Spray Theory and Applications

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Lecture - 20
Linear stability analysis- Kelvin-Helmhotz instability – 3

Hello, welcome back we will continue our discussion of the linear instability analysis of a two-fluid interface.

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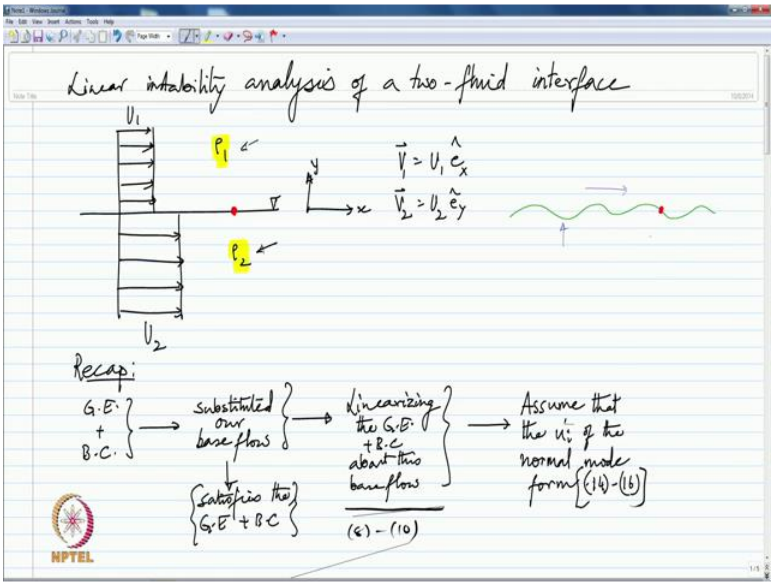

Let us quickly recap what we have done thus far, we are looking at the stability of an interface formed between two-fluids, say I have a fluid of density rho 1, superimposed over another fluid of density rho 2. The fluid of density rho 1 is moving with the velocity U 1 the fluid of density rho 2 is moving with the velocity U 2 - rho 1, rho 2, U 1, U 2, are all constants. We started with our basic governing equations and boundary conditions and we quickly substituted this base flow, given by rho 1, rho 2, U 1, U 2 into the governing equations plus the boundary conditions to check that it satisfies the equations as it does.

Then we introduce a small perturbation to this base flow field, which we called we denoted by the primed quantities U i prime, V i prime and p i prime etcetera. And we substituted the perturbed flow fields into the governing equation and boundary conditions and linearize the flow. So, the first step was, to linearize the flow field about this perturbation about this base flow field. So, that gives us set of linear partial differential equations that will describe the growth or decay of the perturbed quantity. The next step that we followed was to assume a particular form for the perturbed quantity called the normal mode form. We wrote these in equation 14 to 16 and we are now at a point where we know, we have linearized governing equations in equations 8 to 10 and the normal mode form in equation 14 to 16.

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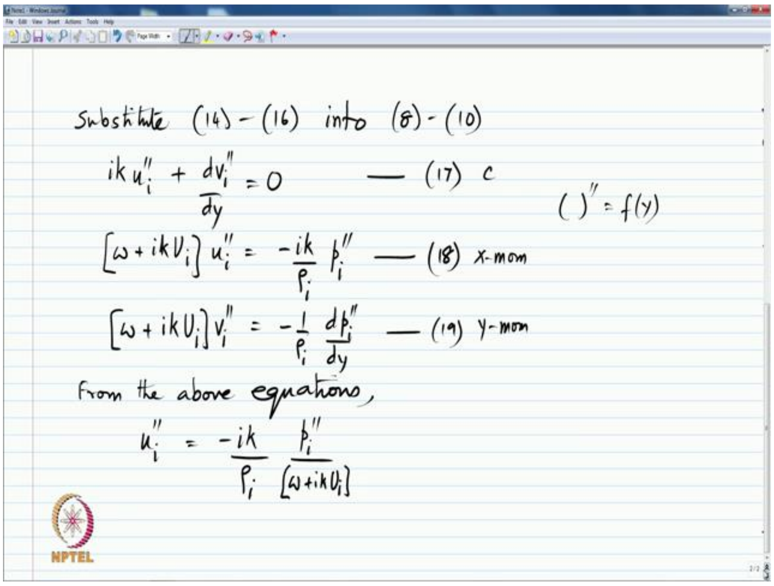

So, we begin by our solution process allowing us to substitute. When we perform the substitution, what we get? These 3 equations - the first one coming from the continuity equation, the second one from the X-momentum equation, and the third one from the Y-momentum equation, just for our reference; the double prime quantities are only functions of y. Just so we recall. And that is reason the derivatives of the double prime quantities with respect to y remained, but with respect to either x or time or simply converted in to an algebraic multiplication with either ik or omega or as the case with the advection term omega plus ik U i.

So, from these equations, we are now able to, we want to get a single ordinary differential equation, if we can for p i double prime by eliminating U I double prime and V i double prime. So, let see what that process looks like.

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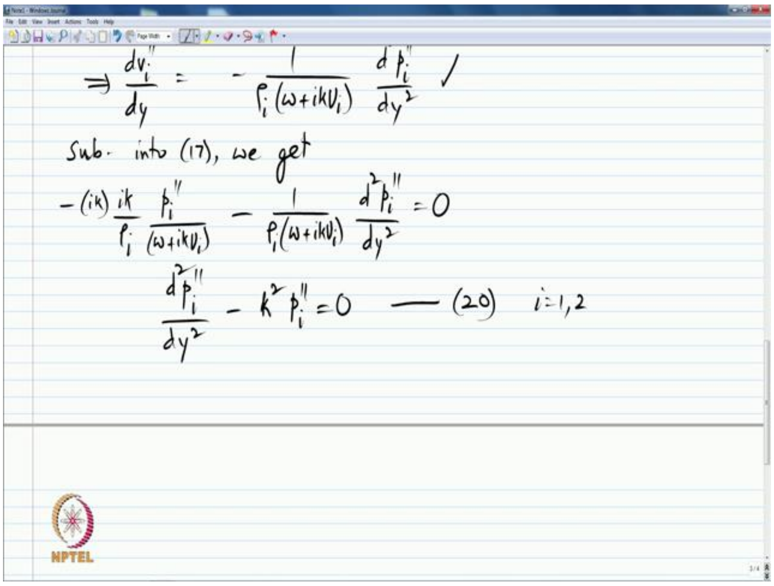

So, we are able to write down U i and V i, U i double prime is minus ik over rho i times p i double prime divided by omega plus ik U i and V i double prime from 19, is given by this. Now, I can use these two equations and substitute into 17. So, in order to do that I need to take the derivative of the equation for V i double prime with respect to y and that is what I have done here. Now when I substitute the equation, substitute this equation and into 17 here which is what we got from continuity. So, in for short, I can write this as - this is our equation 20. Now remember i equals 1 or 2 we have maintained that notation all the way through from the beginning of this derivation. So, essentially p 1 and p 2 are the static pressures in fluids 1 and fluid 2 and they are independent, they are separate.

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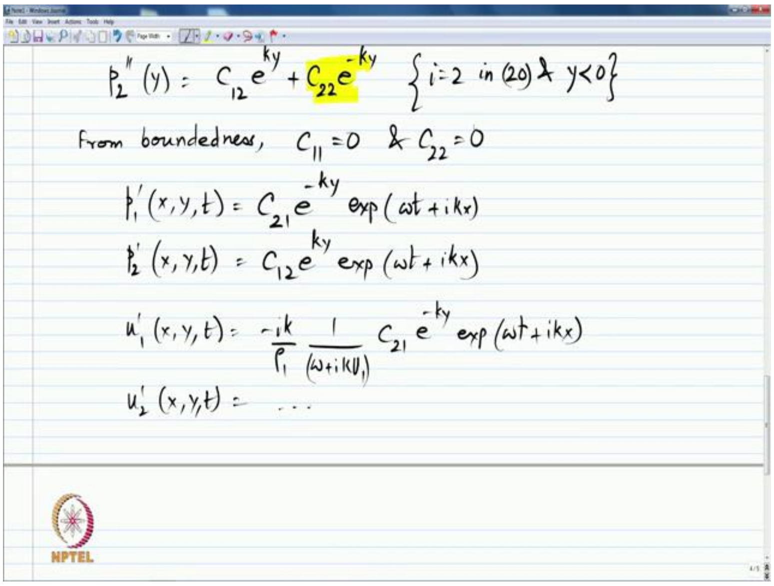

So, I am going to solve for them separately and from there I will find. So, that is our location of the 2 fluids 1 and 2, 1 is located in the region y greater than 0 and 2 is located in the region y less than 0.

So, this is the most general form of the solution of equation 20. Now if you think of fluid 1 for a moment, for all y greater than 0 and remember k is a positive number in our temporal linear instability analysis, k is a positive real number. So, if k is a positive real number then this first term here C 11 e power ky, is going to become increasingly larger as y increases, but the static pressure in fluid 1 especially noting that these are all little p, these are all lower case p meaning they are perturbation quantities. The perturbation quantities have to remain bounded. So, for the perturbation quantities to remain bounded and this to be the solution only possibility is that C 11 has to be 0. C 21 not being 0 is OK, because the e power minus ky function decays a way exponentially for all y greater than 0. So, C 21 does not have any constraint on it, but C 11 has to become 0.

Similarly using similar arguments for p 2 double prime y and noting that fluid 2 is located in the region y less than 0. So, if y is less than 0 k still being a positive number, e power ky is going to decay exponentially as y becomes increasingly negative number. That is y becomes y goes from minus 2 to minus 3, dot, dot, dot, to minus 100 etcetera.

As y becomes an increasingly negative number e power minus ky is already bounded, but that is not the case with e power; e power ky is bounded, but that is not the case with e power minus ky, e power minus ky now becomes unbounded. So, therefore, in order for the whole static pressure to remain bounded C 22 must be equal to 0.

Now, we are in a position to write down the general form of the static pressure in fluid 1 and 2, which is our single prime quantity. From here, now we already have a relationship between U i and p i prime, U i double prime equal to minus ik over rho I times p i double prime divided by the omega plus ik U i. So, using that, I we will be able to write down what U i double prime is supposed to be. And similarly I can write U 2 double prime I am not going to write it down, but you can see that once I know U 1, U 2, I can also use the other equation. I can use this equation to get to what V i prime would have to be.

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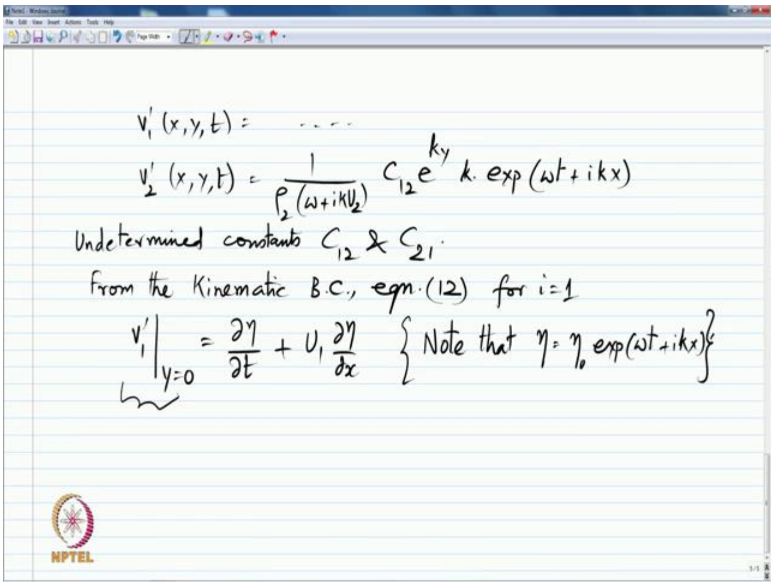

So, now that we have U 1 and V 1, let us also write down and V 1. V 1 double prime would be analogous.

So, now we have the flow fields and we have two undetermined constants C 12 and C 21. Let us quickly take stock of that. And we will use the 2 boundary conditions that we have the kinematic and the dynamic 2 sets of boundary conditions, the kinematic and the dynamic conditions to calculate the value for C 12 and C 21 such that we have a non trivial solution for the problem that is our goal. We do not want C 12 and C 21 to become 0 because if they become 0 then we are left with trivial case that is the base flow alone persists. We want to introduce a perturbation and study whether the perturbation will grow or decay for that purpose we do need C 12 and C 21 to remain non-zero.

So, let us now write down our kinematic boundary condition. Just to recall kinematic boundary condition is basically a requirement that if I have a fluid particle on the interface. So, if I have fluid particles sitting on this interface that, fluid particle remains on that interface at all points in time, so if I have a meniscus kind of like that and if this fluid particle is sitting here, whether the fluid particle is translating, whether the movement of the fluid is to the left or right or up or down, the fluid particle is not detached from the interface. That is pretty much the physical meaning of what we call are kinematic boundary condition.

So, let us write that down what that say is V 1 prime, evaluated at y equal to 0. We have this same normal mode perturbation introduced to eta, eta happens to be the actual physical perturbation that you have introduced to the wave. So, if you have eta to be of having some amplitude eta naught and some temporal frequency omega like we have said before and a wave number k then, the requirement that the y direction velocity in fluid 1. V 1 prime as we approach y equal to 0 from fluid 1 which is basically denoted by this quantity V 1 prime at y equal to 0 is equal to the physical movement of the interface, a physical movement of the interface in an advected sense.

So, dou eta dou t plus U 1 dou eta dou x is basically an advection of that interface with a velocity U 1, U 1 happens to be the fluid velocity the base flow velocity in fluid 1. So, once we figured this out. Now, substituting what we have for V 1 into this, we get k times C 21.

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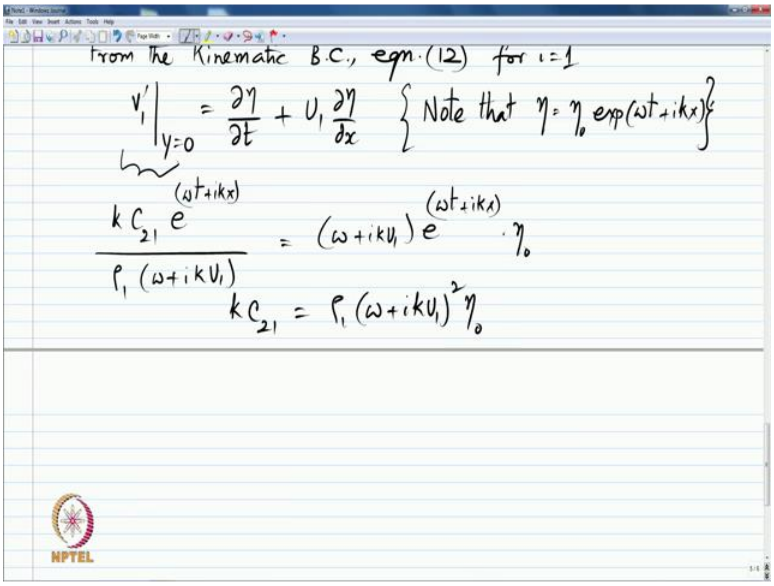

So, simplifying, we see k times C 21 equal to rho 1 times omega plus ik U 1 squared times eta naught.

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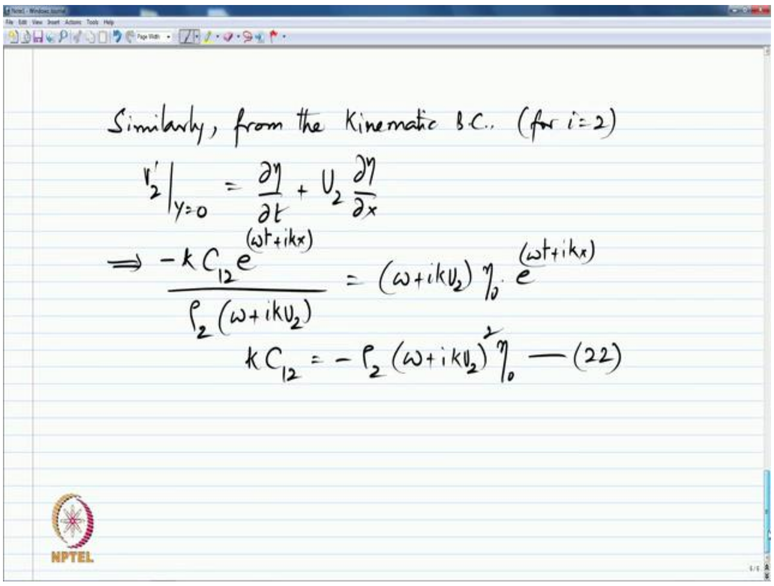

Similarly, though we have another kinematic boundary condition requirement for fluid 2 and from there we have essentially the same requirement, but return now from the sense of fluid 2. Now substituting what we have for V 2 prime in here we get minus k C 12.

Again simplifying k times C 12 equal to minus rho 2. So, we have C 1 C 2 and we have eta naught which is the amplitude, the initial amplitude of the perturbation that was introduced, for non-zero eta naught we want non-zero C 1 and C 2 that is our goal, like we said before.

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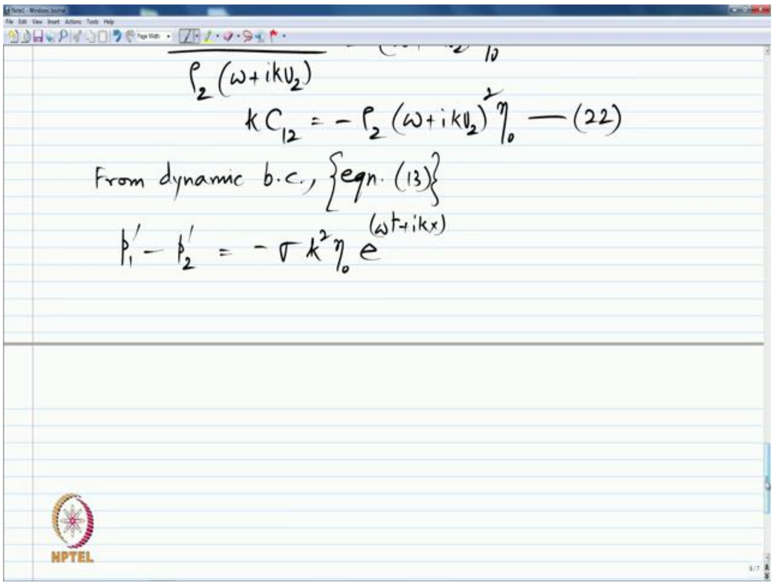

So, that we have one last boundary condition this basically a force balance at the interface we call this our dynamic boundary condition. Let us rewrite it quickly this is p 1 prime minus p 2 prime equals minus sigma. So, if I have p 1 prime and p 2 prime in terms of C 1 C 2 and if I make that substitution, what do I get? C 21 minus C 12, simplifying.

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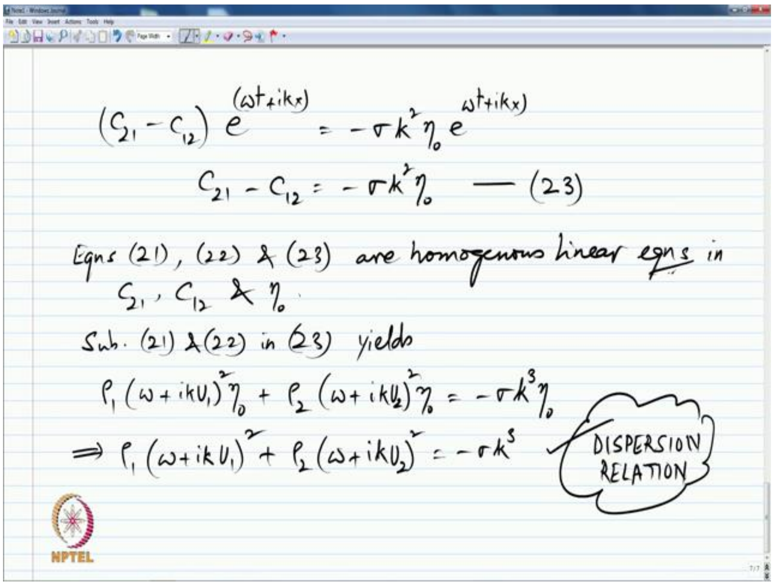

So, we have equations just have a quick recap 21, 22 and 23 are homogeneous equations. And we have looking for; we have 3 equations 21, 22 and 23 in 3 variables C 12, C 21 and eta naught. And we want to find a solution that is non-trivial where neither of these quantities goes to 0. So, there are 2 ways of doing this I can look at these 3 equations and any time I have 3 equations and 3 variables. The only time you have a non-trivial solution is when the determinant of the coefficients goes to 0. That is one way of solving it and that is probably the most general way of solving it. I will restrict myself to a simpler case because I have essentially from 21 and 22, a closed forms solution for C 21 and C 12 respectively.

So, I will just substitute for C 21 and C 12. So, essentially we are able to eliminate C 1 and C 2 by substitution and we find that we have an equation for eta naught and the only time, this equation in eta naught has an non-trivial solution for eta naught is when the rest of the coefficients are equated. So, that gives us this equation here, which we call are Dispersion Relation. This is important. Essentially what we have done is we found a condition that is written kind of nice and elegant here, that rho 1 omega plus ik U 1 squared plus rho 2 omega plus ik U 2 squared equals minus sigma k cubed. For a given k we can use this equation to find omega you notice that I have quadratic equation in omega. So, for each k value of course, given U 1 and U 2 rho 1 and rho 2 you have 2 roots in omega.

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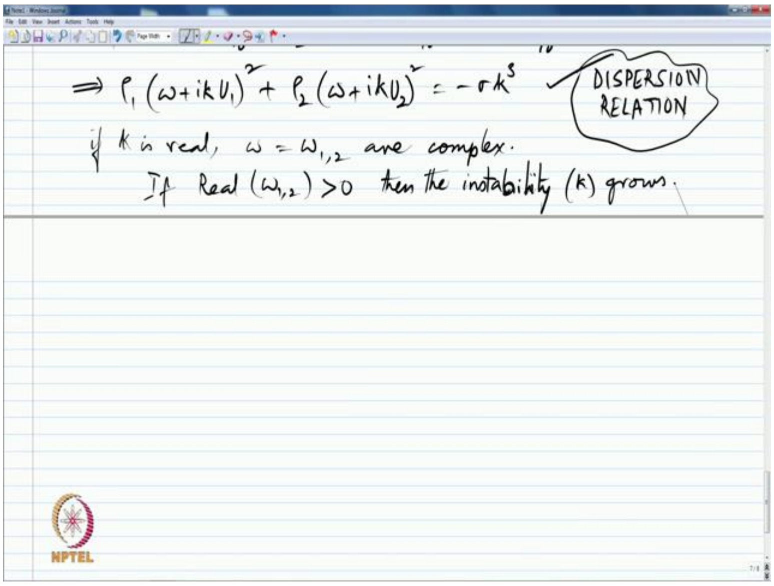

Now, in general if k is real, omega equals omega 1 2 are complex. So, in general I have 2 roots omega 1 and omega 2 which are in general complex. So, if I take, if the real part of omega 1 comma 2 is greater than 0. If the real part of either omega 1 or omega 2 it does not have to be both, if real part of either omega 1 or omega 2 is greater than 0 then the introduced in perturbation of wave number k will grow.

So, if the converse being if the real part of either omega 1 or omega 2 is less than 0 then instability corresponding to that particular wave number k, decays. For the decaying condition, the real part of both omega 1 and omega 2 must be negative, if either one of them is positive then the instability is likely to grow.

So, we will now use this our based dispersion relations, simplify it to see if we can understand this as you see rho 1, rho 2, U 1, U 2 given, we have a quadratic in omega.

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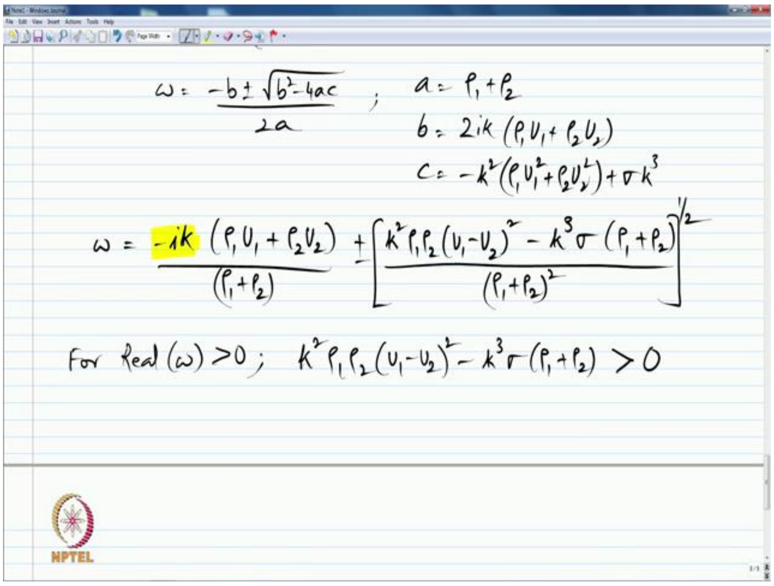

In general, I can write this equation down after expanding out the squares in this form. So, this is simply the dispersion relation written out as a quadratic of the form, a omega squared plus, b omega plus c equal to 0. Now we are able to solve, this equation closed form because we have a formula and in here I know a equals rho 1 plus rho 2, b equals. I am going to not do the algebra here we will just write down the solution you can check this at your convenience.

So, this is a closed form equation that would yield us to omegas - omega 1 comma 2, the 1 corresponding to the positive sign and 2 corresponding to the negative sign and the term under the radical of course, coming from our discriminate b square minus 4ac. Now, you notice that you know we remarked that if the real part of omega is positive that is when you are likely to have an unstable perturbation. So, in this situation, if you notice the first part here, has an imaginary is purely imaginary which means this part can never be real, all the real part can only come from the discriminate under the radical.

So, for omega to be greater than 0 or for real part of omega to be greater than 0, we know k squared times rho 1, rho 2 has to be greater than 0.

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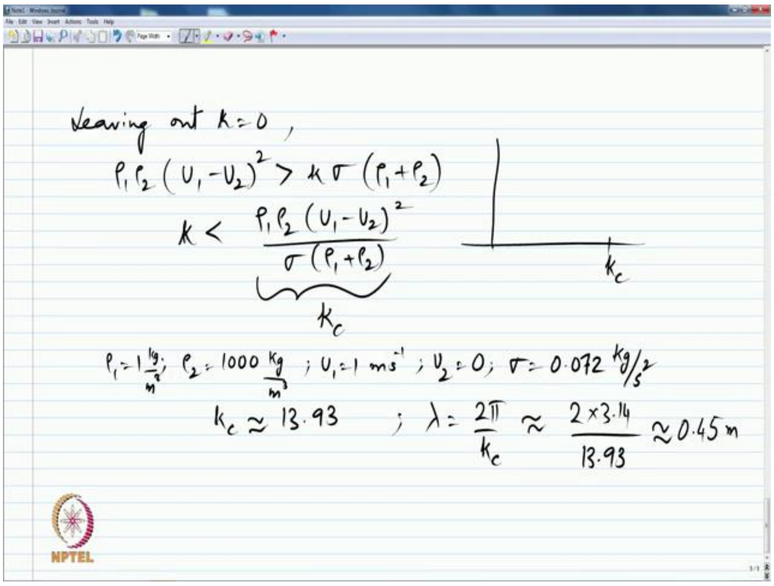

So, leaving out the case, where k is which is trivial we are able to show that rho 1 rho 2 times U 1 minus U 2 the squared has to be greater than k times sigma times rho 1 plus rho 2. I want to draw your attention to 2 observations here, first thing is that if I inter change the subscripts 1 and 2 in this equation. The equation does not change. So, rho 1 becoming rho 2, and rho 2 becoming rho 1 with U 1 becoming U 2 and U 2 becoming U 1 - equation is not changed. That basically means that the condition for instability is not governed by the absolute values of the densities or the velocities and that we have no preferential direction whether 1 is superposed on 2 or 2 is superposed on 1 which should be obvious.

The second point I want to draw your attention to is that it is also symmetric in the velocity. So, whether I simply replace U 1 with U 2 or U 2 with U 1 the only quantity that matters is U 1 minus U 2, the magnitude of the relative velocity U 1 minus U 2 not the absolute values U 1 or U 2. So, based on this we can set either U 1 or U 2 to 0 without loss of generality essentially assuming that the frame of reference is moving with a velocity with one of fluids, either U 1 or U 2 and the condition still remains unaltered.

So, if I rewrite this, this is the condition for instability that all k for a given rho 1, rho 2, U 1, U 2 and sigma, all k less than this would have a positive value of the real part of omega. So, if this quantity is what we will call our k cut off, it is always good to have some numbers. So, let us just pick some set of realistic numbers, all these are in SI units. For this case I can quickly calculate k cut off to be about 13.93 or if I want to convert back to lambda. So, essentially if I have a breeze of 1 meter per second blowing on water I have assumed rho 2 to be 1000, rho 1 to be 1. So, if I have the breeze of 1 meter per second blowing on water at about water which is at rest, U 2 is 0, I am expecting waves whose wavelength is roughly about half a meter and you will notice that as U 1 increases k cut off increases, which means lambda cut off decreases.

Now, remember this is not; this is the lambda cut off that is you will not see waves shorter than 0.45 meters. That is the meaning of lambda cut off, k cut off being 13.93. So, this is essentially information that we get from linear instability analysis which was purely an analytical calculation, but leads us to draw some inferences in the physical space of the kinds of waves that we would observed if we had a particular flow condition, velocities and densities.

We will stop here and continue in the next lecture.