Spray Theory and Applications
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Lecture – 17
Linear stability analysis – Introduction

Let us continue our discussion of spray formation, and underlying the idea of spray formation is the concept of where you have bulk liquid inside the nozzle, bulk liquid is liquid that is still in some connected form in it is entirety that breaks up into drops slightly outside the nozzle. So, we want to see the process and the forces involved that cause these cause this bulk liquid to break down into drops.

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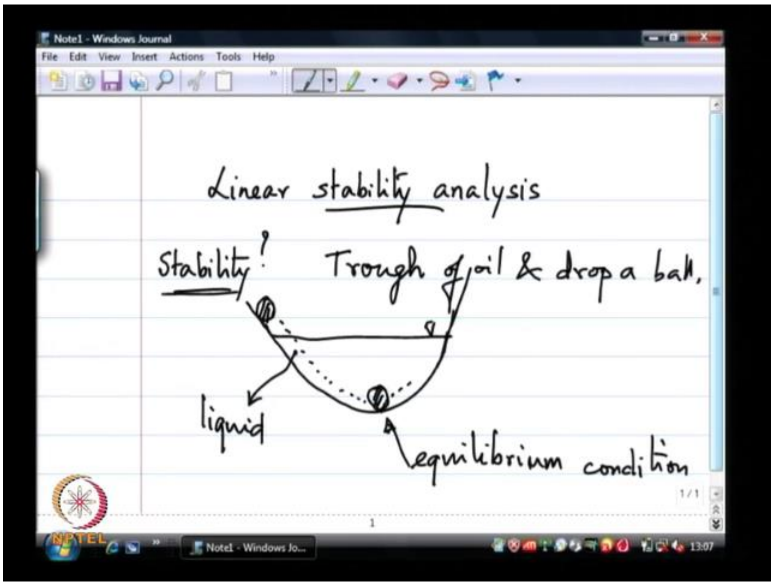

So, this we are going to investigate using a theory called linear stability analysis. Before we go much further, let us understand this word stability, what does stability mean? If this just simplest way to understand this is using a mechanical idea of stability, where let us say I take a ball in a trough, let us say the trough is filled with some liquid, we all know from common observation that if I leave a ball on the side of this trough, this is going to roll down may be even over should a little bit, but eventually it will come to rest right at the bottom.

So, if I take a trough of oil, then this is going to come to rest and this is what we call the equilibrium condition.

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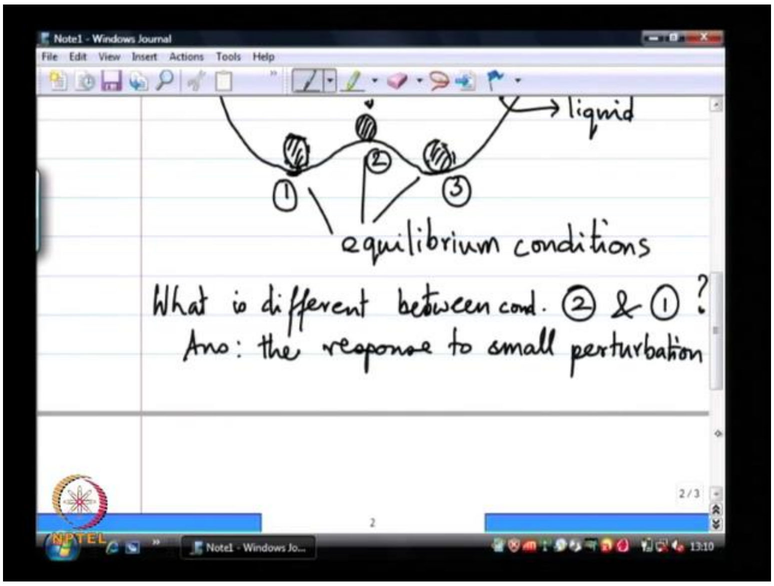

Now, if I take another trough, that is also and if I drop this ball here and let us say this is all liquid, it is possible that it could come to rest here or if I am very lucky it could come to rest here, where it over shoots and then you can clearly see it has to happen by chance or let us just say I gently placed it over there that is quite possible. Now these two are both equilibria, so both of these. So, previous was a case where there was only one equilibrium solution, one equilibrium condition whereas; here is a case where there is more than one equilibrium condition. In fact, if I want to be precise there is three equilibrium conditions.

So, what is the difference between this equilibrium condition and these 2 equilibrium conditions and the one where the ball is sitting on top? The fundamental differences, that if I was to take the, I will number these equilibrium conditions as 1 2 and 3 and, I am asking this question, what is different between 2 and 1. I am going to leave 3 out because 3 is the same as 1 for now.

So, what is different between 2 and 1 as far as this a between conditions 2 and 1. So, the difference is in, the answer is in the response to small perturbations, if I took a ball at condition 1.

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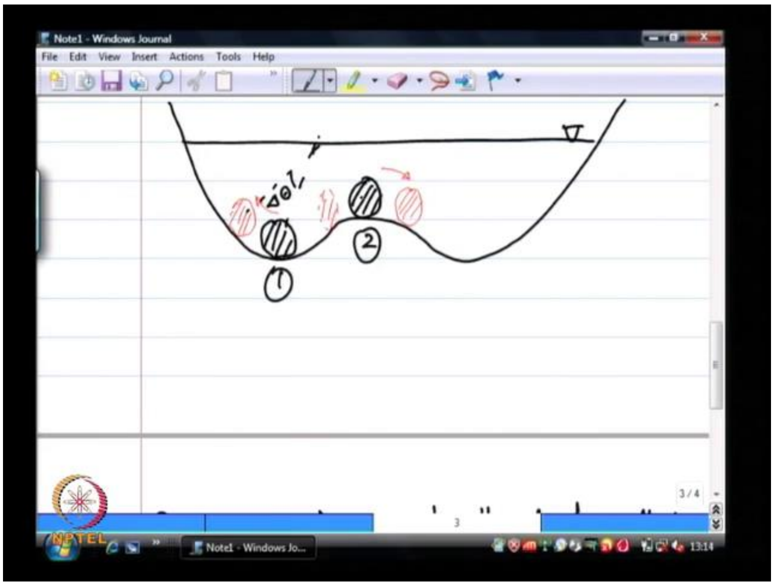

I will redraw the figure here just for clarity, if I displace it slightly, what the ball will do is essentially trace it is way back towards equilibrium condition 1. If I apply the same logic to this, if this is the direction of displacement, the ball is going to or the ball is going to move away towards 3. If I displace the ball on this side it is going to come back towards 1.

So, the nature of the perturbation is going to dictate the new equilibrium solution it is going to seek whereas, in case of 1 whether I perturb it to it is left or to it is right it is going to a comeback to 1, it is going to attempt to comeback to 1. Now the reason you have to understand to 2 things here, 1 is an equilibrium condition, 2 is also an equilibrium condition. And if I perturb the system from it is equilibrium position essentially the tendency to come back to the same equilibrium position or to go to new equilibrium position is the only distinguishing factor between 1 and 2.

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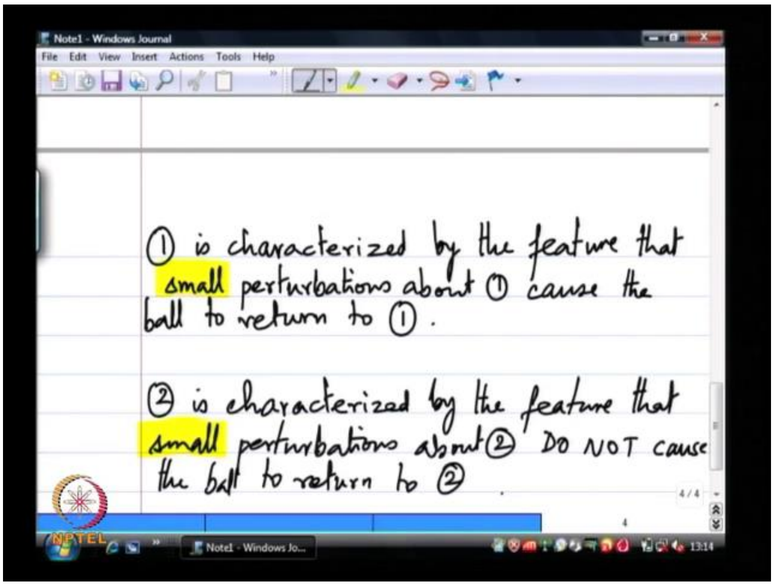

Therefore in the case of 1, 1 is characterized by the feature, 2 is characterized by the fact. Now it is very important to understand one thing here, these are only true when you are dealing with small perturbations think of what I mean by small. If the perturbation let us say if I take this sort of an angle, if that perturbation where to become large, whether it is in 1 direction or the other let us say if I perturb it to where I move the ball to 2 exactly to 2 clearly it is not coming back to 1. So, what I had written in the previous slide will no longer be valid if delta theta is on the order of the angular displacement needed to take it from 1 to 2.

If I go further then it goes to 3, which is also a case where the statement is not true where as for small enough perturbation. So, really speaking in a mathematical sense small means infinitesimal small. So, we are only looking at perturbations that are infinitesimal small in comparison to the mean state, that if we want to see if I just move it to it is left or to it is right slightly what will it do. So, this is, if I take several equilibrium solutions of let us say in this case the equilibrium is simply a force balance, where the horizontal forces and the vertical forces are all in balance then that is they are all in balance had only 3 points in that trough and 2 of those 3 points are alike in their stability characteristics, which in this case are 1 and 3 and the point 2 is different in it is stability characteristics. So, minds you they are all equilibrium solutions, it is just at 1 is stable 3 is stable 2 is unstable. So, what we are going to conclude from this 1 and 3, 2 is an unstable equilibrium condition.

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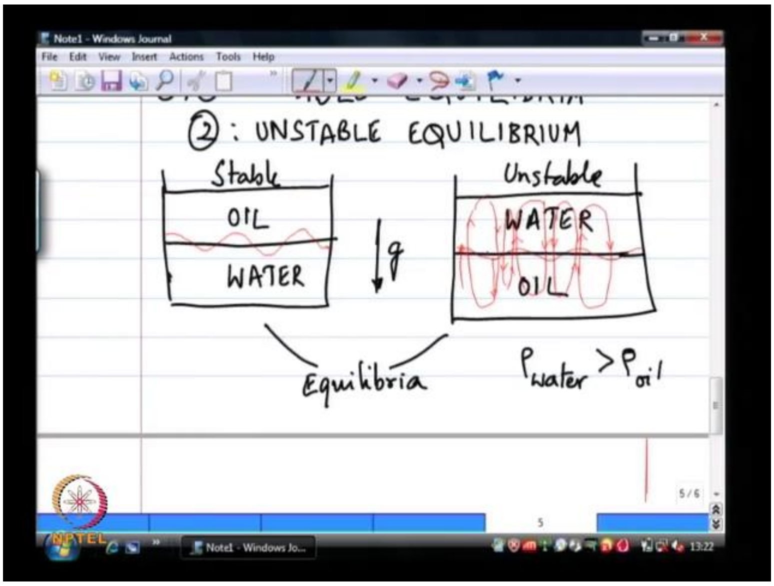

So, now what is this mean for fluid mechanics and what does it mean for sprays in atomization. If I take a condition, again where let us say I will take a trough I will take a simple case, where I will place oil above water. So, just I have a beaker of oil sitting above water in a general in a gravitational field just like that, what do we learn from this? Oil, it is lighter than water. So, it is pretty much, this is an equilibrium condition.

Now, if I ask the question, if I gently place water above oil and I ask you the question is this equilibrium condition, the answer is yes. So, both of these are equilibrium even though density of water is greater than the density of oil both of these are equilibrium conditions. How do I know that? You can in fact, the simplest way is you can actually experimentally do this, you can take a little bit of oil in the bottom of a beaker and very slowly pour water on top, not really pour it is almost like you are placing water on top and you can place a nice thick layer of water on top of oil, it is you will be able to see it.

But if I was to take this beaker of oil and water, and give it a slight impulse, just give it a slight jerk, the oil and when the oil is on water, the oil and the water meniscus both sort of undergoes some oscillations, but essentially come to rest in the same configuration. Whereas; when you take water on top of oil and you give it a slight jerk, overtime the water would come down and the oil would go up, you can see this experimentally those of you that have never done this it is a very simple experiment to do - go in to your kitchen today and do this experiment. Try to recreate this unstable equilibrium condition, and this stable equilibrium condition and see for yourself that is indeed and equilibrium condition unless the meniscus is perturbed.

So, what this little like shake or a jug doing to the beaker? Essentially what I am doing is I am taking this meniscus and perturbing it just like that, now the top meniscus is also perturbed, but that is less important in comparison to the water oil meniscus. So, as soon as this is perturbed the fact that right around here I have an opportunity for the heavier fluid to come down and the lighter fluid to go up is going to set up a motion in this direction that is going to cause it to become go to this stable condition of oil on top of water.

So, every one of these little perturbations is going to set up a roll like that, now I did not draw these intermediate rolls here, but essentially this is also a roll there, a roll there. So, as soon as I perturb the meniscus slightly, I will create these rolls what I mean by roll is a condition where the heavier fluid wants to come back down to the bottom of the trough and the lighter fluid wants to rise up because that is the more favored equilibrium condition.

So, the question of whether fluid mechanic system is stable or unstable, is determined by whether a perturbation of let us say in this case the meniscus is going to die out and cause the meniscus to return back to it is original shape and condition or like in this condition will the meniscus grow. So, think of what would actually happen if you had the meniscus growing.

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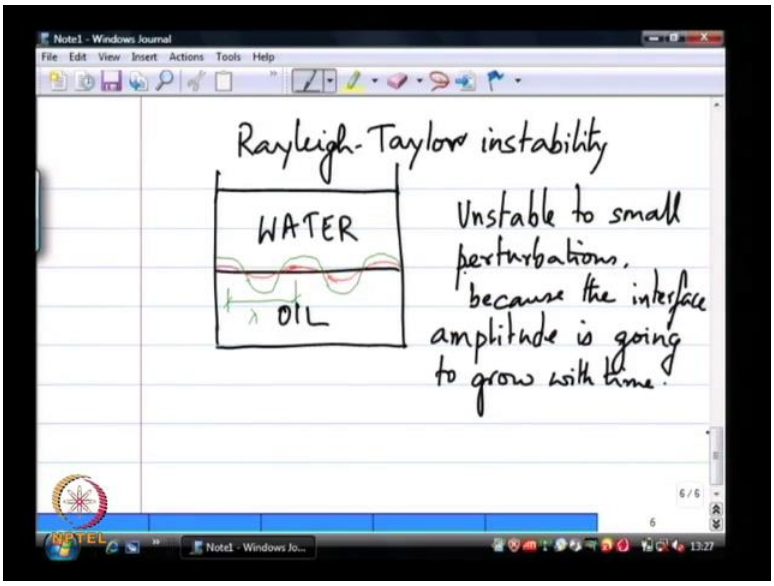

So, if I impose a small perturbation on this meniscus, overtime like because the oil wants to rise up naturally and the water wants to come down. The meniscus shape is now going to become altered and go to that condition. So, imagine I took this nice and flat meniscus with the water on top oil on the bottom and I put let us say in the simplest sense a sinusoidal perturbation. A sinusoidal perturbation of this meniscus of some wave lengths, so let us say now I can also make this some wavelength lambda and that wavelength lambda remaining the same, the amplitude of the perturbation will grow. In the infinitesimal small time, as soon as I have let go off this meniscus from this perturbed shape because water wants to come down and oil wants to go up in the immediate time following t. So, t is slightly greater than 0.

What is the behavior going to look like? The shape of the meniscus is not going to be altered drastically. In other words the functional form let us say in the case being sinusoidal, except the amplitude will grow slightly. So, just by looking at whether the amplitude would grow or decay, in a short interval of time following the perturbation itself, we are able to characterize whether the perturbation is stable, whether the system is stable to small perturbations or unstable to small perturbations. So, this is the case where the system is unstable to small perturbations.

How do I know that? Because, the interface amplitude is going to grow with time; now this inter this kind of instability was is not something that is new it was discovered back in the mid 1800 and it goes by the name Rayleigh-Taylor instability. Now I mean we are only going to talk about this in a physical sense for now, but we will come to the mathematics shortly.

The rate of growth going from the red line to the green line that you see here, the rate of growth is going to depend on lambda itself keeping everything else the same. If lambda was short or long would cause the rate of growth to be different. And that is the 1 characteristic that makes the discussion of stability of fluid mechanic systems different from the stability of the mechanical system example that we discussed earlier. There the only perturbation possible is in the angular coordinate theta that we discussed. So, a slight delta theta and the force as a function of delta theta is attempting to bring it back towards theta equal to 0, the 1 equilibrium condition that we are after, right.

So, whereas, in the case of Rayleigh Taylor instability of superposed fluids I can impose infinite different types of perturbations of different lambda, of different wavelengths and if each wave length has certain kind of growth characteristics then what sort of a perturbation do I really, should I really looking at. So, let me go back rephrase what I said in a more physical way. In the previous example, only 1 delta theta is possible whereas, here if I want to perturb this meniscus as a sinusoidal function there are an infinite number of sinusoidal functional perturbations possible of that flat, initially flat meniscus.

But each perturbation has a different growth rate associated with it. The rate of growth of the meniscus is different for different lambda. So, what is it that I, in other words I could also have a certain lambda. Let us say for example, if I put a very small lambda perturbation on it. You will see from the mathematics later on that in fact, it will be stable to that perturbation. Just because it is stable to 1 perturbation does not mean it is stable to all perturbation. So, in other words we have to sort of qualify our discussion of stability from their earlier example, and say that for a fluid mechanic system to be called a stable equilibrium system. That system would have to be stable under all kinds of perturbations.

So whatever be lambda, if the meniscus was to return back to in this condition, with water on top and oil on top then you can say that this is a stable equilibrium condition. So, with oil on top and with water at the bottom it does not matter how you perturb the meniscus, as long as the perturbation is small in comparison to let us say the thickness of water and oil, as long as the amplitude of perturbation is small, it will return back to the condition where oil is on top, water is on bottom, the meniscus is flat and everything is at rest. That is the equilibrium condition that I know there and that will come back, it will come back to that condition irrespective of what kind of perturbation I gave, whereas for this particular situation there is a range of lambdas for which oil wants to reverse go back to the top water wants to come to the bottom.

There are lambdas outside that range, for which it will come to this condition, but that does not mean that the system is stable, it is has to be called unstable because there is at least 1 lambda for which the system will go to another state. So, how do I know which lambda is equal, is actually important. So, the way we approach the answer to that question is we say that I do not know what lambda is possible, because if take water on top, oil at the bottom and I jerk this container slightly, I do not know what sort of a lambda I imposed on the meniscus.

So, we are going to assume that somehow the meniscus has a way of sampling all the possible values of lambda from 0 to infinity. Now if the growth rate is different for different lambdas, then there is at least 1 lambda which has the maximum growth rate that is going to somehow grow at the fastest rate in comparison to the rest. So, even though initially all values of lambda are possible we can look at least 1 situation where the lambda causes the growth rate to be much higher than all it is all the other values of lambda.

There is one value of lambda for which the growth rate is higher than all the other values of lambda. This growth rate, this lambda is eventually going to dominate as far as the dictating the shape of the meniscus itself is concerned slightly later on after the perturbation has been introduced. So, what we will see is that there is one value of lambda that will survive this competition even though I start out with the meniscus which is purely random, so I have like lots of different lambda superimposed on there, one particular value of lambda would out complete the rest in the rate at which it grows.

So, that is essentially the sort of physical understanding of the theory of linear instability analysis - perturbations are small, the meniscus retains it is shape at least in the harmonic form. So, if in this case I impose sinusoidal perturbation, the sinusoidal nature of the meniscus shape is not destroyed in the immediate times following t equal to 0, but the amplitude grows. So, if I have lots of different lambda each lambda is associated with the certain amplitude and the amplitude corresponding to that lambda grows at a rate as though it is the only superposed disturbance. So, we are going to break this problem down in to a linearzed version of the problem, where in solutions to every given lambda can be superposed to find the solution to a general kind of a perturbation.

So, let us go back now, and see what are all the different kinds of instabilities that we see in a fluid mechanic system and we will try to give some physical arguments for the kinds of instabilities.

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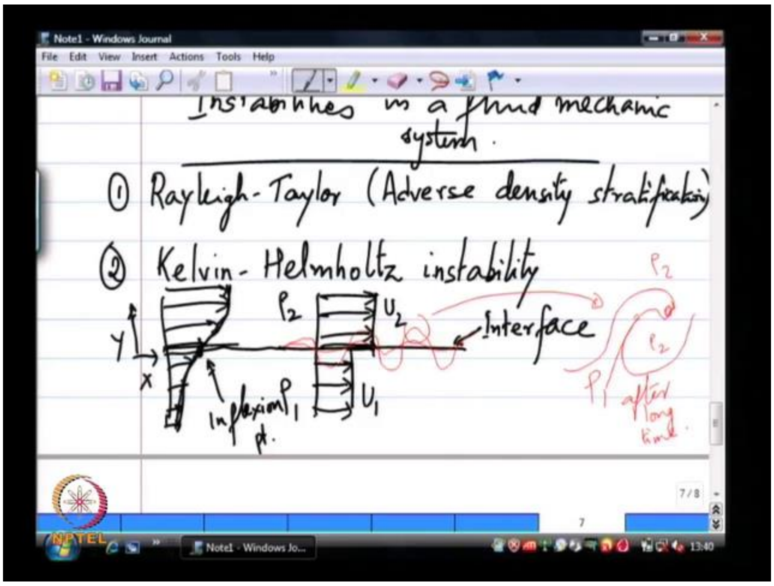

The first one, we just saw moment ago is called Rayleigh Taylor instability. This is essential due to adverse density stratification. That is the heavier fluids sitting on top of lighter fluid is not completely the most stable of situations, the lighter fluid wants to go up and the heavier fluid wants to come down therefore, this is cause for an instability. The second is what we will call Kelvin Helmholtz. So, what do you mean by this Kelvin Helmholtz instability? Let us say I have a fluid of density rho 1 and rho 2, the fluid here is moving at a velocity U 2. The fluid here is moving at a velocity U 1 and this is the interface.

So, the interface because, the U 2 is different from U 1 there is a relative motion and that relative motion is cause for the meniscus the interface to become unstable. So, overtime I could get, I am going to just draw some little cartoons that are representative of what happens. So, if this meniscus is slightly perturbed, then there is again a certain range of lambda for which this perturbation would want to grow. In fact, over time you can see some sort of a like waves, where you know this kind of a trough evolves into a wave that this is now fluid rho 1 and this is rho 2.

So, the meniscus, you can see some sort of a rolling up of rho 1 fluid into rho 2 region. This is after a very long time, we are not really concerned about that for now, but all I want to know is will the meniscus grow or decay. A more generalized version of this U 2, U 1 situation is U 1 is a situation where I have a smooth transition, it does not have to be an abrupt step like I describe with U 2 and U 1.

So, if I have some velocity profile, U as a function of Y what happens to the to this velocity profile if the fluids were to continue to flow. The answer to that is that you will find that because this kind of a velocity profile has an inflection point. That inflection point is sufficient to ensure that this velocity profile will be unstable. So, where do I actually see something like this? So, think of an atomizer.

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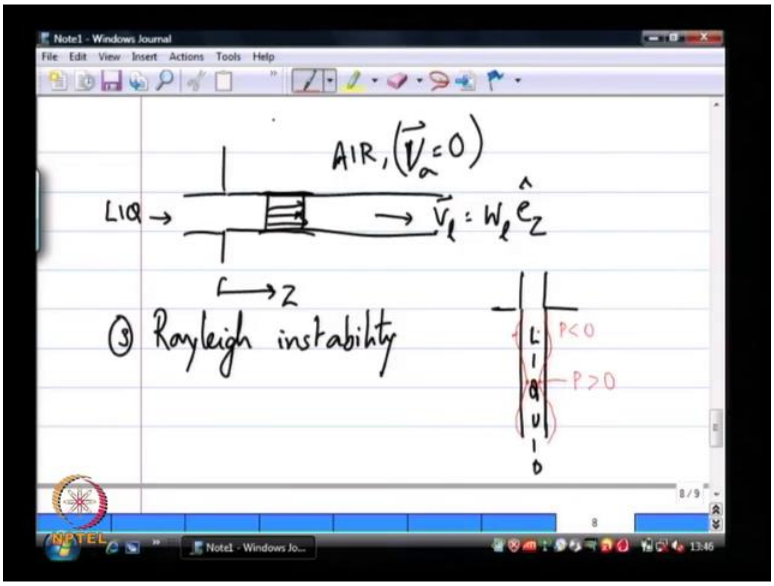

So, now I am injecting liquid through a little hole, I have created a cylindrical column of liquid and this is air outside, the air velocity is 0, the liquid velocity is some w l e z.

So if I was to draw the velocity profile and the air is at 0 velocity. So, this meniscus here is subject to this relative velocity between the fluid moving fast and the outside air being at rest. Where did we encounter this kind of a physical situation? This is your diesel injector. So, a diesel injector has this kind of a velocity profile, we want to see that if this velocity profile if this was a nice thick pencil of this liquid coming out to the diesel injector and into perfectly quite air where there was no source of perturbations at all that pencil of liquid would remain exactly the way it is. But if I have some sort of perturbation either in the air or in the nozzle or any other source I want to know if those perturbations will grow in amplitude or decay in amplitude if they grow in amplitude that is the case where I could cause the interface to ultimately break up.

If they decay in amplitude; that means, I am going to create what looks like a fire fighting nozzle. So, if I want, if I am a fire fighter earlier instability analysis like this is extremely important for me to know the operating conditions under which any perturbation I introduced to this flow is not going to cause into break up if there is such a possibility. So, I can just shoot a let jet of liquid towards the target. So, this is simple example of Kelvin Helmholtz instability.

The third is what we call Rayleigh instability, for this I will take a let us say a liquid column, I am going to create now liquid that is just to make the distinction with Kelvin Helmholtz instability, it is easy to understand this in the context of nothing moving. So, if I somehow created a column of liquid in micro gravity. So, I have a cylindrical column of liquid, that is stretching, that is infinitely long this infinite column of liquid is left like that would remain like that forever, does everybody look at, I want you to imagine - a long column of liquid like a pencil of liquid that is left in micro gravity with no end caps. So, I am looking at an infinite column of liquid on either side if this infinite column had a nice smooth cylindrical interface there is nothing that causes in to a break up.

In other words the pressure inside because of this curvature of the meniscus the pressure inside would be slightly greater than the pressure outside, but that inside pressure is the same everywhere inside this cylindrical column and the outside pressure is also equal everywhere outside. So, there is no net force acting on any element of fluid for it to transition to another state whereas, if I took this infinite column of liquid and perturbed it slightly just like that.

Now I have created, because the curvature here, is smaller the radius is smaller, curvature is larger because of that the pressure here, would be higher than the pressure here. Assume for a moment initially the pressure everywhere was 0 inside the liquid and slightly negative outside just to make sure the force is balanced out, we will see this mathematically in just a moment. I just want sort of introduce you to the physics of how these different kinds of instability work. If I take a column of liquid, if there was no perturbations on the column of liquid it will essentially stay like a cylindrical column forever.

Whereas if I now impose small disturbances because I have introduced curvature differences on that cylindrical sheet. The curvature here is different from the curvature there because of that the surface tension force here is different from the surface tension force, where I have a crest. So, where I have a crest versus where I have a trough, I have different surface tension forces because of that under certain, for certain lambda values this amplitude of the disturbance could grow. And this growth is entirely determined only by surface tension forces nothing else, it is because everything else is quite no gravity, no movement anywhere I could take a cylindrical column of liquid and if I impose all different kinds of lambda the rate of growth of one particular lambda would be faster than rate of growth of all other kinds of lambda.

So, you will see that quickly, that value of lambda out completes everything else. So, assuming initially you had a chance to impose all the different values of lambda you will see that because of this competition helping one particular value of lambda that shape, mode shape will show up as the shape of the cylindrical interface. This is the reason why when you have a faucet that is dripping it creates a very uniform drop size. So, anywhere that you see order anywhere that you see sort of a uniformity in a length scale the length scale is a easiest for us to perceive, but even time scale would be quite, if the same argument would also be applicable to time scales.

So, any physical system that you see out there that has that shows sort of a repeated pattern either in length or time chances are that, that sort of a system is governed by a state, that was initially unstable and when several different values of lambda where, several different perturbations each of a different lambda was given to that system one particular wavelength outcompeted everything what you see finally, with your naked eye is the result of that competition and having only one particular value of lambda. There is some very simple, but classic examples of this. On some days when you look at the sky you see clouds arranged in almost like striations with the spacing being very uniform across. The spacing between 2 rows of cloud you will see on some days is very repeatable. So, this is like a repeated pattern in the distance between like 2 rows of clouds. That is a case Kelvin Helmholtz instability.

There is also case like that in fact, something like that is also governed by Rayleigh Taylor instability. Another case is waves on an ocean or waves on a repulse on a lake, the spacing between a pair of repulse on a lake on a quiet nice day with where there is a gentle breeze flowing the repulse are very uniformly spaced. The reason for that is there is no source, out there that is exactly giving repulse at that wavelength this is just that disturbances of all wavelengths are available in the air around in the water inside etcetera.

But one particular value of lambda outcompetes in the rate at which it grows. So, what you see if I took a perfectly flat meniscus of a lake with no repulse at all and gave all the different possible wavelengths one after the other, the one that has the highest growth rate would show up, quickly in time and that is what we perceive. There are many examples like, this in a where like for example, if I took water on top of oil and you perturb it is slightly it could be like a little tap on the side of a beaker, you will see that you will see holes formed on the meniscus through which the water is jetting down towards and on between a pair of holes you will have oil rising upwards and the spacing between these is very repeatable and very uniform.

This is actually reasonably simple experiment to do with a glass beaker water and oil in a kitchen. I strongly recommend you do it, you can see how repeatable that pattern is without really having any kind of a control over it. So, it is not like I have to impose a certain kind of disturbance, I have to just give a small impulse, an impulse if you look at it in the Fourier space is nothing but superposition of every possible wavelength. So, I can take every possible wavelength and deliver it to the system and one particular wave length is chosen by the system. We can actually go well beyond all of these mechanical systems. Take the spacing between leaves on a tree, leaves on a little branch you will find some kinds of trees have a very uniform spacing between the leaves on a simple branch. It has nothing to do with some controlled freak inside the branch that is causing leaves to sprout up exactly at that interval. A system like that wherever, you see a natural system with repeated patterns it is in general characterizable by this kind of a theory, alright.

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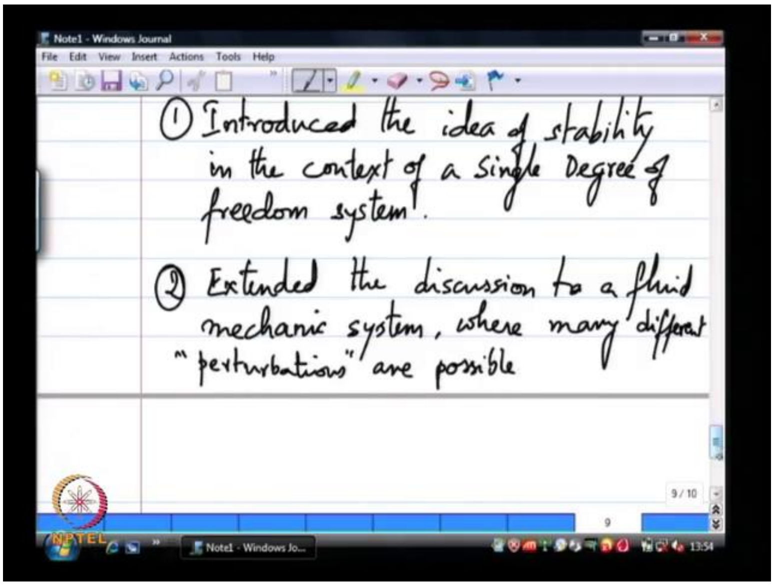

So, the let us just quickly recap what we learn today. The first thing we did is we introduced the idea of stability, the ball in a trough is what in mechanics we call a single degree of freedom system; there is only a one degree of freedom for the ball. And then we extended the discussion and then finally we discussed a few different flow systems and the related instabilities.

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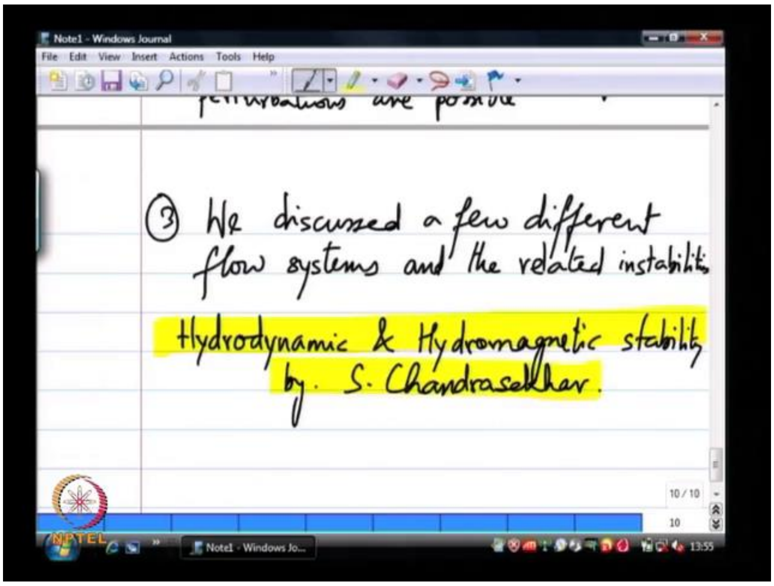

An excellent reference in this regard, is what is called Hydrodynamic and Hydromagnetic stability; it is a book by S. Chandrasekhar. This would be a very good research to read.

Thank you.