounded, but the solution leading to disappear towards 0.

Likewise when i equal to 2, this happens to be the case with the inside fluid now we use the fact that the solution has to be bounded inside the liquid jet inside the fluid.

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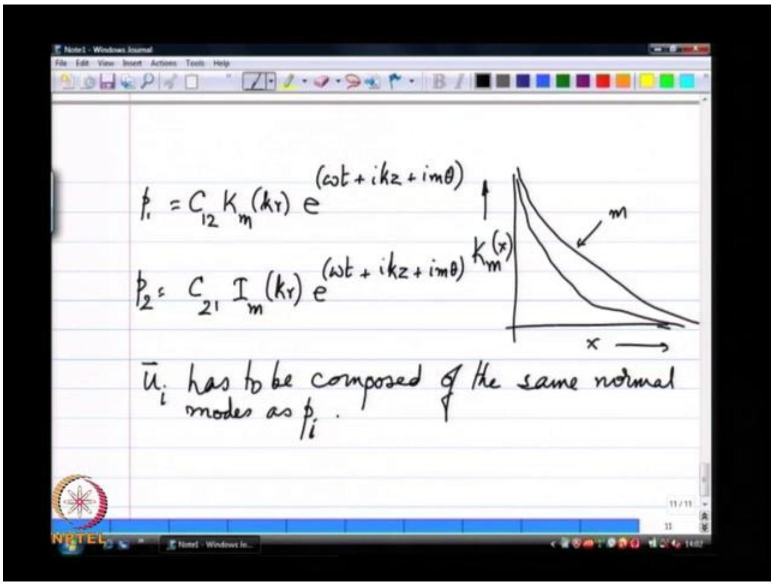


So, just to be clear if I draw graph of k m of x as a function of x, this is a functional form. So, at x as x becomes smaller and smaller the value of k m of x becomes larger and larger this is called the k m is calling the modified Bessel function of the second kind, and i m is the modified Bessel function of first kind.

So, i m as this kind of a property k m as this kind of a property, so when i equal to 2 if k m k r happens to as r become small k m k r becomes unbounded, which means that C 22 has to be equal to 0. So, from here I can write the full pressure in the outside fluid now once I find p 1 and p 2, what I also know is that U i has to be composed of the same normal modes as p i.

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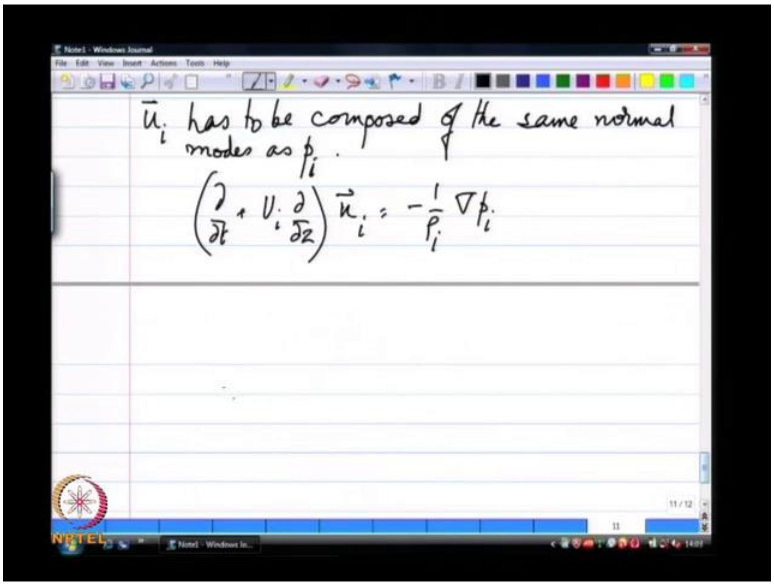


So, if I you can this is again this is also not an assumption, I can go back to the original equation number 2 which said d dt.

So, if I take the solution for p i that I have substitute in this equation essentially take the gradient of p i and said term wise, how these have the parts of U i corresponding to time and spatial variation have to equal, you end up getting exactly this. So, this is a characteristic of any linear problem that the response of the linear system is always the same as the response in the forcing function. So, what we want to do is use that fact. So, U i is also is also of the same functional form in r theta z and time.

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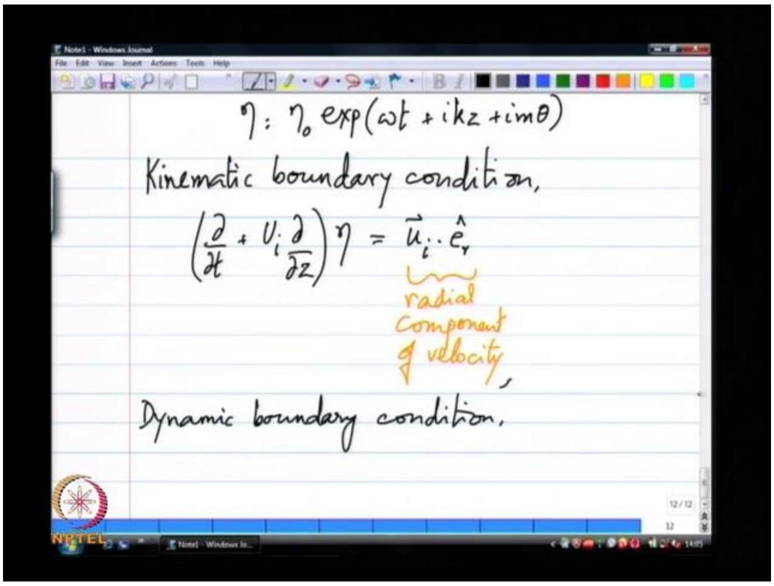


Now so essentially that gives us an analytical solution for all of U i and p i and then finally, like we said we have two sets of boundary conditions, one is called the kinematic boundary condition, that says d dt plus U i d dz acting on the interface is equal to U i dot e r this is my way of saying that this is the radial component of velocity that I am concerned about on this side, also eta is some perturbation just like of the same exact form has the one we have before.

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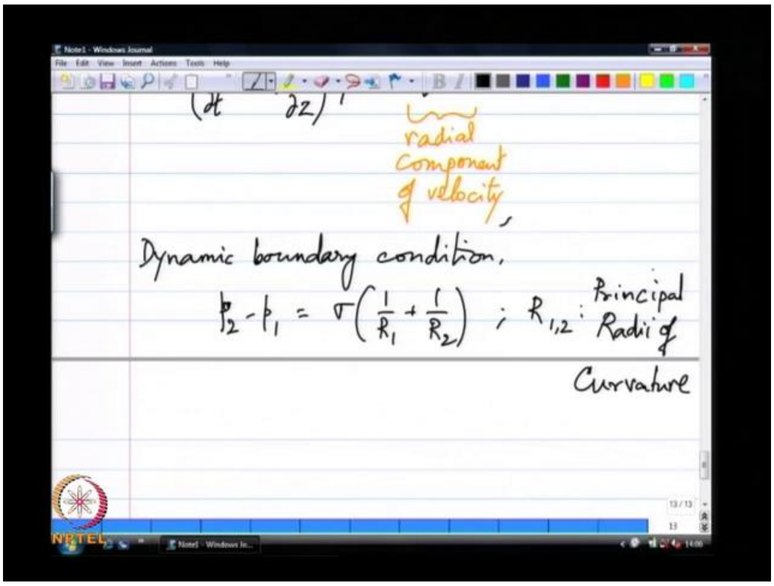

And then we have the dynamic boundary condition, p 2 minus p 1 now these are the differences in the perturbation pressures alone, and these are functions of the principal radii of curvature.

We will continue with this discussion in the next class, where we will start from here and work our way to the dispersion relation for a cylindrical jet.