Spray Theory and Applications
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Lecture - 18
Linear stability analysis- Kelvin-Helmhotz instability -1

Let us continue our discussion of Instability Theory, specifically Linear Instability Analysis.

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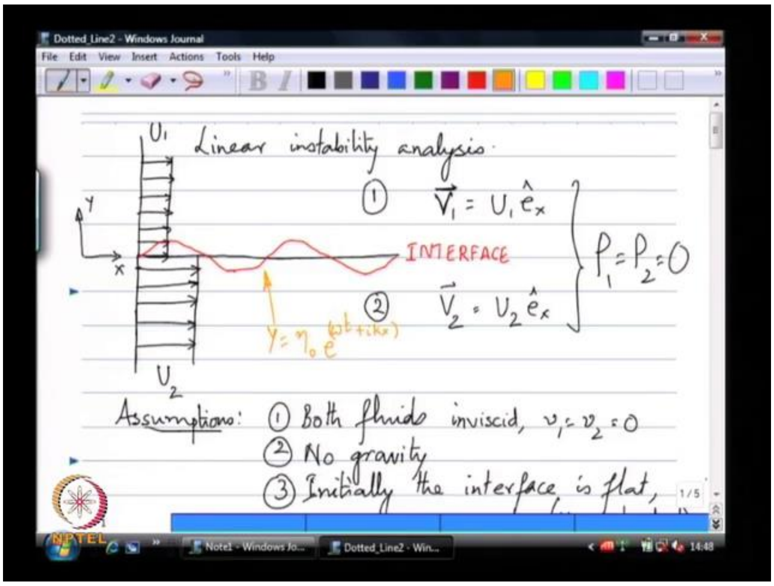

What we are going to do today, is we are going to go through formally look at the stability of stratified fluids. So, I have an interface between a fluid 1 and fluid 2; fluid 1 moving with a uniform velocity U 1, fluid 2 moving with a uniform velocity U 2. I need to introduce a coordinate system to study this problem. So, the velocity vector in 1 and call this V upper case is basically U 1 ex. And the velocity vector in 2 is U 2 ex.

The y component of velocity in both these cases is 0. I am going to assume nu 1 equal to nu 2 equal to 0. So, this is saying 0 viscosities. So, let me formally write down the set of assumptions. And initially, the interface is flat; this is the unperturbed as we will see in just a moment, at the position y equal to 0. So, this is my interface.

Now in general, each of these fluids 1 and 2 can have a velocity in the x and y direction. So, we will just also assume that the flow is two-dimensional. So, there is no flow in and out of this plain of the paper neither is there a possibility of any flow. So at the moment there is no flow in the y direction, but there is a possibility that things could start to flow in the y direction if they choose. But there is no flow in the z direction right now then there is no possibility also of the flow to be initiated in the z direction; that is the meaning of two-dimensional flow.

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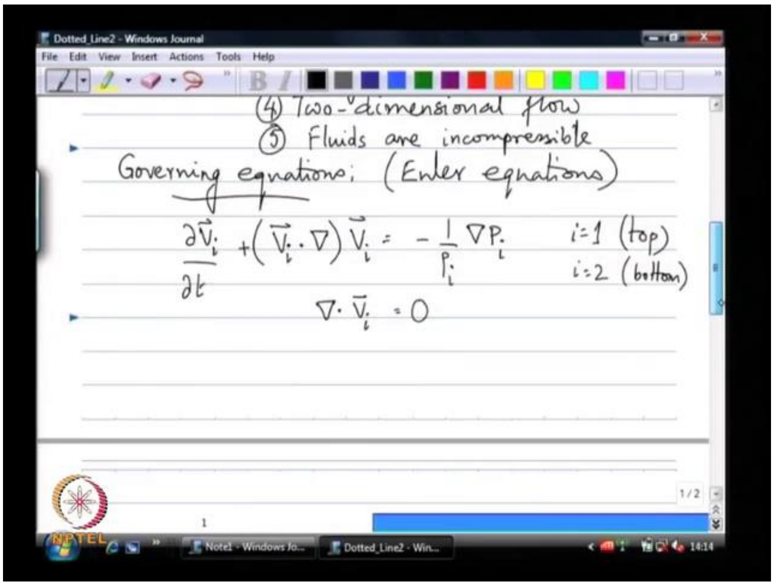

So, let us write down the governing equations and the boundary conditions for this flow problem. If V i is a velocity field and P i is a pressure field then this is the; so i equal to 1 is the top fluid, i equal to 2 is the bottom fluid. So, this is simply what we very often called the Euler's equations. I am going to write this out in a full Cartesian form. So, the x, and of course I did not mention the continuity equation to go with this that says del dot V i equal to 0; which means let me go back I have one more assumption that one fluids are incompressible. So, rho 1 and rho 2 do not change with x and y or any other independent variable.

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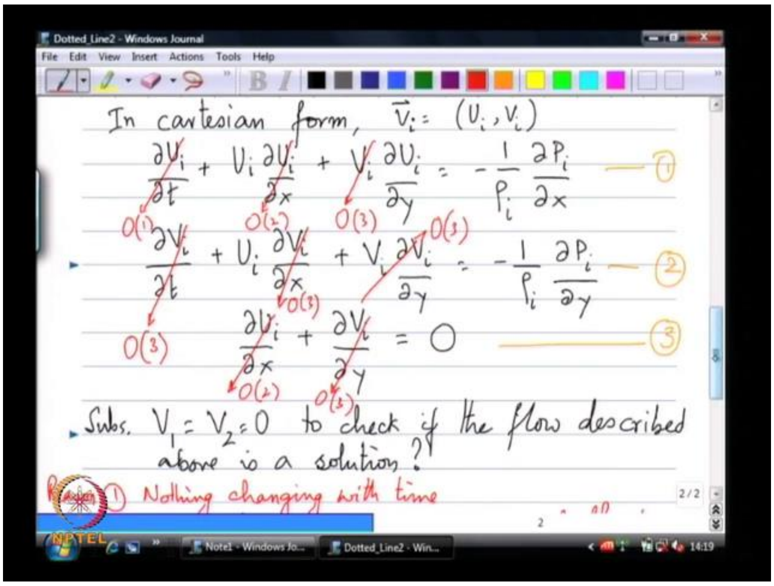

So, if I write these out in Cartesian form the vector V i is composed of components capital U i and capital V i. So, V i being the y direction component in general, so this plus U i; let me write this clearly. So, this is the equation I call this is 1, this is 2, this is 3.

So let us first you know, we have we started with a certain flow configuration that fluid 1 has a uniform x velocity of U 1, fluid 2 has a uniform x velocity of U 2. Fluid 1 and 2 both have no y component of velocity, so if I substitute to check if the flow described above is a solution, what do we find. If I said V i to 0; now I also know U i is constant in time. So, from that if d u i dt is 0 not changing in time, this is 0 because it is not change in x, this is 0 because V i is 0. Let me write those down call this is due to 1, 2, 3.

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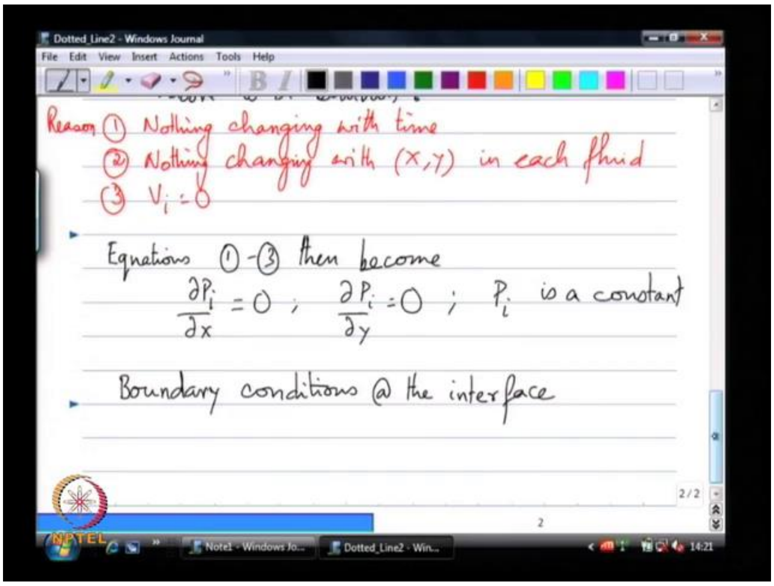

Reason 1, is nothing changing in time. Reason 2; we are solving each of these equations, for each fluid you have one set of three equations describing the U 1, V 1 for that fluid U 2, V 2 for the other fluid, so nothing changing with x and y in each fluid. And the third is of course saying, V i is 0. Likewise, V i is 0 so I can say this is due to 3; V i is 0 this is due to 3. So, this is 0 due to 3. This is 0 due to reason number 2.

So, what have these simplified to? Say I do not know what P i was supposed to do; all I know is a flow field. So, essentially what these now becomes is that dP i dx equal to 0, dP i dy equal to 0. So, all that says is P i is a constant. The pressure in each fluid everywhere in the domain is a constant, it is not dependent on x and y. That is basically what we are learning from saying dP i dx is 0 and dP i dy is also 0.

Now, I still have not talked of the boundary conditions, let see if what the boundary conditions are. I do not want to go into the detailed boundary conditions right now, except to say that look I have a flat meniscus; for a flat meniscus to remain flat for all times, you can clearly see that the pressure in fluid 1 and the pressure in fluid 2 have to be equal.

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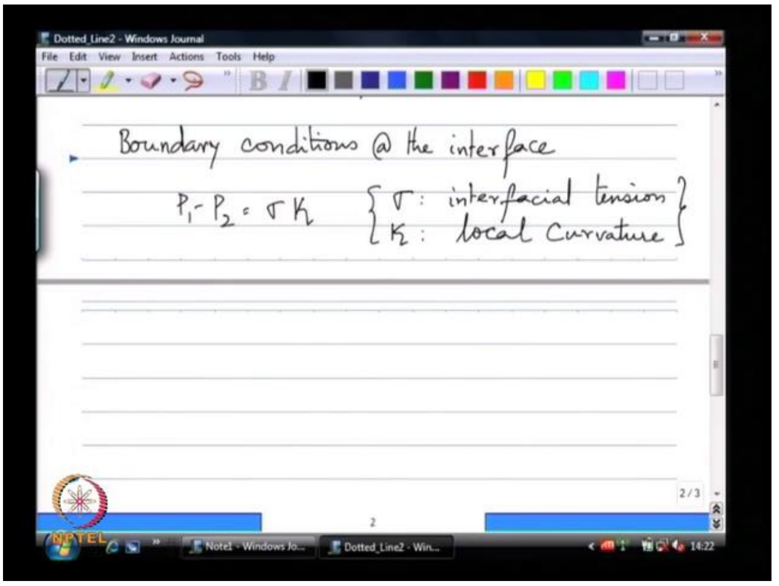

So, the boundary condition at the interface can be written as this. The simplest of boundary conditions says that P 1 minus P 2 equal to sigma times kappa. Where, sigma is the surface tension or most specifically we will call it interfacial tension and kappa is a curvature. Since the unperturbed interface is flat it is easy to see that kappa has to be 0 everywhere.

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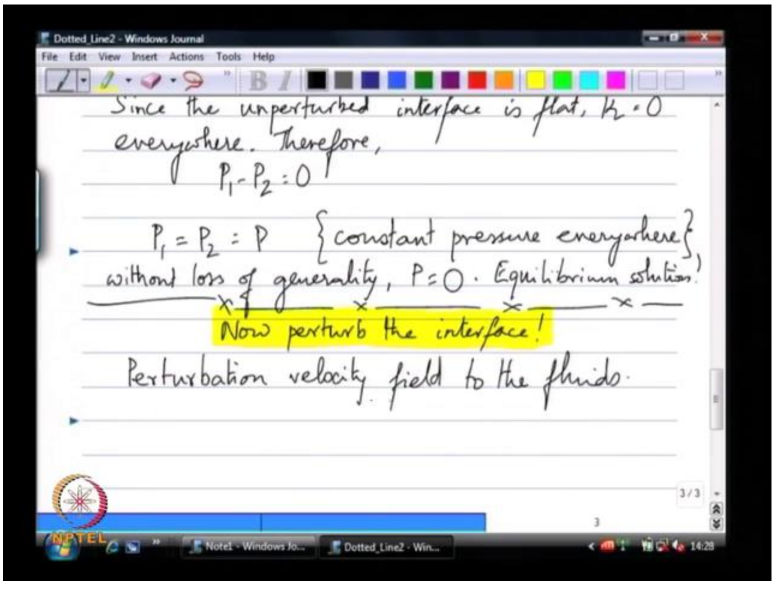

Therefore P 1 minus P 2 equal to 0. May now know P i is a constant, the governing equation said P i in each phase is a constant. The boundary condition says P 1 minus P 2 has to be 0. What they says is P i everywhere is equal to P 2 is equal to sum P. So, physically it is just constant pressure everywhere. So P is just a number, this capital P is whatever is the actual magnitude of the pressure and that is the same pressure everywhere both in fluid 1 and in fluid 2.

So, without loss of generality you will see this a lot, so where I know that particular number has no significance I can give it any convenient value; in this case I will assign P equal to 0. So, if capital P equal to 0 everywhere and V 1 equal to U 1 ex, and V 2 the vector V 1 equal to U 1 ex and V 2 equal to U 2 ex and P equal to 0. So, V 1 equals U 1 ex V 2 equals U 2 ex and P 1 equal to P 2 equal to 0 is a solution to the governing equations. That is just like saying that it is a system where forces are balanced in, the ball in a trope example that we saw in the last class. So, this particular flow configuration is an equilibrium solution. Is it a stable equilibrium or an unstable equilibrium? That is the question that we now have to answer. How do we go about that?

Now this interface is infinite in the x direction. Essentially, imagine an interface of two superposed fluids that has no start and the end that is the kind of interface that we are looking at. So, if I now take this kind of an interface and superpose a perturbation, so I have these two fluids that are co-flowing, so these two fluids are co-flowing and I want to understand if I perturb this interface slightly will the under the action of these co-flowing fluids top fluid moving at U 1 bottom fluid moving at U 2 and mind we did not make any assumptions on whether U 1 is greater than U 2 or the other way round it does not matter in fact.

So, under the action of this co-flowing fluids does this meniscus amplitude decay or grow that is the only question that we are interested in asking. So, if I keep everything else the same, but perturb this interface slightly; what that does is it introduces a perturbation velocity field. If I now perturb the interface; as soon as I perturb the interface I have introduced a perturbation velocity field to two both the fluids.

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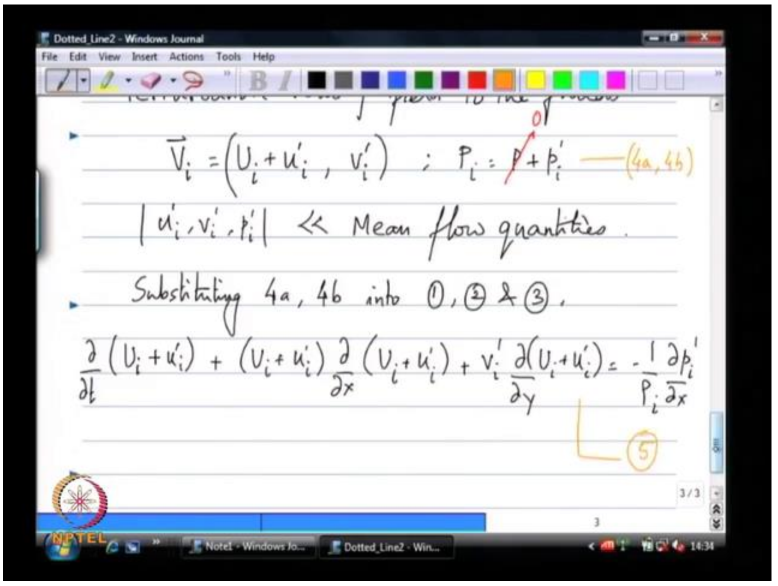

So now if I look at V i, this is still U i plus a little U i; I will call this U i prime for perturbation times comma V i prime. So, this is a vector I am denoting using order pair notation. U i plus U i prime is the new velocity in the perturbed interface configuration, so I have introduce a small additional x component of velocity. Now mind you both the prime quantities U i prime and V i prime both for i equal to 1 and i equal to 2 are both what are called field variables. So, these are velocity fields, they are functions of x and y and time.

So, U i prime and V i prime are both velocity fields which were not there until I perturb the interface, now because I have perturb the interface I want to see what this velocity field would look like. Now, whatever be U i prime and V i prime and let us even say the pressure P i is P plus sum P i prime then P itself is 0, we know that. So, the pressure in each fluid i equal to 1 is the top, fluid i equal to 2 is the bottom fluid. The pressure in each fluid is now given by some small quantity P i prime.

The key point to note here is that U i prime, V i prime and P i prime, the magnitudes are much less than the mean flow quantities. So, they are infinitesimally small. This is our new velocity field, and this is our new pressure field. These new velocity and pressure fields also have to be solutions to our unsteady Euler's equations. Previous we showed that this is an equilibrium solution U 1 and U 2 being constant everywhere and P being 0 everywhere is an equilibrium solution because nothing is changing with time; that is the idea of equilibrium, it is steady in time.

So, if I now come to the perturbed case, U i prime and V i prime could be functions of time. All I want to know is does U i prime and V i prime, do U i prime and V i prime decrease with time or do they increase with time; that is what I am interested to find out. If the primed quantities increase with time from some small initial starting value that is not good for the stability of the interface.

So first of all substituting, I will call this equation 4; I will call this 4a and b. What do we find? The first is the x momentum equation, second is a y momentum, and the third is the continuity equation; we quite familiar with that.

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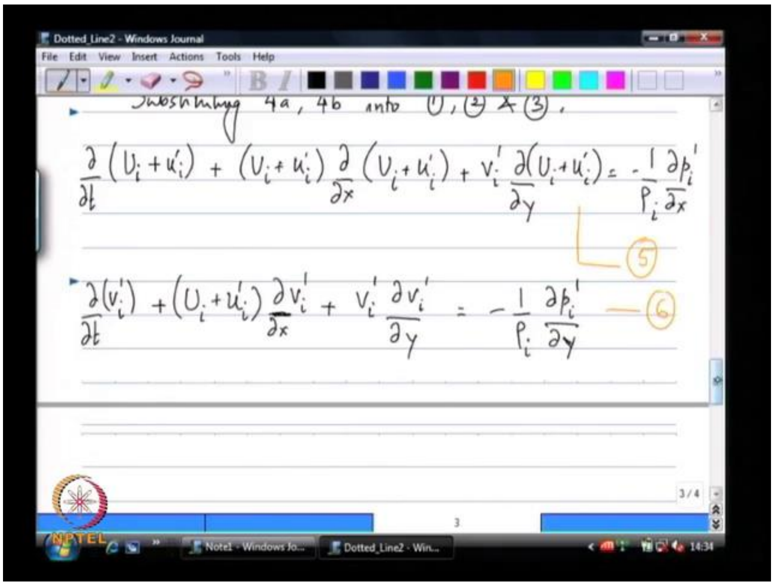

So, this is U i plus U i prime plus U i plus U i prime times d dx of U i plus U i prime plus V i prime d dy of U i plus U i prime equals minus 1 over rho i dP i prime dx. This is the equivalent of equation 1.

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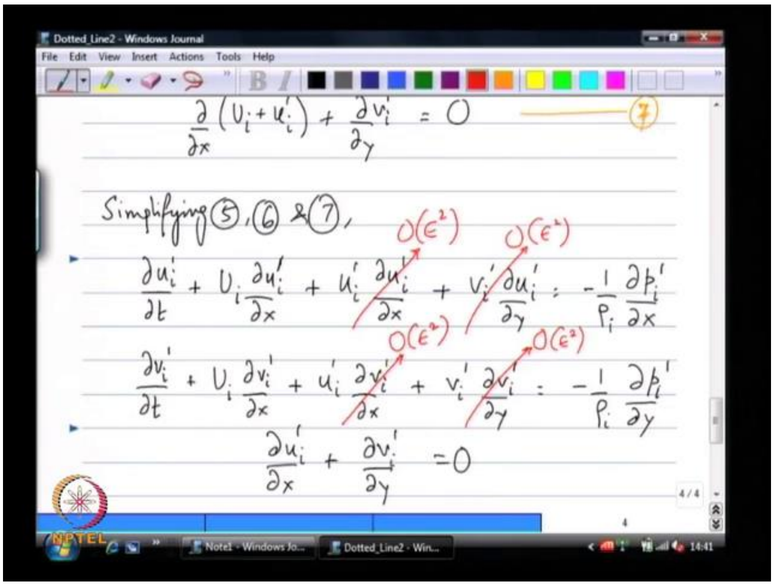

The y momentum equation becomes, the last one is this. Now this capital U i is basically U 1 and U 2, those are not changing with x y or time. The prime quantities are allowed to change with x y and time in fact. So, we can simplify 5, 6 and 7. Using that what we find is first one; I took this term here capital U i plus U i prime times d dx of U i plus U i prime. U i itself is not varying with x, so d dx of capital U i is 0. But, d dx of U i prime is not 0. So, that is left in the calculation still, but I have that term being multiplied by U i plus U i prime.

I just want to write it out as two separate terms. So, I have U i times du i prime dx plus U i prime du i prime dx plus I also have the additional term V i prime du i prime dy equals minus 1 over rho 1 rho i. So what do I find here? I can write the other one as well; dv i prime plus U i dv i prime dx plus U i prime dv i prime dx plus V i prime dv i prime dy equals minus 1 over rho i dP i prime dy. The last term is simply du i prime dx plus dv i prime dy equal to 0.

Now remember U i prime is a small number in comparison with capital U i. So, I want some way to compare the magnitudes of these four terms on the right hand side of the top equation, on the left hand side of the top equation. If there are four terms on the left hand side of the top equation, if capital U i is order 1, so it is like 1 meter per second let us just say. Then du i dt is order U i prime, so that is order epsilon it is like an small number in comparison to 1. This the first term is order epsilon, the second term is order epsilon because it is capital U i is order 1, U i prime is order epsilon. So, the second term as a whole is order epsilon. The third and fourth terms are order epsilon squared.

So, if U i; if these quantities are all order epsilon, epsilon is some small number then what it automatically means is that these two terms have a two prime quantities multiplying each other; which means that the magnitudes of these two terms that I have scratched out in this particular equation are very very small in comparison to the terms that I have left. So, the error that I will end up making by completely scratching them out of my equation is very small and that smallness becomes smaller and smaller as epsilon goes further and further towards 0.

This is the idea of an infinitesimal perturbation. It is so small that you can ignore epsilon squared in favor of order epsilon, and it is as small a number as possible essentially. Now if I give that much of a perturbation is that perturbation further going to grow or decay, that is all I am interested in. I just want to know a direction to the growth that is all. So, I can ignore these two terms which are now order epsilon squared in favor of the other three terms that I have remaining. I can do the same thing with this. The third equation both the terms in there are order epsilon so I really cannot through away anything.

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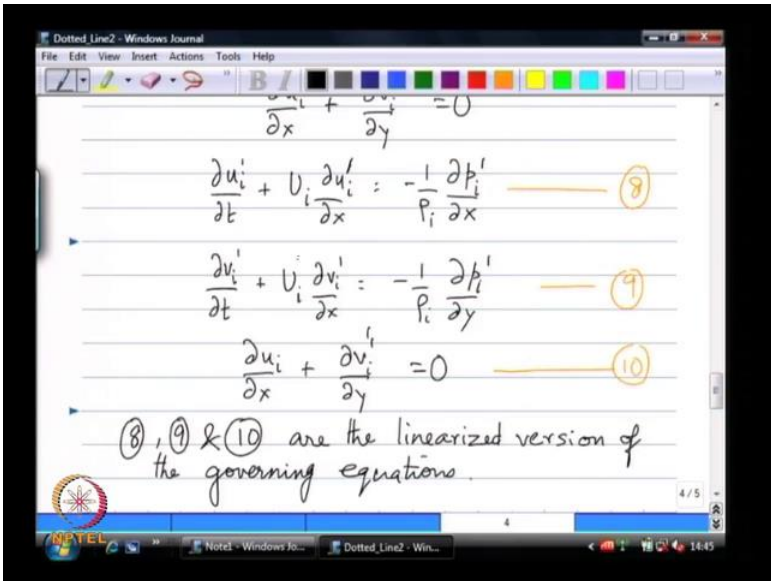

So what I can now do? I can rewrite the equations without the terms that I scratched out. So, completely throwing away the terms I scratched out in that fashion. I can number these equations. Now 8, 9, 10 have are important. Now how do I know that they are linearized versions of the governing equation; I want to make sure that you completely understand this idea of linearized version.

If you look at these equations, there are three equations. The three unknowns are; U i prime, V i prime, and P i prime. I have one set of three equations for the fluid 1, one set of three equations for fluid 2. There are no terms in these equations that are like U i's prime squared or V i prime squared. Even though our original Euler's equations have terms like ud udx, what we have done is we have linearized the complete Euler's equation about a state that I have already know is a solution to the Euler's equation.

Capital U i being constant both in x y and time. E is a solution to the Euler's equation, we have found that. And we have written a linearized version of the Euler's equation around that ground state fuel of U i being some number like U 1 is a number U 2 is a number, so the capital U i in the equations 8 and 9 is just a number that you already know. The only unknowns are the prime quantities U i prime V i prime and P i prime, and these are field variables they are functions of x y and time. Therefore, these equations since they do not involve any terms that are non-linear in the prime quantities this set of governing equations are linear. So, I have a system of linear partial differential equations describing the prime quantities.

So now, how do I go forward from here? We started to say that we are going to introduce a perturbation to the interface. And the simplest form of perturbation is a sinusoidal perturbation. We will see may be later on why a sinusoidal perturbation is introduced, but essentially what we want is a set of harmonic functions; a set of orthonormal basis functions of the linearized version of the governing equations that I can use to take any arbitrary interface shape and express as a series of those orthonormal basis functions. This is simply going back to solutions of linear od's in fact not even liner pd's.

So, our idea of coming up with a basis function for this particular geometry that gives me set of orthonormal basis functions, that gives me a set of basic functions in which I can expand any arbitrary interface shape. For this particular case it happens to be sinusoidal function. So, I will express this as a functional form y equal to eta 0 times e power omega t plus ikx. So, here I will write this out little later on.

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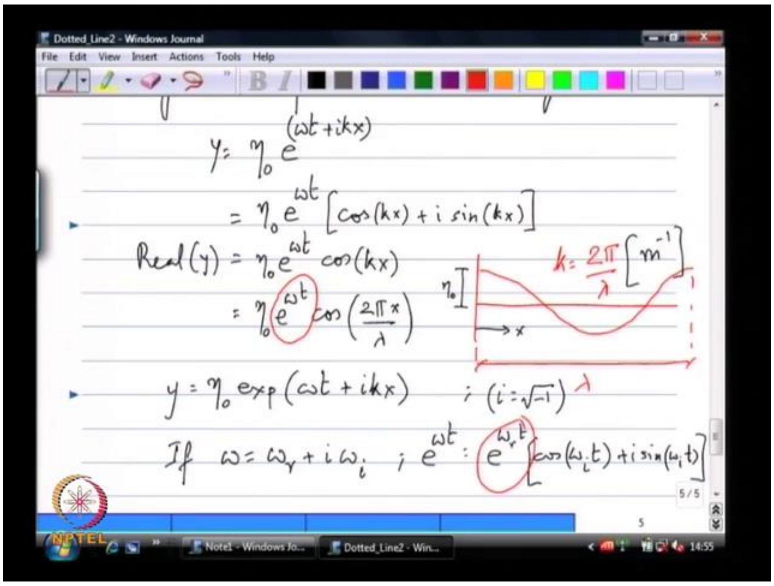

Now where did I get this functional form, I want to make sure we understand that. If I expand this out I have eta 0 e power omega t times e power ikx, which I can write as cos kx plus I sin kx. If y is a real number then essentially the real part of this is nothing but e power eta 0 e power omega t cos kx.

So, I have allowed k; k is like is a wave number. We will see in just a moment what that is if I take, essentially if lambda is the wavelength then this can be written as eta 0 e power omega t cosine of 2 pi x over lambda. So, when x equal to lambda the phase angle of this wave is equal to 2 pi and x equal to 0 the phase angle is 0, the phase angle goes smoothly as a linear function of that distance from this point here. And if I define k to be equal to 2 pi by lambda, k is a wave number and has units of per meter.

So, we will work in this wave number space, it is easier to work in the wave number space then it is to work in the wavelength space. So, we will define this wave number k equal to 2 pi by lambda and 2 pi by lambda times x in the cosine is the way to describe sinusoidal or a co-sinusoidal in interface. Sin and cosine are only off by a phase angle of pi by 2. And on infinitely long sheet in the x direction it does not matter at the end of the day. So, here is a description of this interface now what is eta 0? Eta 0 is that amplitude.

Now, like I said the only feature I am interested in is if I impose a disturbance of wavelength lambda or now wave number k is that, disturbance go in to grow in time or decay in time that is the only part that I am interested in. And we use this function to capture that behavior. If omega, let us just say omega is a real positive number. If omega is real and positive then e power omega t is a number that grows as t increases. If omega is a real negative number then e power omega t is a number that decreases as t increases.

So, for in the short period of time around t equal to 0 or I want to know is if I give a certain lambda does the omega for the system come out to be a positive real number or a negative real number. In fact, I will go back to now; so this is the physics of what we are going to look for. So, rather the mathematics of what we are going look for, but I find it easy instead of sins and cosines to work in this exponential notation, with imaginary complex numbers. So, we know that i is this number square root of minus 1, and e power ikx is simply cos kx plus I sin kx.

So, we are not going to really make the distinction between real and imaginary numbers, we will just say that omega is a complex number; omega could be a complex number, but all I care is whether the real part of omega is positive or negative. So, let us take this if omega is a complex number like that e power omega t is e power omega r t times cosine of omega it plus i sin omega i t. So, really speaking I can let omega be any kind of can be a full complex number, the real part of omega is going to basically determine if the disturbance is going to grow in time or decay in time. The imaginary part of that omega is only going to cause a sort of a ripple effect; omega i is like imaginary part it is going to cause a wave to be transported in time, but the amplitude of the wave is not affected by that, amplitude is only affected by this e power omega r t.

So think of it this way, k is a real number I am going to force k to be a real number which means I am giving a real perturbation to the interface. I perturb this interface with some waves of total wavelength lambda or wave number k and I let go of this template that I use to create this wavy interface. As soon as I have let go of this interface is the interface amplitude this eta 0 going to decay or grow further, that is the only part that I care about. And that is mathematically conveyed to me by whether omega r is greater than 0 or less than 0.

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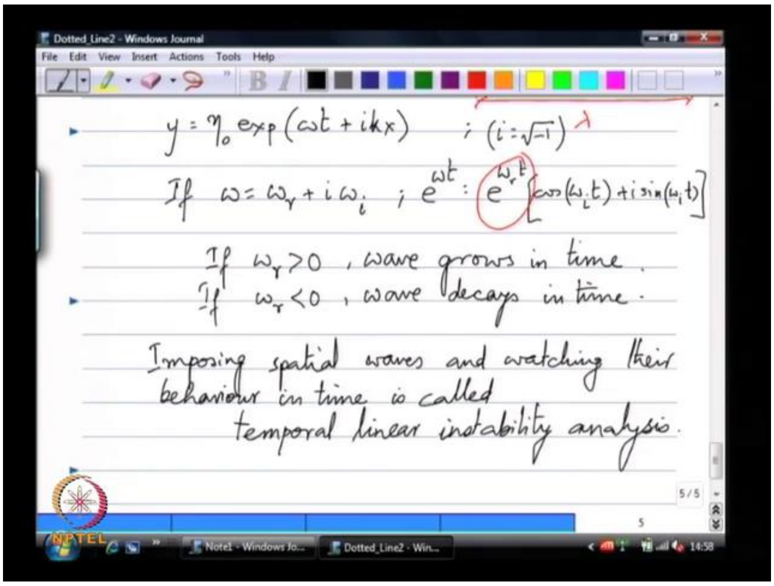

Essentially, this idea of imposing spatial waves and watching; called a temporal linear instability analysis.

So, just you quickly recap we started with a full Euler's equations; showed that the mean state is a solution. Then we perturbed the interface using the prime quantities and wrote a set of linearized governing equations in the prime quantity; which are these equations 8, 9 and 10. And then we said that in order for me to solve them I am going to make this assumption. This specific interface shape assumption and from there we are going to look for solutions of whether the real part of omega is positive or negative.

We will continue this in the next class.