Spray Theory and Applications
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Lecture - 21
Linear stability analysis procedures

Good morning again, towards the end of the last class we had got so far as to derive the dispersion relation for the simple case of one fluid flowing over another fluid that may also be moving in general.

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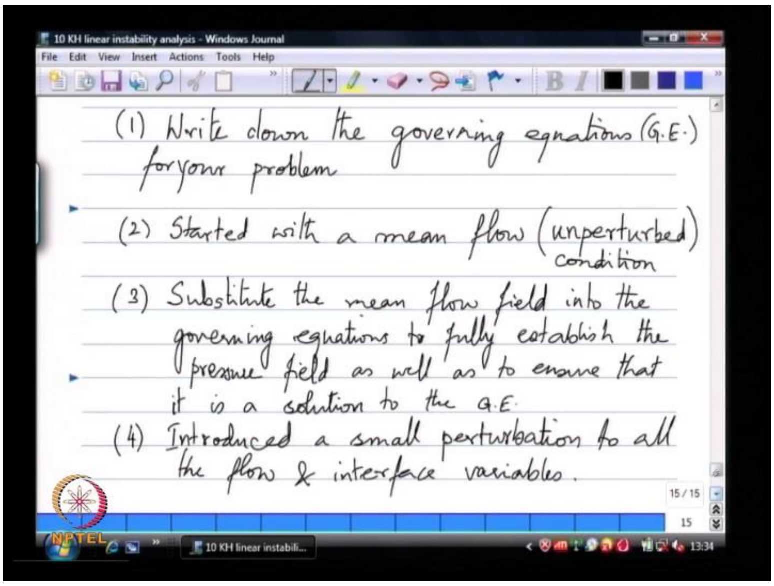

So, let us quickly recap the process of this linear instability analysis right from the beginning. The process started with first, write down the governing equation; and then we started with a mean flow we also call this in several instances the unperturbed condition or more accurately the pre perturbed condition.

Then we substituted the mean flow field into the governing equations to fully establish the pressure field as well as to ensure that, it is a solution to the governing equations. So, essentially this is a check where we make sure that, what we had a for our mean flow field does indeed satisfy the governing equations. And then, we developed or introduced a small perturbation to all the flow variables flow and interface variables will write it in a general way, and then we substituted the perturbed flow field into the governing equations.

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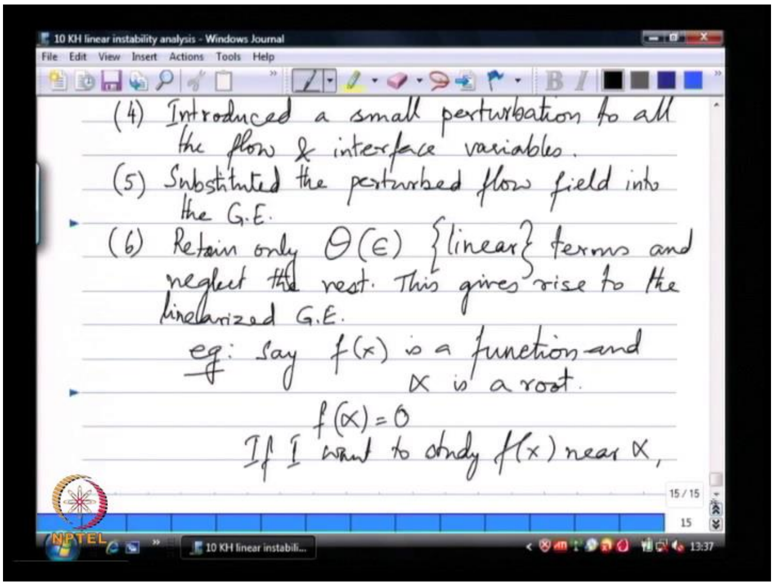

The next process is, retain only order epsilon, which is basically linear terms and neglect the rest. This gives rise to the linearized governing equations, what at this stage what we have is a set of equations that the perturbed quantities obey up to order epsilon. The perturbed quantities behave as per those equations because you have the mean quantities are already satisfied in the governing equation. So, I will give you an example here say I take a function f of x and alpha is a root, if alpha is a root of the function; that means, clearly f of alpha equal to 0. If I want to study the behavior of this function near alpha, what do I do?

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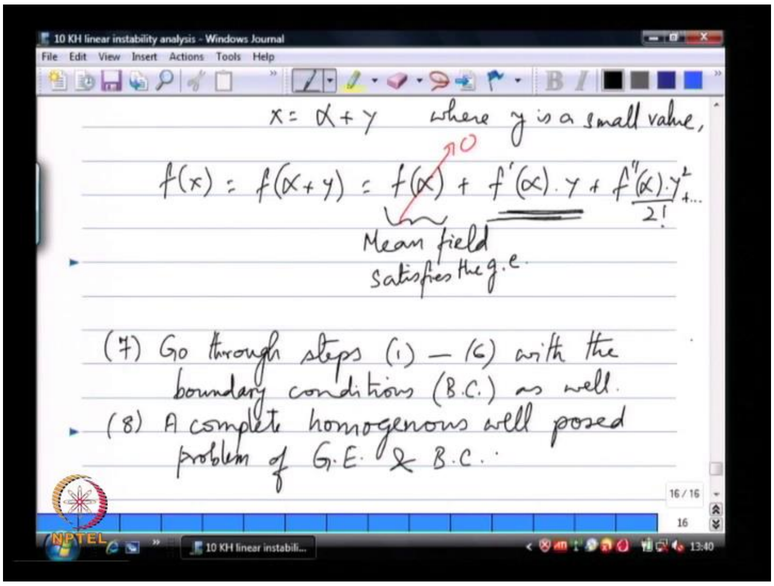

I define x to be equal to alpha plus y where y is a small parameter is a small value, now I can take f of x which is equal to f of alpha plus y to be equal to f of alpha plus f prime at alpha times y plus f double prime at alpha times y squared over 2 factorial etcetera. So, since f of alpha is 0 because alpha is a root. So, this is basically where we are saying the mean field satisfies the governing equations. So, essentially what we have is f of x can be replaced by f prime alpha times y which is a linearized form of y which is a linearized which is where the function is been linearized in the neighbourhood of alpha this is exactly what we have done.

Now, the growth or decay of this function or the slope of this function let us say near alpha is data is basically a prime alpha we know that from simple mathematics. So, this is f prime alpha determining the behaviour near alpha equal to 0 as far as slope is concerned whether the slope is positive or negative etcetera. This is for a simple mathematical for an algebraic or transcendental function for a differential equation the process is exactly similar you have the linearized governing equations that determine the growth and what we want to find is the set of eigenvalues of those linearized differential equations which is basically what omega is. So, this is step number 6, step number 7 once we have the linearized equation we go through, we went through as a matter of fact with the boundary condition as well. So, what we have now a complete homogeneous.

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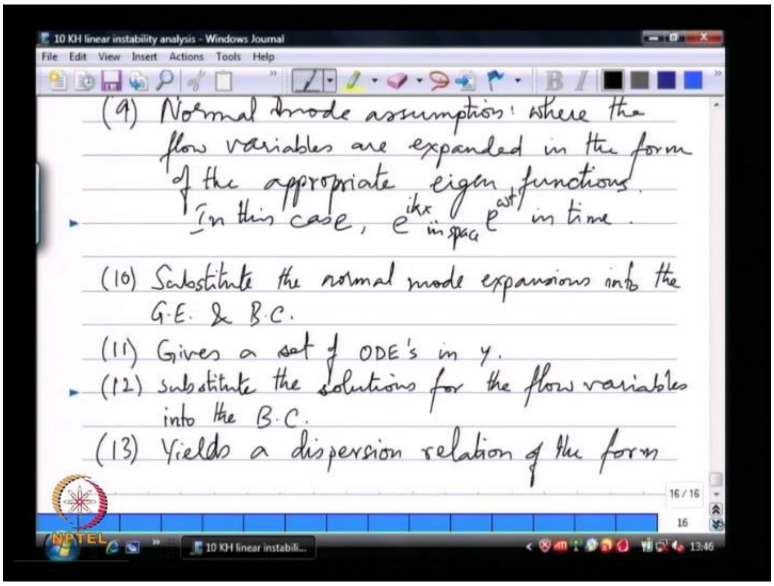

The next process we went through was to make the normal mode assumption, where the flow variables are expanded in the form of the appropriate eigen functions. In this case there are sines and cosines e power ikx, that was the spatial from of the eigen function and then exponential in time. So, the linearized governing equations are still partial differential equations, but in the in x y in time by the time you complete step 9 you can substitute these into this gives a set of ordinary differential equations in y. So, that gives us the behavior of these quantities in y.

Now, if you take any arbitrary wave that you impose on the surface since we are dealing with a linear problem any arbitrary wave can be decomposed into sines and cosines this is basic you know fourier series soar to say. And since we are talking about again a linearized version of the full problem, the behavior of each sin and cosine can be superposed to yield the behavior of any arbitrary wave this is simple. So, if I have a linear governing equation for any problem and I know the behavior due to let us say one forcing function I have the, I know the behavior due to a second force in function the behaviour due to both the forcing functions acting together is simply the summation of the solution; due to each of the two forcing function acting individually without the other this is. So, essentially what we are saying is if I have some arbitrary wave on the free surface I can treat that as being a super position of several sinusoidal components, and if through this process I study how each individuals sinusoidal components is going to grow, I can then end up predicting what the complete what the arbitrary wave that I have started is going to look like in some period of time.

Let us complete this. If I say I have a set of ordinary differential equations in y. So, I know the complete solution. So, substitute the solutions for the flow variables into the boundary conditions and what that does is at yields a dispersion relation omega equal to some function of k.

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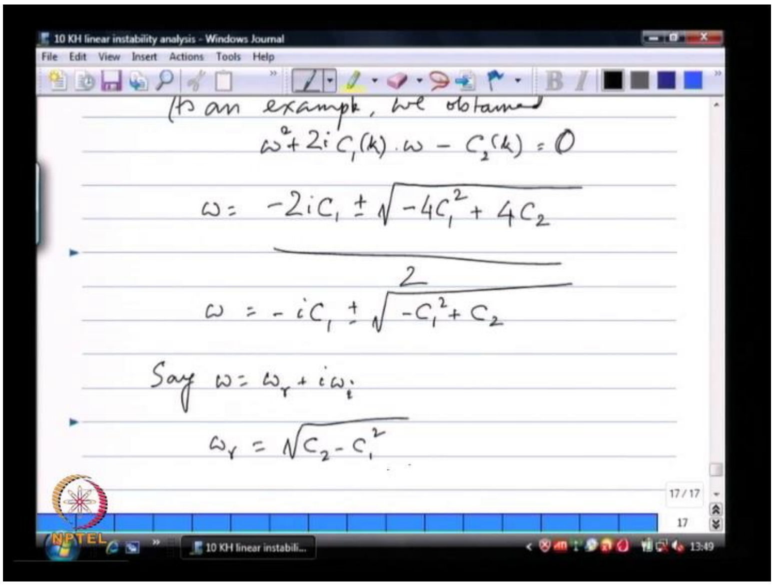

Again as an example what we had was some C 1 of k omega square actually we did not have C 1 of k, we had omega squared plus C 1 of which is a function of k times omega plus some C 2 of k equal to 0 this is the kind of function that we got which says that omega equal to. In fact, as it turns out for our specific case.

This was of this form 2i times C 1 of k that I just want to use that because what that does, is this minus 2 i C 1 plus or minus under the radical minus 4C 1 squared minus 4C 2 divided by 2. So, this gives me minus i C 1 plus or minus C 1 squared minus C 2. So, first of all remember this C 1 and C 2 are functions of k and this omega has two parts the imaginary and real part; for the real part to be non-zero, what is under the radical should be positive. So, this real the real part is non-zero only when minus C 1 squared minus C 2 is a positive number.

In fact, let us not be confuse here I think now this was minus there was a negative sign there and there rest of it was a positive functions. So, really speaking you could write it this way just to see a case where it could be positive were used to sort of dealing with positive number so, will leave it like this.

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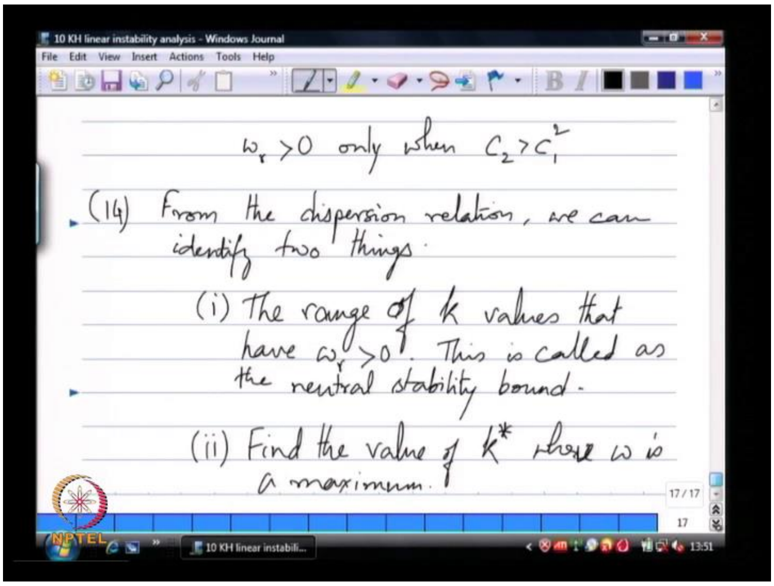

So, omega r is square root of C 2 minus C 1 squared and omega r is greater than 0, only when C 2 is greater than C 1 squared we know what those functions are from earlier. So, essentially we have a dispersion relation from the dispersion relation we can identify two things; one is the range of k values that have omega r greater than 0. So, all the k values in this case for example, all the k values that have C 2 greater than C 1 squared C 2 of k being greater than C 1 squared C 1 of k the squared will have will be part of this range of this values this is called as the Neutral Stability Bound. So, you are establishing the bounds of k values where the growth rate is exactly 0 or you are establishing the range of k values that could that have positive omega r, second thing you could do is find the value of k, I will call this k star where omega is a maximum.

So, if I go back to the same equation, actually where omega r is a maximum again for the example if omega r is square root of C 2 minus C 1 squared.

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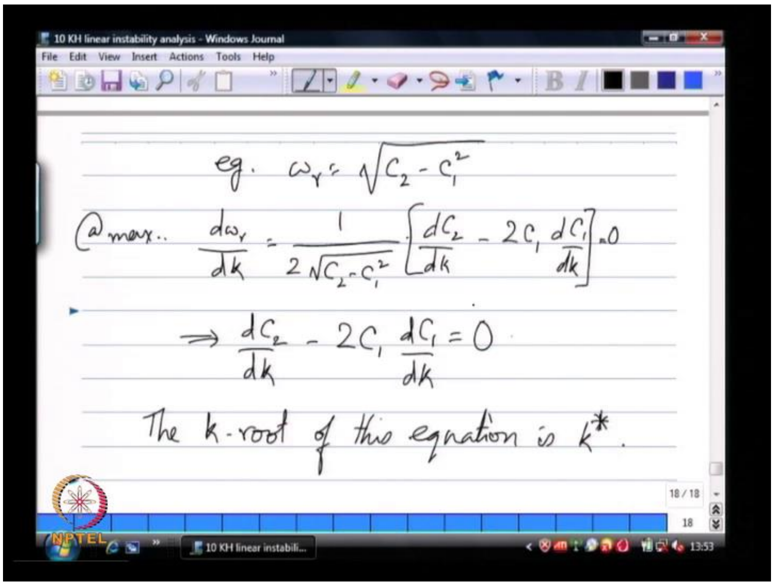

If I take d omega r dk that is 1 over twice square roots C 2 minus C 1 squared times dC 2 dk minus 2C 1 dC 1 d k this is equal to 0 at the maximum point. So, this implies dC 2 dk minus 2C 1, dC 1 dk equal to 0. The k root of this equation is k star. So, remember essentially this is the derivative of whatever is under the radical with respect to k and if you set that equal to 0 that gives you a particular value of k for which the growth rate would be a maximum when the derivative vanishes is when the growth rate has reached up a maximum value in the k space.

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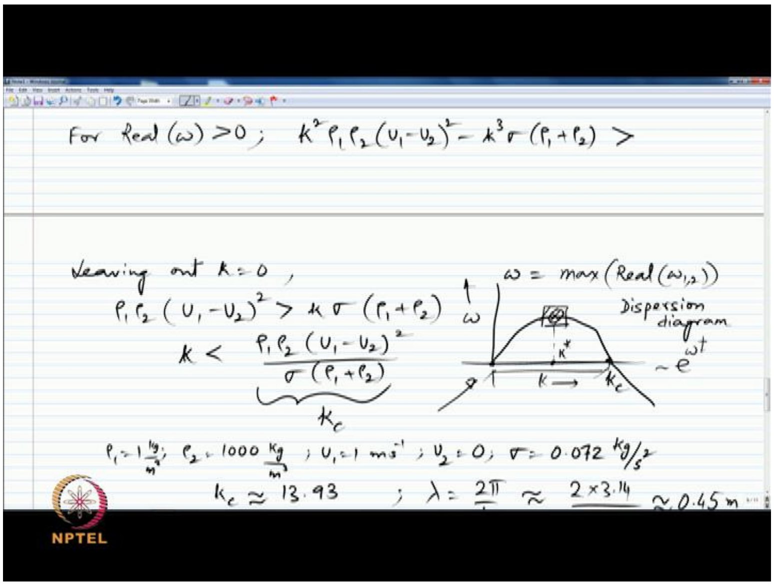

So, let us go back to this earlier form just to see what it looks like, it is now a two-fluid interface is the building block of all atomization system. So, all shear induce atomizers rely on high speed air for example, interfacing with a generally low speed liquid stream causing atomization. So, the interface between two-fluids understanding the physics associated with the corrugation of the interface between two-fluids is essential to studying atomization. So, that is the purpose of what we have been doing we went through the whole linear instability analysis calculation over the past couple of lectures and we now arrived at dispersion relation which in this particular instance is can be written out explicitly as shown here.

So, omega is minus ik times rho1 U 1 plus rho 2 U 2 divided by rho 1 plus rho 2 plus or minus a term under radical. Now we went through and discussed the issue associated with k cut off, that is the value for a wave number above which all omega is both the omegas have real parts that are negative. That is what we signal with this k cut off. Now if we plot the maximum real omega and will call that our omega, so maximum real part of both omega 1 and omega 2; if we plot omega verses k this plot is often called the Dispersion Diagram. What we already seen is that for, k greater than the k cut off omega will all only be negative.

So, this part of a curve has already been seen and as you see from the close form of the dispersion relation when k equal to 0, omega takes on only one value 0. So, it has to naturally pass through there and we will find that the actual shape of the curve is something like that. So, there is a whole range of k values where the maximum part of real omega 1 comma omega 2 is greater than 0 which means all this waves, all the wave associated with this wave numbers, if introduced to the interface would grow exponentially remember all our growth is of the form e power omega t. So, if all of the waves would grow exponentially in time some waves have higher growth rate than others as was as one can see from this dispersion diagram it is natural to expect that the wave associated with this particular wave number would grow faster than any other wave.

Primarily because it is we are looking at exponential growth of these waves in time, this is like an nice simple physical wave of understanding; why one choose the wave number with the maximum growth rate to determine the actual source of instability. Another way which is associated with croup velocity is as follows; that we let us come to that in a moment.

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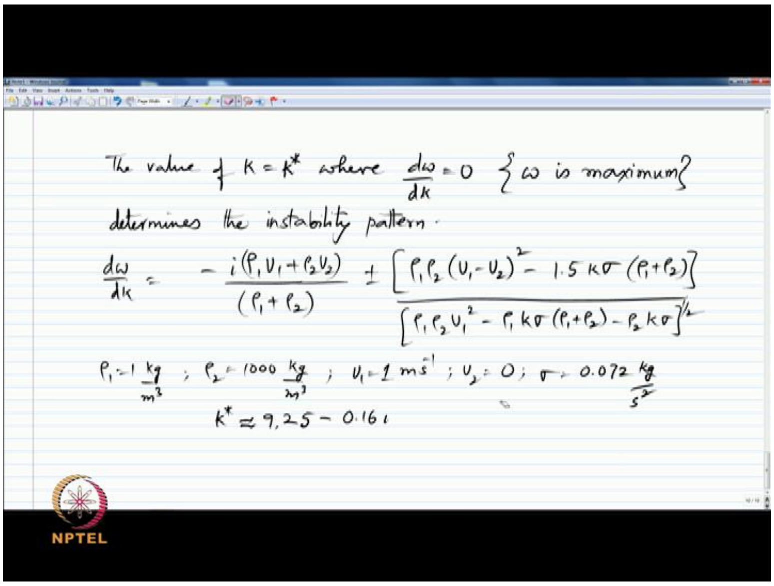

So, essentially what we have found now is that the value of k, the reason is the wave disturbance associated with the wave number k star outruns all other disturbances because we have looking at exponential growth rate. So, if I take if we take the dispersion relation that we have given here and differentiated with respect to k. So, again it is always useful to substitute some simple numbers will go back to the same numbers that we had before rho 1 is 1, rho 2 is a 1000.

From here we find k star is approximately 9.25 minus 0.16 i. So, what this tells us is that the wave associated with this particular wave number has the maximum growth rate and if we simply convert just quickly convert this lambda star and I am only going to take real part.

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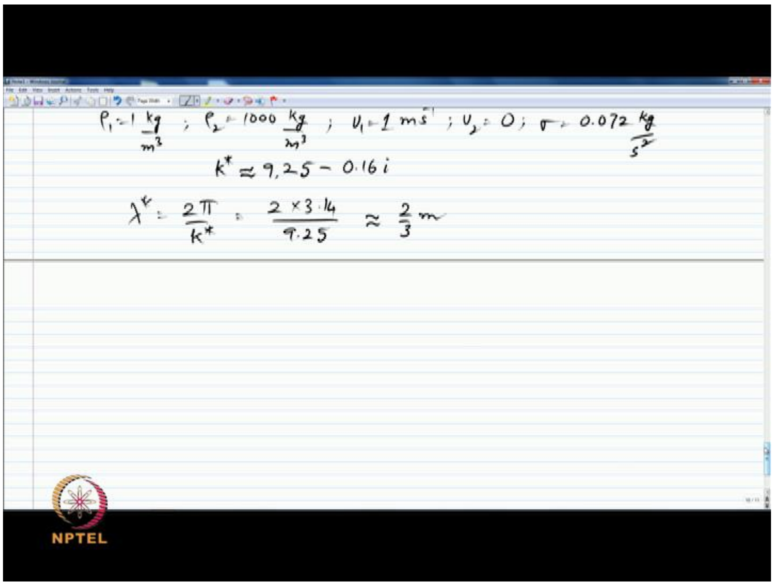

So, it is roughly about 2.3, two third of a meter. This particular wave length is what is expected to have the maximum growth rate of all the wave lengths that are even unstable, which means that if I were to do an experiment the and imagine the experiment had access to all wave lengths of instability the all wave numbers of instability and all of them grow with their respective omega as was shown in the dispersion diagram here.

What this means, is that the wave number with the maximum growth rate is likely to show up in the experiment even in a very short period of time, because all other waves would not grow as fast as the, growth as fast as the wave associated with the maximum growth rate and this is interesting information that again I keep insisting on this repeating this point that or remember all of this purely analytical treatment of the problem. So, from starting with the governing equations and boundary conditions we are able to estimate a wave length that is likely to show up in a experiment where you have a given rho 1, rho 2, U 1, U 2 and sigma. You are able to predict wave length that is likely to manifest from a linear instability analysis.

And this is the power of this analysis technique and I want to I cannot emphasize this enough that one gets to realistic values, and these have been validated in experiments in many different kinds of experiments is a matter of fact that the predictions obtained from linear instability theory matches well with experiments. Now that agreement with experiment must be taken with the small pinch of salt primarily because the non-linearity associated with the growth process has been ignored, we are dealing with a linear instability calculation the real experimental observations agreeing with these theoretical predictions may be somewhat fortuitous.

But it cannot be discarded as purely being for fortuitous because this agreement has been shown in many different instances not restricted to atomization alone. So, the kind of a power that this technique brings to any kind of a studying the instability, studying pattern formation in many different physical systems is quite remarkable as a matter of factor. So, the objective of linear instability analysis is to get this dispersion relation and then use it to find k star and k n as functions of the flow quantity. So, this is essentially the utility of linear instability analysis and I also know the range of the one wave length that will dominate the process. So, given this is like a characterizing the natural response of that system.