Spray Theory and Applications

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Lecture – 19
Linear stability analysis- Kelvin-Helmhotz instability -2

Good morning, we are going to continue our discussion of linear instability analysis. Towards the end of the last class we had derived a set of linearized governing equations. So, let us start by looking at the linearized governing equations and our first job is going to be to define a set of boundary conditions that can go with these linearized governing equations.

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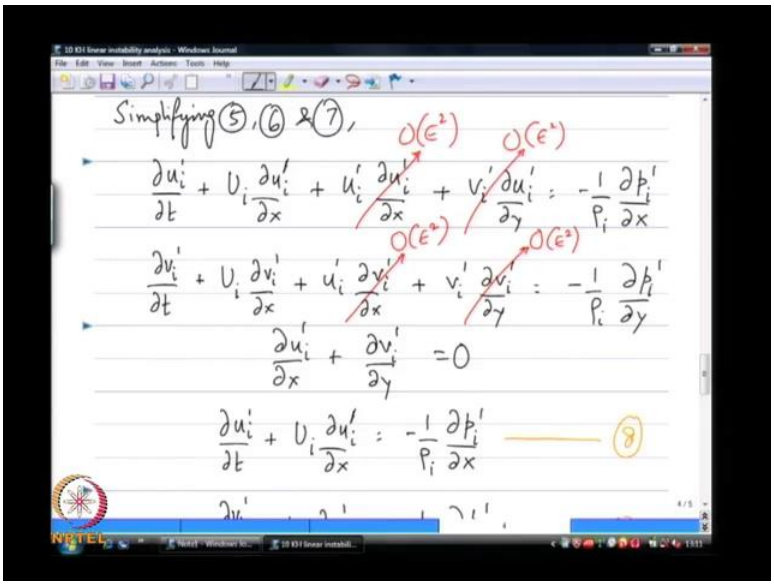

So we make that our job for today. We took the complete two dimensional Navier stokes equations and sorry Euler's equations and after scratching out the terms that are going to be order epsilon and epsilon after scratching out the terms that are order epsilon squared and only keeping the terms that are order epsilon, epsilon being the order of the perturbation quantity which are the primed quantities.

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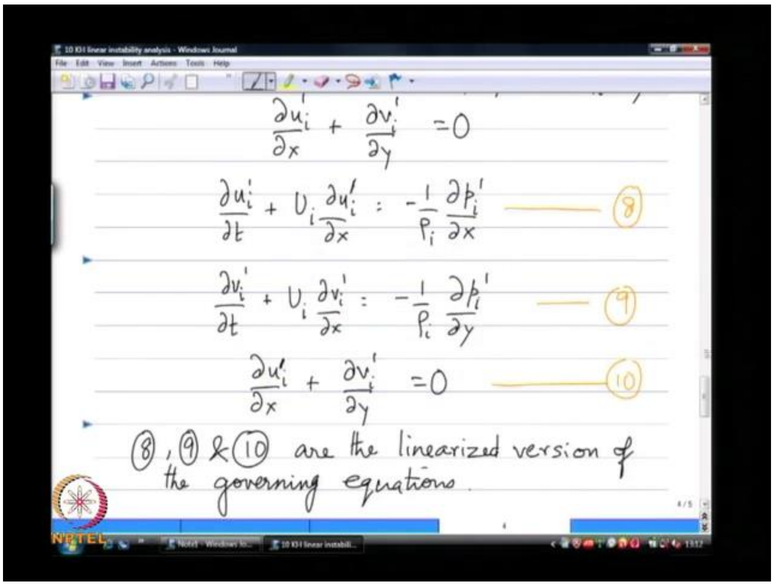

So, essentially only the, the part that is linear and the prime quantity we now have a set of equations that will tell us what the behavior of the prime quantities is in times and space. Now, from these, this is basically like saying these are the equations that govern the growth or evolution of the disturbance as the evolution grows in time, but as what we want is always go back to our analogy of ball in a cup.

So, if I take the ball and move it slightly to the left, from it is bottom most equilibrium position I can write a set of mechanics equations that describe the movement of that ball from this perturbed position. So, that essentially saying the force imbalanced due to the slight perturbation and the position of the ball in that cup is going to result in an acceleration and if that acceleration happens to point towards the previous equilibrium position that is a stable equilibrium that was, that is our essential understanding of stability.

These equations 8, 9 and 10 here, essentially govern the growth of those perturbations or really not even the growth the change of those perturbations in time and space x and y, now what we want to do is impose sinusoidal perturbation on the meniscus itself, on the interface between the two fluids.

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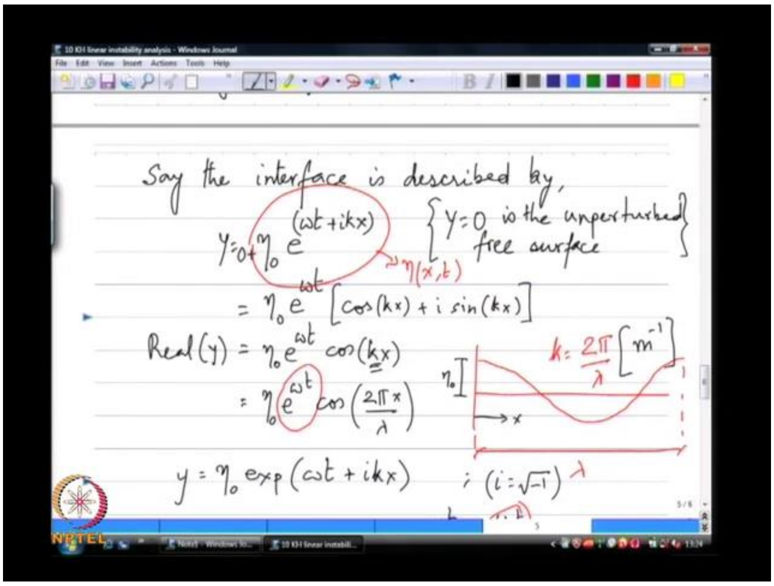

We started to say that we are going to make this meniscus have a function of the form cosine 2 pi x over lambda times e power omega t times eta 0. The 2 pi x over lambda essentially tells us the functional form of the wave that we will impose on the free surface and watch it, either grow or decay in time and their growth or decay is going to be determined by the real part of omega. If omega r is greater than 0, then the wave amplitude grows in time because the amplitude is this eta 0 times e power omega t. And the imaginary part of this wave number of the growth rate essentially creates another sinusoidal motion in time.

So, if I take a wave and if the wave is, say for example, simply going up and down in time just like that, that would involve no imaginary part to the omega if the wave is also doing this, like all waves do there is a motion of the crest in time to be the left or to the right there is a wave speed associated with that crest, that wave speed is given by this omega i and k the wave number put together. So, as far as the actual stability whether the thing grows in amplitude or decays in amplitude is only determined by the real part of omega and then we made one small distinction that said.

Say for example, k in this equation here is real meaning k is a real wave number and omega is allowed to be complex and that in turn gives us the real part of that complex number gives us the actual growth of the amplitude in space.

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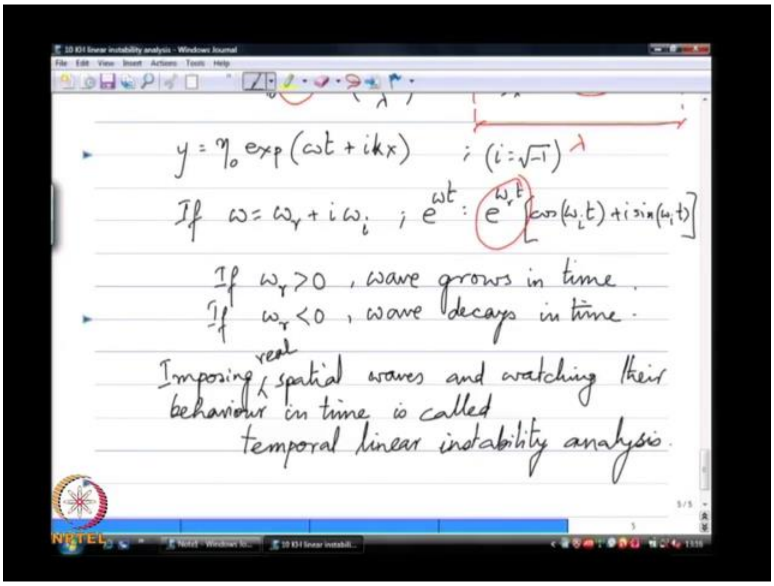

So, this kind of a an instability analysis is called a Temporal Linear Instability Analysis where we impose real spatial waves or even put that, we will impose real spatial waves and watch their behavior in time. The other possibility is to impose a wave in time say for example, at some inlet location I have a small perturbation that I can add to the time component and watch what that small time perturbation does over a space.

We leave that out of our argument for now, but essentially this is our discussion, now we have chosen to impose a disturbance on the meniscus, remember just for clarity I will say y equal to 0 is the unperturbed free surface. So, on top of that unperturbed free surface we have added this disturbance. So, essentially you might think of this as 0 plus this just as everything else that we have done was the mean flow plus the small perturbation in this case, the mean interface location is y equal to 0. So, eta 0 is an order epsilon quantity.

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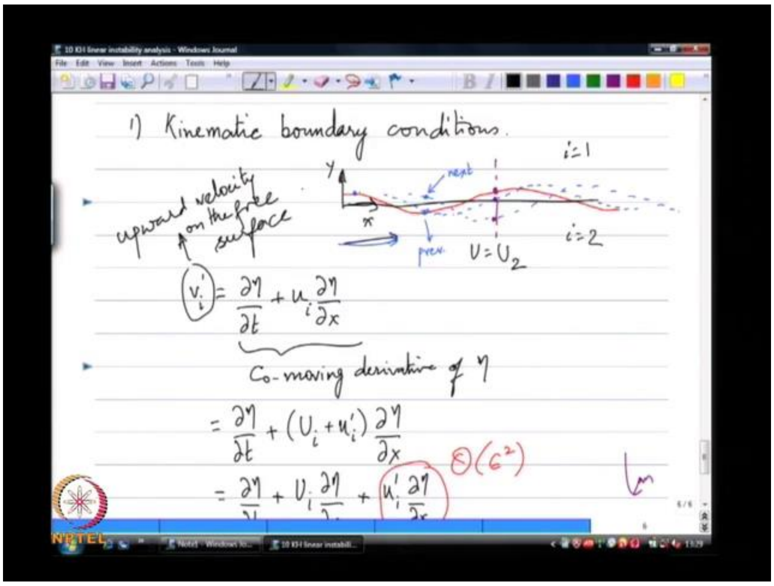

So we will need to be aware of that. So, now we need the next order of business is to define a set of boundary conditions, before that we will quickly make a list of all the variables that we have u i prime v i prime p i prime each of this for i equal to 1 or 2 and eta 0, eta 0 is the amplitude of the disturbance. So, as far as u i prime and p i prime is concerned we have three equations in the form of 8, 9 and 10, now remember these are still subscripted in i which means each one each of these for i equal to 1 and 2 gives us a total of 6 equations, for the 6 variables the eta 0 is an interface condition it is not necessarily an unknown quantity it is what we are imposing. And we just want to watch it is growth in time or this is like an initial amplitude of the disturbance let us be more specific.

The first set of boundary conditions is what we will call kinematic boundary conditions. So, let us quickly see what that is, I will redraw our sketch from the earlier page here, just take the condition the mean flow condition that U equals capital U 2. So, now, I have imposed the surface waves on the free surface and the liquid if I take a liquid particle right on that free surface. So, this is a particle that is on the perturbed free surface as this fluid flows this way, the wave itself is going to be moving with the fluid because of that the fluid has; because a fluid particle that I say this is the next time position of the same wave what was here is now here, just because the wave moved. This is the previous time position; this is the next time position. Now as the wave reaches it is crest and comes back down this particle which is at that free surface is only going to go up and down.

So, the first set of boundary conditions which I called the Kinematic Boundary Conditions, basically say that the y direction velocity of the particle is given by the rate of change of eta plus u d eta dx. So, this part here is the co- moving derivative of eta. So, co-moving derivative of eta and what we say is even if eta is not a function of times. So, think for a moment remember we said eta is this, this is our eta as a function of this whole thing as a function of x and t. So, even if eta let us say this omega was 0, just by a wave moving now at a velocity u i. You get an up and down motion which is v i.

So, if I ignore the first term here called d eta dt, the partial derivative of eta with respect to time if I ignore that for a moment and only take the second term on the right hand side here u i times d eta dx is saying that the rate of movement of a particle in the upward direction is essentially governed by the for the x direction movement of the velocity and the slope of the interface. So, if the interface has a certain slope and it is moving this way then a point on that meniscus is essentially like it is being pushed upwards with a velocity u i times d eta dx, d eta dx is the local slope of that interface. If on top of this if the wave amplitude itself is growing that is further additive y direction velocity.

So, I get two kinds of y direction velocity at that free surface, one is no growth in the amplitude just movement of a sloped meniscus in this case a wave v meniscus and the second kind is where the growth, there is you know like this is the meniscus it is just growing in time. So, the trope is coming down and the crest is reaching for higher values of y that in itself means that there is a y direction velocity, that part of the y direction velocity is represented by the first term on the right hand side. If I had this wave on the free surface and if I was simply observing a point on the free surface, that point as this wave moves that spatial location shows an upward y velocity that is represented by the second term on the right hand side. The total y velocity at any point at any x location on the free surface is given by the some of these two.

Now there is still one issue, if I take the U i and do our perturbation plus the mean flow plus perturbation substitution that part there again happens to be order epsilon squared.

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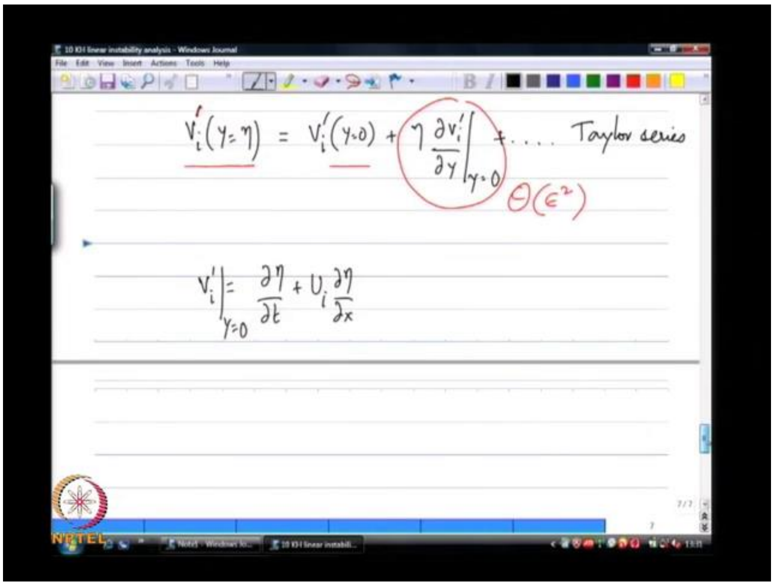

So, what we end up is v i prime at y equal to 0. So, the actual velocity now will come to this in just a moment. Now remember we are talking of the fluid velocity on the meniscus write there, this fluid particle used to be there now it is here and now it is down here. So, at this x location there is a negative y velocity because slope is negative, with no d eta dt being positive. Now ideally I am evaluating the v i velocity on the meniscus. So, this velocity in this equation is the upward velocity on the free surface, which is if I was to represent that mathematically v i at y equal to eta, eta is the free surface itself v i at y equal to eta can be written as v i at y equal to 0 plus eta times dv i dy evaluated at y equal to 0 plus this is simply a Taylor series expansion.

If you notice again, this part here is order epsilon squared. So, just as we simplified the right hand side of this equation to only have order epsilon terms. If I simplify the left hand side to only have order epsilon terms then v i at y equal to eta can be replaced by v i at y equal to 0 or v i prime at y equal to 0. So, ideally we want v i prime because v i prime is the only upward velocity there is no mean flow that is in the upward direction. Therefore, this is again ideally I should have v i equal to d eta dt plus u i dx and v i is equal to 0 plus v i prime. So, we can drop this 0 plus v i prime because we have now sort of looked at this enough to know that the mean flow is 0. So, if I take the last part this v i prime is actually the perturbation free surface velocity at the free surface at y equal to eta. But that can be substituted by taking v i prime at y equal to eta is equal to v i prime at y equal to 0 to order epsilon.

If I have to include the higher order term you know sometime later that is up to order of epsilon squared, but as far as a linear instability analysis is concerned I am only going to keep terms that are order epsilon.

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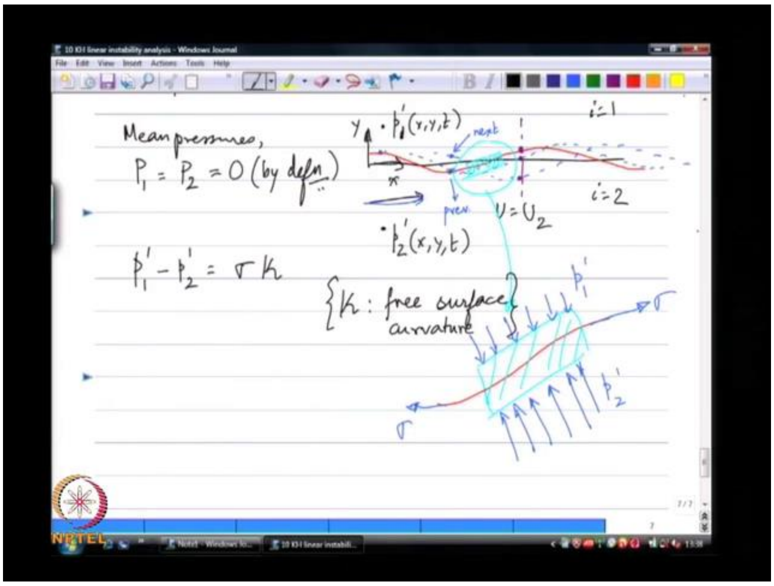

So, we have the governing equations 8, 9, 10 to go with, I have this boundary condition a 8, 9, 10 to go with that I need this boundary condition is get to that, I need this free surface condition 11 and this condition 12. Now mind you equation 12 has two possibilities for i equal to 1 and i equal to 2, meaning I have the same meniscus d eta dt, but depending on the fluid I am in the upward velocity of that fluid is determined by the balance from whichever direction i take that blue particle argument. So whether this blue particle was in the, i equal to 2 fluid or i equal to 1 fluid makes no difference as long as the free surface. I am able to imagine the free surface is composed of 2 fluids 1 on top of the other and whichever fluid I look at it does not matter this condition has to hold.

Now, the second set of conditions is called Dynamic Condition, what do we mean by that, if I go back to the same diagram. If I imagine for a moment that let us look at the mean pressures capital P i or will be specific capital P 1 is equal to capital P 2 equal to 0 by definition that is where we left off. So, we found that for a flat free surface capital P 1 is the mean pressure in the entire fluid, fluid 1 capital P 2 is the mean pressure in entire fluid P 2 this pressure cannot vary with x and y we found that from the equations and then once I know they are constants and they are both equal to each other, I have the freedom to said that to any value we said that equal to 0.

So, now that I have a curved interface the red line here in this figure, the perturbed versions of these pressures do not necessarily have to be equal. So, if I take p i prime a p 1 prime is the pressure on the fluid above p 2 prime is the pressure in the fluid below, that is what already appeared in the governing equations in the governing equations 8, 9 and 10. So if this p 1 prime and p 2 prime are now functions of x, y and time if I take an arbitrarily small element of fluid just like that and draw a free body diagram of it. So, I will expand that out, there is pressure p 1 prime on there, p 2 prime on the other side and there is a surface tension force. So, because I cut out a small element of the fluid the rest of the fluid exerts of a force on this small element due to surface tension.

So, if I write a force balance here, we can see that p 1 minus p 2 sigma times kappa. This kappa is basically the fluid is the meniscus curvature, this free surface curvature essentially dictates the difference in the forces between p 1 and p 2. So just to understand this we will go back to just a simple curved free surface if I have a pressure p 1 acting here and the pressure p 2 acting on the other side you can see that this infinitesimal piece.

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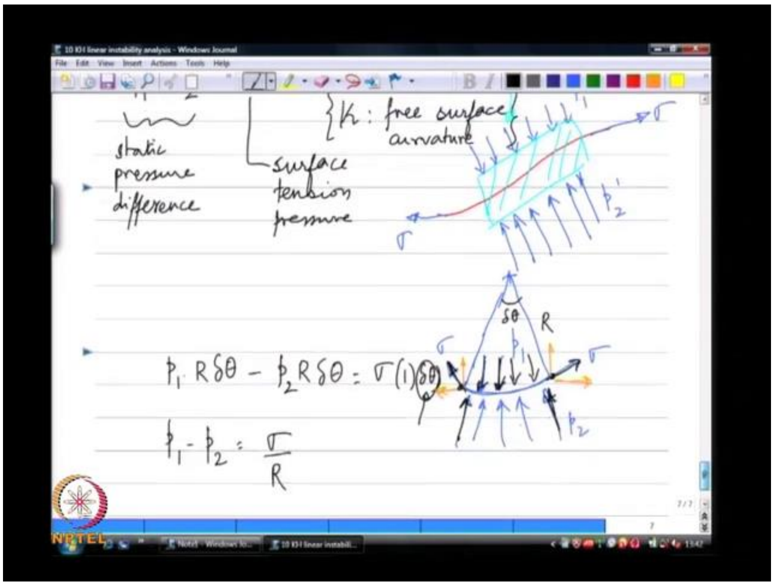

So, I am now looking at a little infinitesimal piece of fluid if I take this infinitesimal piece of fluid and let us say it exerts the small angle delta theta. This infinitesimal piece of fluid exerts a net, there is a net force at this point in the magnitude p 1 times R delta theta that is the outward positive minus p 2 times R delta theta which is the inward pressure force you are assuming the meniscus is 1 unit width in and out of the plane of the chock board this equal to sigma by R times the total meniscus. So, essentially sigma over R is equal to sigma which is the surface tension force itself, times acting over an angular one times delta theta which is the width of the which is the length of the free surface. So, it is over and which, so essentially this sigma force acts on this end here and this end that is one unit long.

This delta theta comes from the fact that, if I take a vector sum of this in this direction and this direction likewise in this direction and in this direction. The horizontal forces cancel out, the vertical component of this surface tension force is the only force that contributes to the pressure differential especially as this delta theta becomes smaller and smaller. In the limit of delta theta going to 0, the horizontal components of these surface tension forces becomes smaller and smaller and the vertical components are what add up to the pressure differential p 1 minus p 2. So, in this note I, we can see that sigma by R in a more generalized notation if R is if I have a circular boundary or a spherical bubble R is simply the radius of curvature. So, we will generalize it to say p 1 prime minus p 2 prime equal to sigma times kappa which is the radius of curvature.

So, this is my force balance conditions the right hand side having surface tension pressure and this is static pressure difference, if I take the free surface itself I know the equation of the free surface that is.

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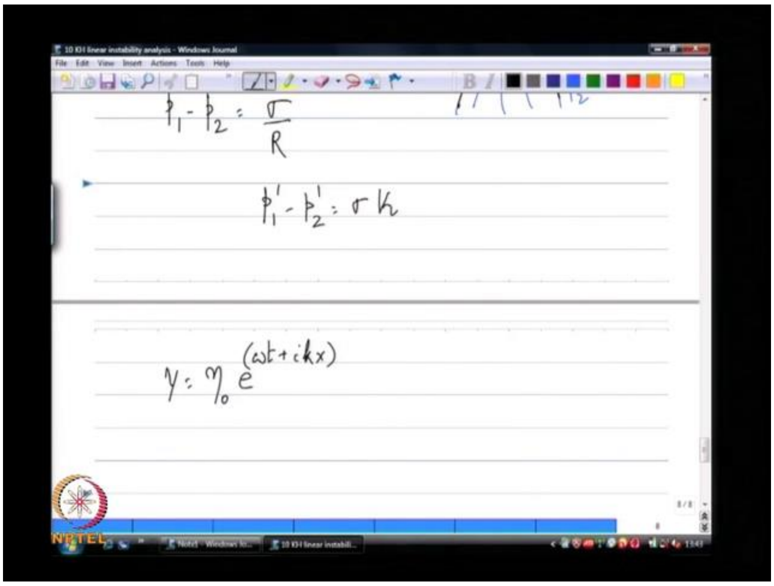

I will re-write this equation one more time here just for our referral y equal to eta 0 e power omega t plus ikx this is the equation of the free surface itself.

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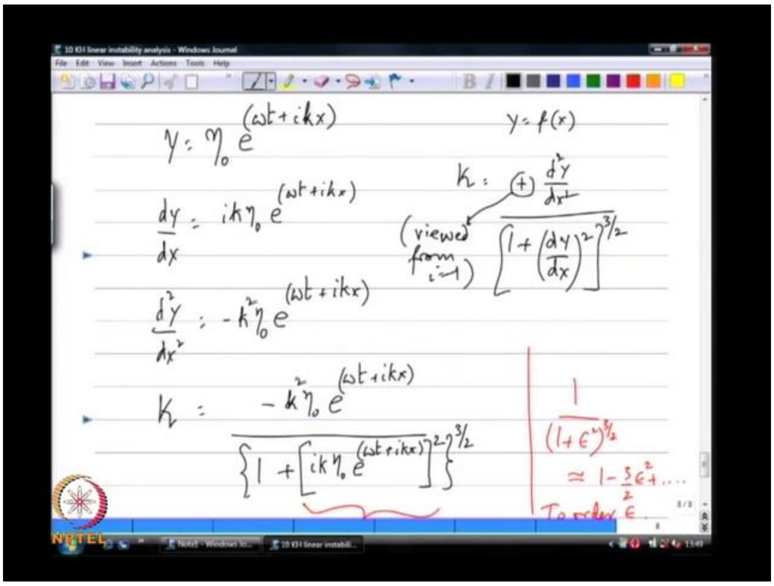

Which means for a given curve if I know y equal to f of x,curvature kappa is given by d square y dx squared by 1 plus dy by dx squared raise to the power 3 halves with a negative sign only to account for the direction of these curve of the curvature. So, this negative sign says wherever your free surface is concave make the curvature negative, wherever the free surface is convex concave and convex viewed from fluid 1. Let us say, wherever the free surface is concave as viewed from fluid 1, fluid 1 is on top wherever it is concave as viewed from fluid 1 we will can this curvature negative; wherever it is convex as viewed from fluid 1, we will call it as a positive curvature. It is simply a matter of convention. If this negative sign was not there we will essentially invert this p 1 prime and p 2 prime.

I am going to show you how I do not ever remember this I want to show you how you cannot remember it either, but get to the correct answer. So, now, given this if I want to calculate the curvature kappa from this equation I need these derivatives. So, let me go ahead compute the derivatives dy dx is ik eta 0 times e power omega t plus ikx, another derivative on that is minus k because ik times ik gives me minus k. So, if I make these substitutions kappa then becomes minus k eta 0 e power omega t plus ikx divided by 1 plus ik eta 0 e power omega t plus ik x the squared this whole thing raise to the power 3 halves.

If you look at this term, this is order epsilon squared. So, I can use 1 over 1 plus epsilon squared I will just call, make this simplification on this. So, this I can approximate it to 1 minus 3 halves epsilon squared plus higher order terms. So two order epsilon, the denominator is equal to 1, because the next term after one is order epsilon squared. So, I can come here and say to order epsilon kappa equals minus k eta 0 e power omega t plus ikx. So, for the given free surface wave v shape this is the curvature at every point. So, I told you how I do not remember this, I want to show you how I end up not remembering and still getting the right answer it is this fact that, if just as a check at x equal to 0 the curvature is minus k eta 0 e power omega t.

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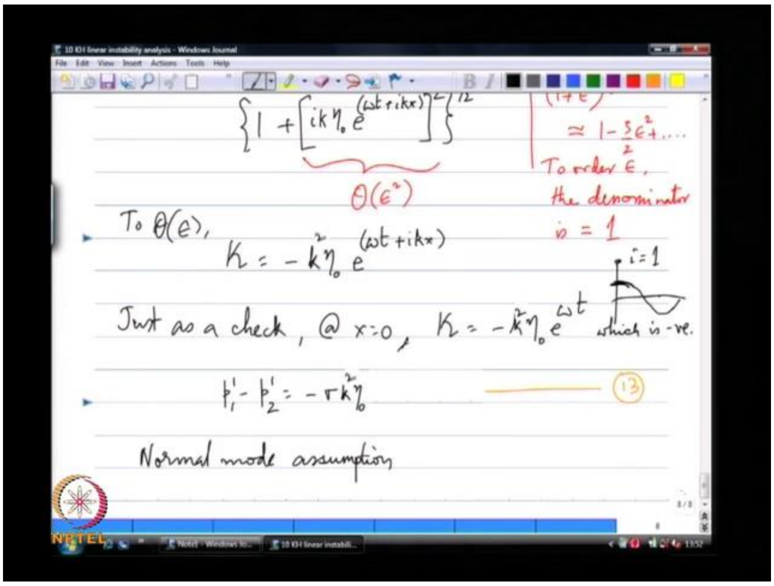

So that is a negative number, which means as viewed from and the shape of the meniscus here the real part would end up looking like a cosine wave we saw that. So, as viewed from fluid 1, this part at x equal to 0 is concave meaning you have a negative curvature which means kappa is your d squared y dx squared is minus k which means this was wrong this should be plus the fact that I tried to put a negative sign there if I did take that negative sign into account this would end up being positive here because d squared y d x squared itself is minus k eta 0.

So if I want kappa to be minus k eta 0, then this is plus d squared by dx squared. So, this sign here gives me viewed from curvature viewed from fluid 1, if I want to write the same exact equation for curvature viewed from fluid 2, I would end up putting making that a negative sign. So, we now have another boundary condition here if I take. So now, if I simplify our dynamic balance pressure condition, p 1 prime minus p 2 prime is minus sigma k eta 0 times e power omega t plus ikx.

So, just to recap one more time these are our variables I have u i prime, v i prime, p i prime for all i equal to 1 and 2 and eta 0, if the is the initial amplitude. Now, if I impose an initial amplitude that is, sinusoidal on the interface then as long as the equations are all linear. The only spatial variation that you can expect in the x direction, so the interface is sinusoidal in the x direction the only variation of all other quantities in the problem that you can expect in the x direction will be the same function sinusoidal will be the same will be of the same sinusoidal form.

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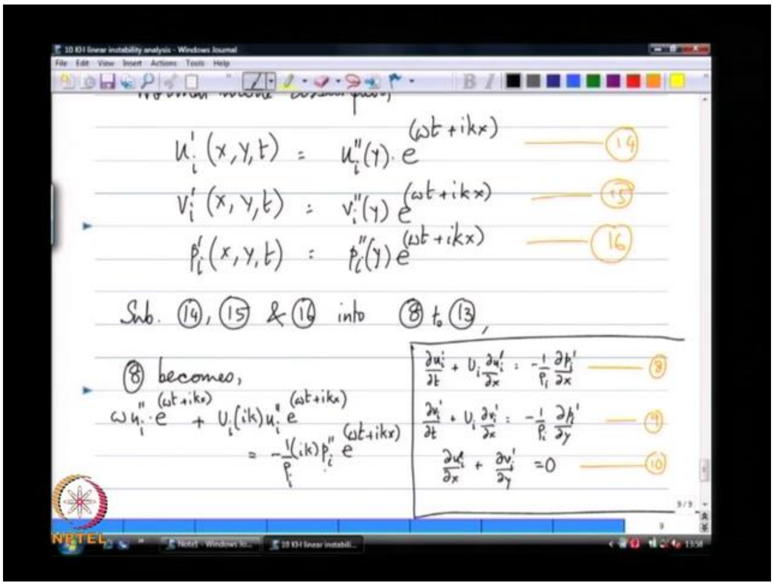

So, we can make us we can make what is called a Normal Mode assumption, that all u i prime which is now a function of x y and t, is of the form u i 0 times e power omega t plus ikx; where they free multiplicative factor to the exponential u i 0 is purely a function of y and e power omega t plus ikx is basically saying the functional form of the variation of u i both in x and y, x and time are exactly the same as the functional form for the impose disturbance. This is the quality of only linear problems. So, if I have a linear governing equation partial or ordinary the frequency whether it is in space or time is not altered in the response. So, I have an imposed disturbance which is my input disturbance the output disturbance let us just call which is the imposed disturbances manifestation in the other fluid mechanic quantities is also in exactly the same frequency.

I do not want to go into the details of why this happens, but let us just take that to be the case, but the function does still have a variation in y. Likewise I can do the same thing with this, this is not v i to the power 0 u i 0, u i 0 y is simply I will put this in parenthesis or let us just switch notation just to avoid any confusion we call this U i double prime which is only a function of y. These are called Normal Mode assumptions.

So, all the modes of u i, v i and p i are assume to be of the same they are of the same mode as the imposed wave. So, if I now substitute we will call this 14, 15 and 16 into 8 to 13. So, first of all 8 becomes if I go back to equation 8 b u i dt plus u i d u i prime dx equals minus 1 over rho i dpi dx. So, just for the sake of clarity I will copy these we will place them on the side here, but then we will delete them dui dt dui prime dt is omega times ui double prime times e power omega t plus ikx plus u i d u i prime d x is u i times i k times u i double prime times e power omega t plus ikx this equals minus 1 over rho i dpi dx which is ik pi double prime e power omega t plus ikx.

What I should notice is that what used to be a partial differential equation is now become an algebraic relationship between u i double prime and p i double prime.

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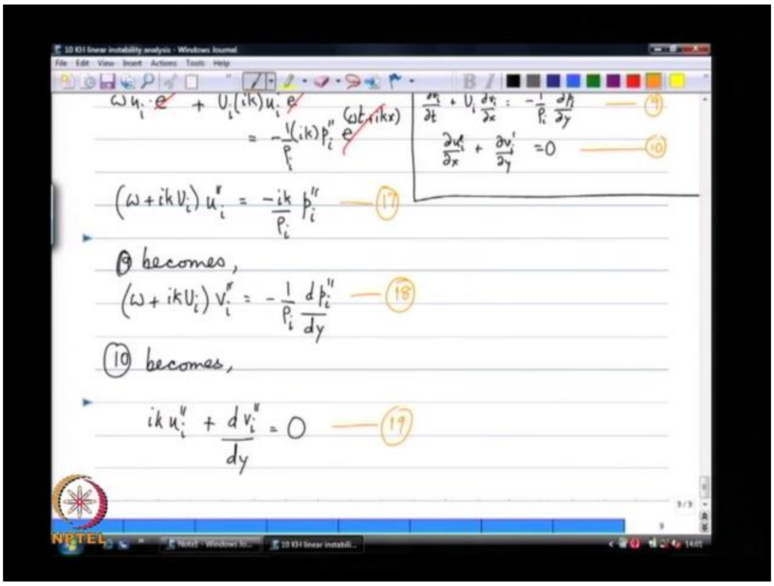

If I simplify this, I have get omega plus ikui u i double prime equals minus ik over rho i pi double prime, the exponentials all cancel out and what I have is this, we will call this equation 17. If I continue on and write down the equations for the other two without going into this same level of mathematics substituting is a same process 9 becomes show omega plus ikui v i double prime minus 1 over rho i dpi double prime dy.

So, we have now a set of ordinary differential equation system for u i, b i, p i in the form of 17, 18 and 19 look at these are ordinary differential equations because I only have p i the double prime quantities are only functions of y all there all the variation and x and time is already been absorbed. So, what I have between 17, 18 and 19 is three quantities u i, v i, p i and three equations this forms a system of ordinary differential equations.

We will take up from here in the next class.