Introduction to Finite Element Method
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Lecture - 9

In the last class we were talking about, if you remember we are talking about, Mises criteria. We called this, as I told you in last class, as failure criteria in the earlier classes and actually they are yield criteria and so on.

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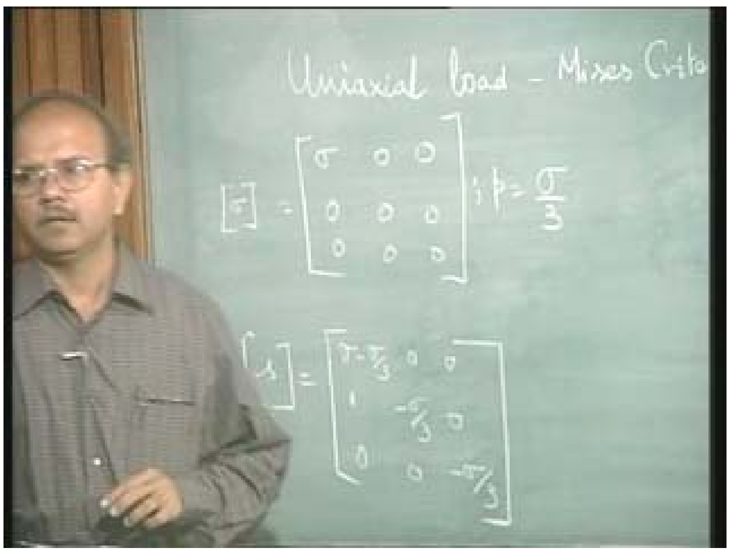

We had come to a stage, where we said that we will find out what this k is?

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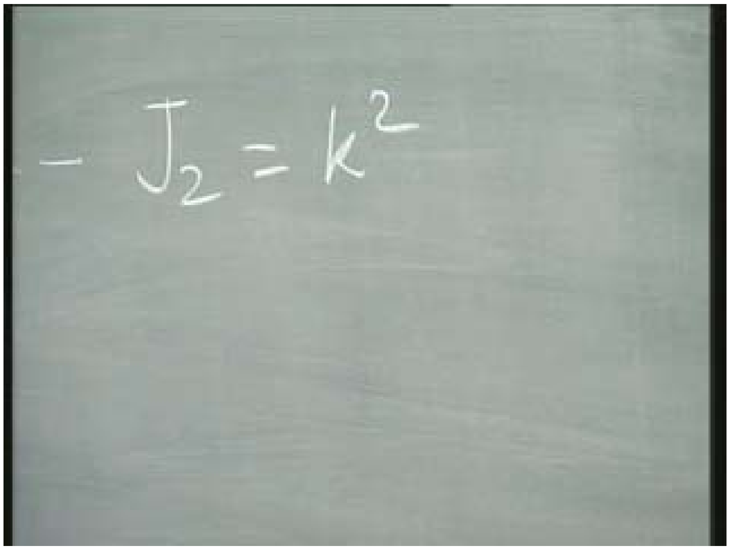

We said that Mises criteria or the yield function such that it can be written in terms of J_2 and hence we call this as J_2 flow theory and said that J_2 is equal to k squared. We said that our main aim is to find out what this k is? That is where we stopped in the last class. We also said that we can get that k from a uniaxial specimen. When I conduct a uniaxial test, then the stress tensor can be written in this fashion where sigma 00 000 000 signifies what we call as the stress tensor. Please note that this has to be split into a hydrostatic part and the deviatoric part. We said, we called this as additive decomposition and the pressure p is determined by one third the trace of this matrix which happens to be sigma, in this case. So, p is equal to sigma by 3. From this, we can write down the s matrix straight away by subtracting p from sigma. That is where we stopped in the last class, and now writing that down, s is equal to sigma minus sigma by 3 00 0 minus sigma by 3 0 00 minus sigma by 3.

What is the next step? The next step for me is to find out what is J_2?

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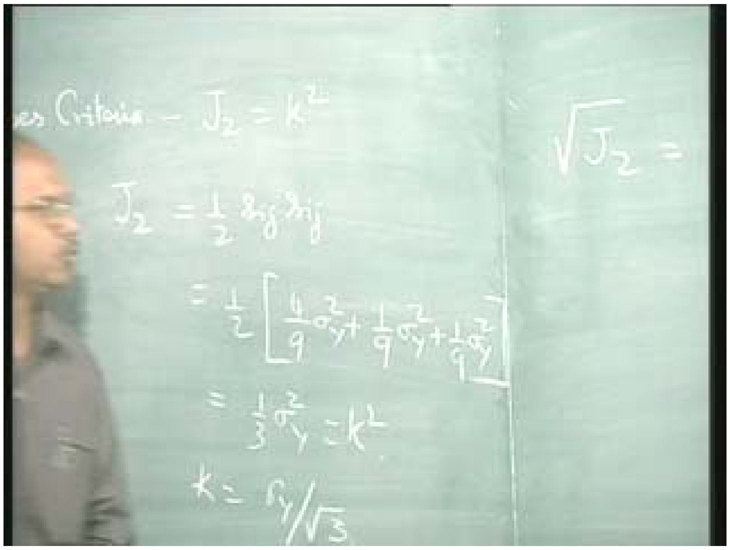

What is J_2? Half s_{ij} s_{ij}; remember that there is a summation on i and j. In this case what happens to this s_{ij}? It is just half into s_{11} squared plus s_{22} squared plus s_{33} squared, because others are zero. That happens to be, what is it? That happens to be, this is 2 by 3 sigma, so, 4 by 9 sigma squared plus second term, one third, so, 1 by 9 sigma squared plus 1 by 9 sigma squared and that happens to be 1 by 3 sigma square; 6 by 9 into 1 by 2, and so on. We stop for a moment here and get back to my previous stress tensor.

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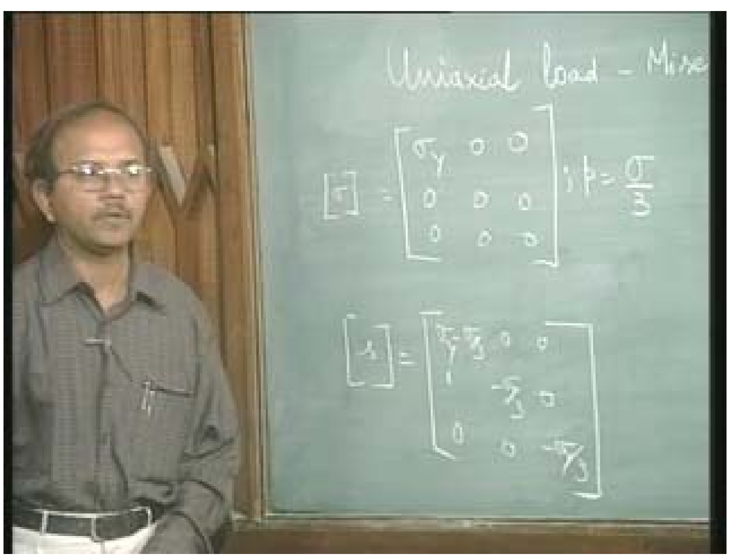

We very well know that yielding now occurs in a uniaxial case when sigma reaches sigma_y or the critical sigma in a uniaxial tension is sigma_y. I have to put here to get my critical k or the k, which is of interest to me by substituting for sigma as sigma_y. So, I put here sigma_y and hence I get J_2 to be sigma_y square by 3. What should that be equal to? k square and hence k is equal to sigma_y by root 3.

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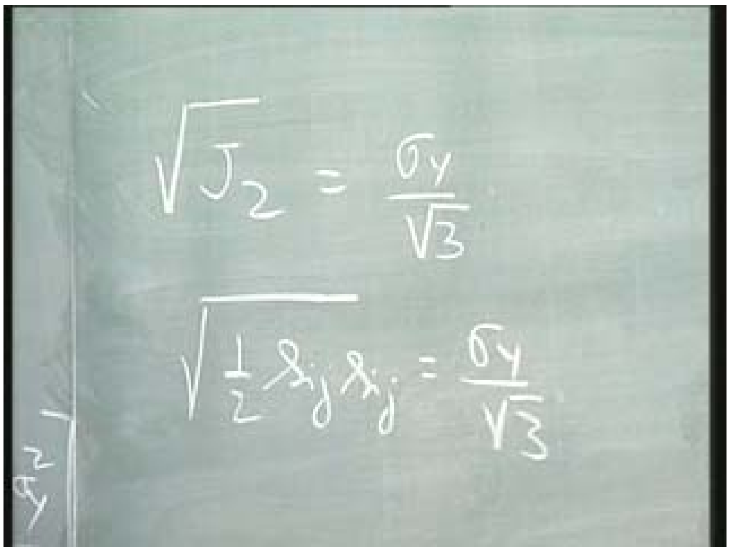

Now, I can rewrite rather my Mises equation as J_2 is equal to or root J_2 is equal to, what happens? Sigma_y by root 3 that means sigma_y by root 3 and what is root J_2? That is half s_{ij} s_{ij} is equal to sigma_y by root 3.

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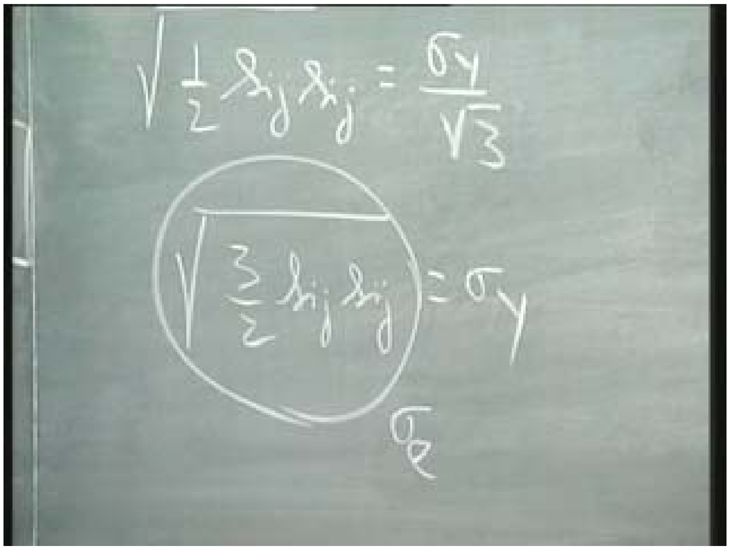

Just bringing that root 3 the other side, I can write this equation as 3 by 2 s_{ij} s_{ij} is equal to sigma$_y$. Now, I have a combination of the stresses, which can be compared directly with sigma$_y$ or I get one number from the 6 numbers to compare it with the material property called sigma$_y$ and hence I call this, whatever is inside this or whatever the root symbol or whatever is in the left as sigma$_e$, equivalent, because it is equivalent to a uniaxial stress, equivalent or Mises stress and so on.

Equivalent stress also means it is the same as Mises stress. Hence in a three dimensional case, I calculate this value, compare it with sigma$_y$ and tell you whether there is going to be yielding or not and that is what we have been seeing in all our earlier work or slides. Any question?

Now, we move over to another quantity called strain.

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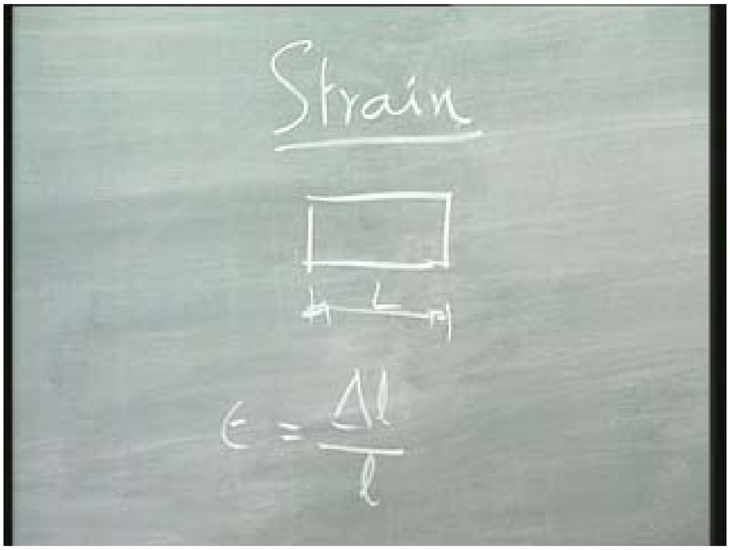

I have a very similar problem. Again, when I look at the strain, in your earlier classes you would have defined strain on the basis of change in length by original length. So, probably the only thing you would have studied or you would have known is that if I take a bar whose length is l, you would have said that if delta l is change in length of this bar, then delta l by l, you would have defined as epsilon. Please remember that in our earlier discussions on stress, we retained the fundamental definition of stress but made it more rigorous by taking a point, a plane and so on. That is what we did.

We will do exactly similar thing when we define now what is called as strain. Why is that becomes important is, because I have a body where the strain distribution can be different. What is length? There is no length to a three dimensional body. I cannot define a length to it and I cannot define a change in length to it and hence my definitions have to be very precise when I go to a three dimensional situation. No doubt, I will use this kind of a relationship, but it would vary as you are going to see now. We will get back to the definition of strain, now.

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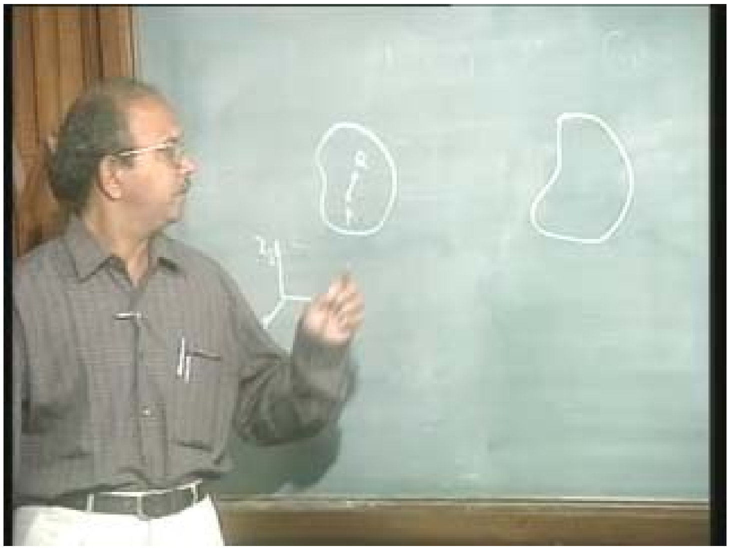

Let us put a body. Let me say that there is some coordinate system for this body. This body, let us say, after application of load is displaced in this fashion. We will come to this in a minute, what this means and so on. Let us complete the definition of our problem. Let us say that I am watching this point P. But since our definition of strain just now says that we are interested in change in length by original length, let me take another point Q which is very close to this P and look at this line element, which is joined by P Q. Both of them are very close, infinitesimally close.

Let us now watch this line element and see what happens to this line element? Let us say after deformation this body goes to this position. There is usually confusion between displacement and deformation. Let us try to understand a bit and this is only a nicety-?, people may not follow it so very closely. When this body, I have a duster, for example. This duster is just displaced and kept here. It is just displaced and kept here. Is there a strain in the duster? No, because this is a rigid body movement, as it goes to this place; it is just a rigid body motion. As a body, as a rigid body it is not deformed. As a rigid body it moves from one place to the other place.

On the other hand, I have a duster and start pulling it. As I pull it, definitely there is going to be a deformation to the body and this deformation is what causes strain. This is intuitivity. What do you mean by deformation, actually? What do you mean by, what you understand by deformation? You understood these two terms; I displaced it, I deformed it. There is a relative displacement or in other words the displacements are not uniform throughout the material particles of the body. Yes they are also displaced from one position to another position. When I pull it, there are some particles which have moved, but the movement of these particles may not be uniform. When I do all sorts of things to this duster, the movements of these particles are not uniform. This non-uniformity of displacement is what you would have intuitively called as deformation; deformed and we know that this is what causes strain.

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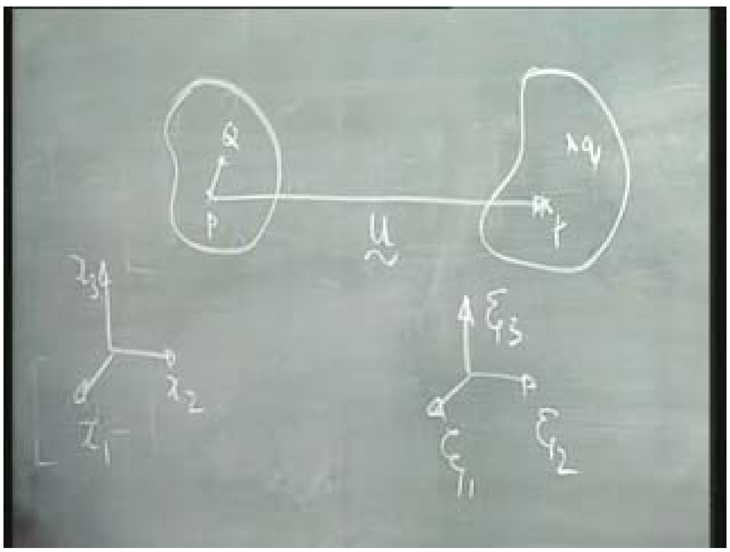

If the displacement of P and Q has to be the same to another point say small p and small q and this displacement vector is called by, say, a vector u and if this Q also happens to go to this q by the same displacement vector, then there will not be any strain. That is what is going to cause or that is what is going to be our basis for further definition of our strain. Look at this carefully. Let me say that I put a coordinate system for this deformed body. The coordinate system need not be x_1 x_2 and x_3.

Let me call this coordinate system as xi_1 xi_2 and xi_3. Do not worry about it, but just say that my coordinate system can be xi_1 xi_2 xi_3. What does it mean? It means that this P, this P which has coordinates say x_1 x_2 and x_3 now after application of load, boundary condition, whatever it is, at the end when the fellow reaches a equilibrium, has a coordinate which is given by xi_1 xi_2 and xi_3. They can be related by a vector say u and Q and P, sorry, Q and the small q can be related by another vector say u plus du.

This Q has moved to this small q through another vector u plus du. This is what we have been harping on; du has to be present. If du is absent, then there is no strain nothing happens. So, u varies in this whole geometry. Let us say that we can write down the coordinates of the new or position of the body, of the deformed body, in terms of x_1, x_2 and x_3.

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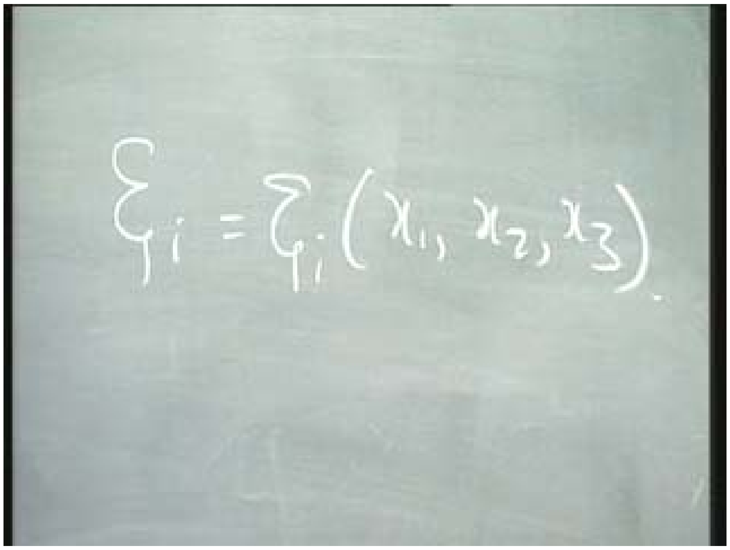

Note this carefully, this is going to be slightly confusing, but I will just explain that. xi_i is equal to xi_i x_1 x_2 x_3.

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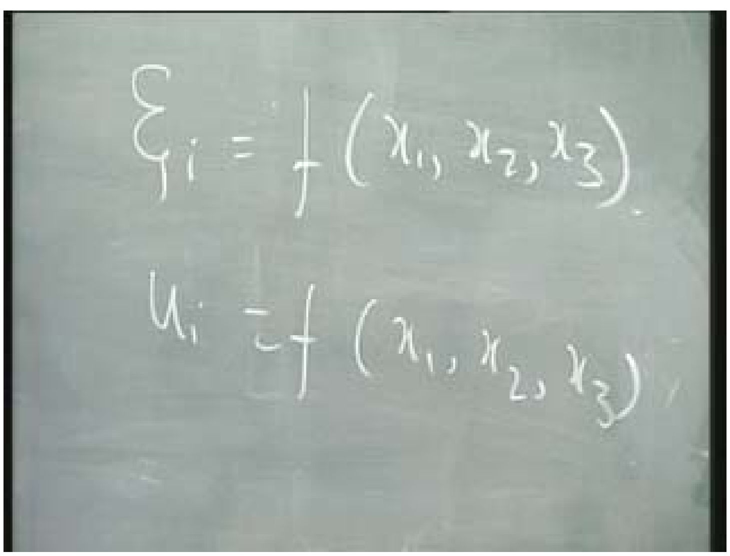

It is customary in your earlier classes to have written this as some function of, you will write like this and if you want to write u also, you would write say u_i as some function of x_1, x_2 and x_3 and so on. There will always be confusion between this function and this function. So, you will keep on putting f_1, f_2, f_3, f_4 and so on, so that is much more confusing.

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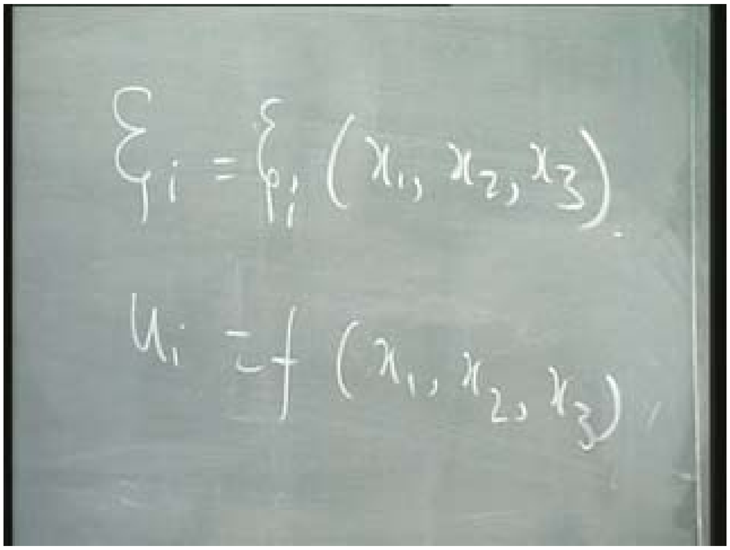

Please note in solid mechanics, this function symbol is replaced by what is there in the left hand side itself and is given like this. What does it mean? It means that it is a function of, it is a function of x_1 x_2 x_3. This xi_i is a function of x_1 x_2 and x_3. What it means is instead of saying xi_1 is equal to say f_1 x_1 x_2 x_3, xi_2 is equal to f_2 x_1 x_2 x_3, xi_3 is equal to f_3 x_1 x_2 x_3 and u_1 is equal to f_4 and so on and so forth, keep on writing like that, that is very much confusing. After sometime you will not know f_4 is for what.

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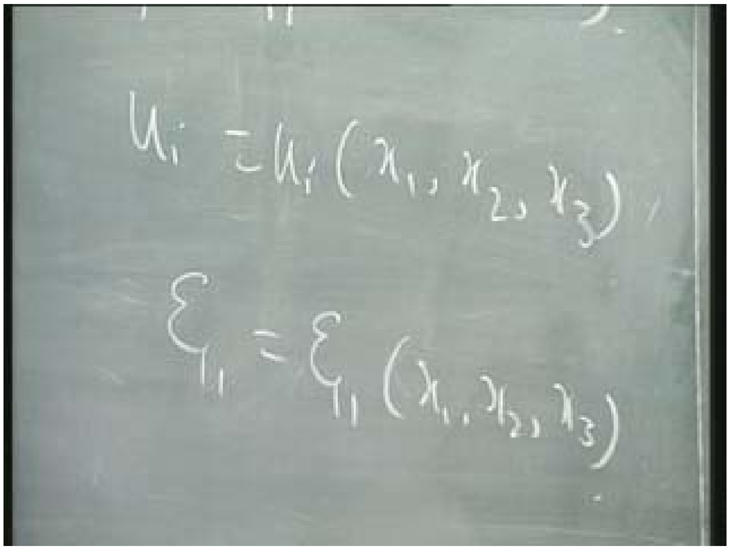

Instead of that if I write say xi_1 is equal to xi_1 x_1 x_2 and x_3, instead of f_1, I will know exactly what this function is? I can write down this as say, this is $2x_1$ plus $3x_2$ squared plus $4x_3$, something like that; that is a function. What is this function? This function is for say xi_1. For xi_2, I can just have xi_2 is equal to x_2. That is another function. In order that I distinguish between f_1 f_2 f_3 f_4 and so on, I write this as xi_1 is equal to xi_1 x_1 x_2 x_3. You understand this. This is symbol for function. Is it clear? Any doubts? So, this is what we are going to follow throughout. This is what most of the solid mechanics text books would follow. Both the symbols will be the same. So, here also, then we will put u_i.

What is this or what is it that I have to get? I have to find out what is the change in length? I am not going to I am i don't want to leave my tradition I do not want to leave my tradition, I still want to keep my tradition there. So, I have to find out change in length and hence I have to find out what is the lengths of the original line segment capital P capital Q and the new line segment small p and small q. Let me write down capital P capital Q.

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That is the original line segment which is here, the length of it to be, square of its length rather, to be dS squared. That is the square of its length. What is dS squared? What is dS squared? Can you tell from this? How do I find out dS squared? It is very simple. I have to define one more; correct, that is exactly what I am coming to. Suppose P is equal to x_1 x_2 x_3, Q is equal to x_1 plus dx_1 x_2 plus dx_2 x_3 plus dx_3 because this Q is very, very close to P and hence dS squared is nothing but dx_1 squared plus dx_2 squared plus dx_3 squared; that is very good and hence that can be written as in our new found symbol, it is written as dx_i dx_i.

Let us call this guy, small p small q length as ds squared. That ds squared is now dxi_i dxi_i and that I am going to write it as say $delta_{ij}$ dxi_i; sorry, excuse me, dxi_i dxi_j. My question is how do I find out dxi_i? How do I find out dxi_i? Let us see whether you understand whatever we have said now, dxi_i. Get back to this, get back to this figure.

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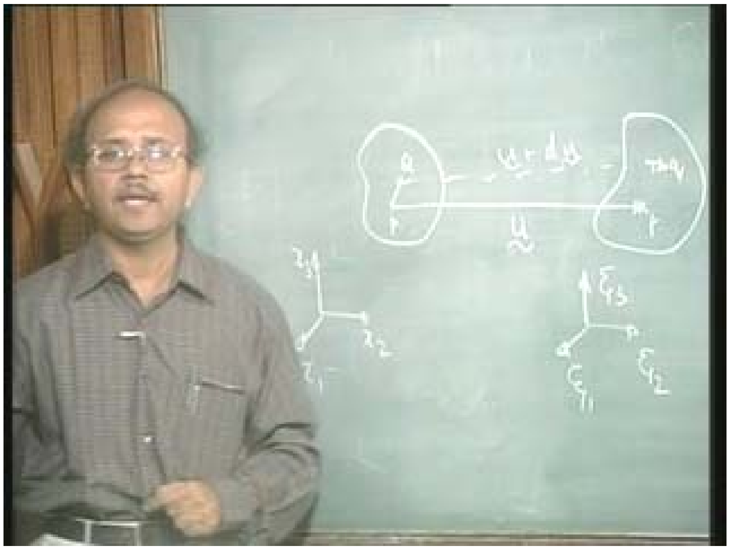

I said that this is u, this will be u plus delta u. Let me say that u plus say du vector. So, what will be dxi_i? I am now tracing q, so, that will be if there is nothing, if there is no change or there is du is equal to zero, what will happen? What will be q_i? Sorry, what will be the position of q, small q? Same; so, what do I write here? dx_i.

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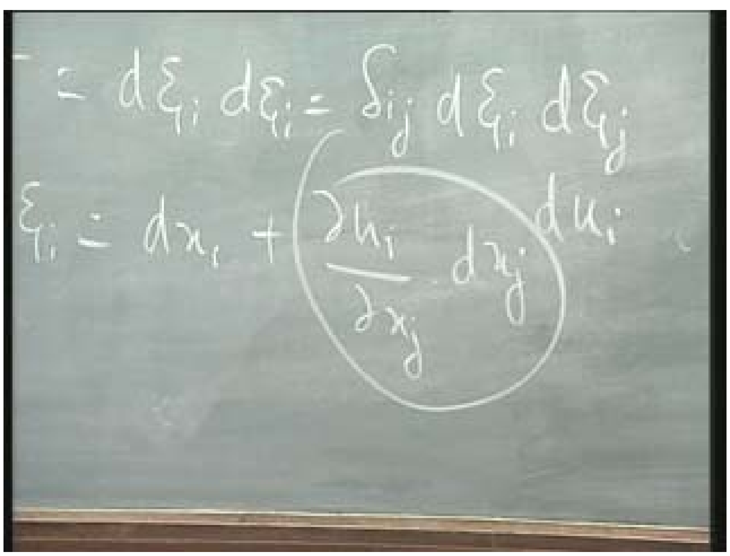

Very good; so, the position will be the same. Now, on top of it I have added du; so, plus du_i. What is du_i? Look at this; dow u_i by, ….., dow u_i by dow x_j into dx_j. This comes out from this expression. So this is du_i.

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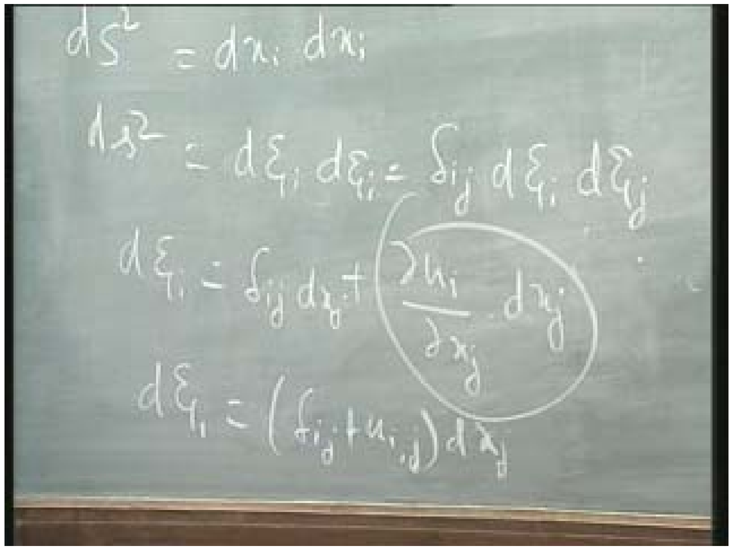

Now, let me write this dx_i as $delta_{ij}$ dx_j. I just write it, because I want to take that dx_j out and write dxi_i; that is equal to $delta_{ij}$ plus u_i comma j dx_j. Any question? Now, what I am going to do is to substitute this into ds squared, this equation small ds squared, that equation which is sitting here. Now, that is going to result in a slightly more complex indicial notation situation, which I am not going to do it and I leave it as an exercise because that is an excellent exercise for a student to do. So, I am not going to do. What I am saying is you just substitute it here in this equation and see what you get and compare it with the result what I am going to write now. I am not going to do that substitution. Nothing great, nothing; no big concept that is involved; only thing is that here you have to systematically write the indicial notation. That will be a very good thing to write and that will be your first problem for tutorial. How are you going to get the result which I am going to write now? Let me write that result.

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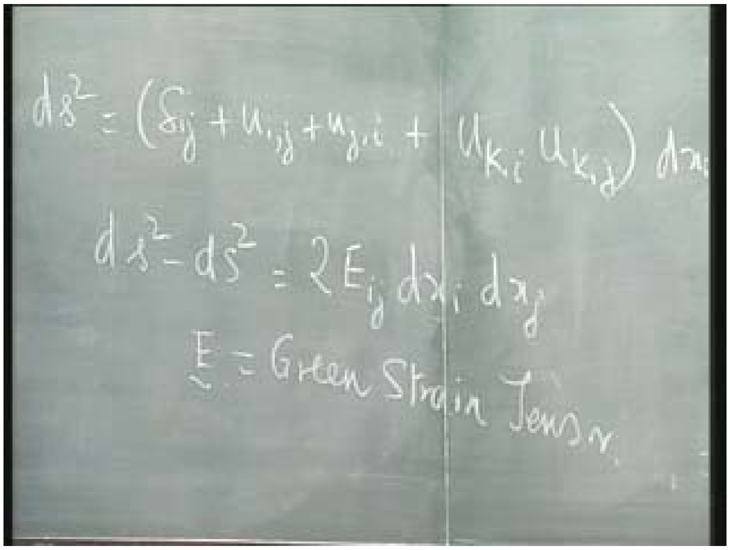

I will remove this, let me keep that part and hence I am going to write ds squared delta$_{ij}$ ds squared is equal to delta$_{ij}$ plus u_i comma j plus u_j comma i. Note this carefully, comma you know, I hope you remember dow u_i by dow x_j plus, let me remove that also; u_k comma i u_k comma j dx_i dx_j. Note this carefully k comma i, k comma j. This term means that there is a summation in k, you know that. I hope all of you remember what is d capital S squared? What is d capital S squared? dx_i dx_i which is delta$_{ij}$ dx_i dx_j; so, that is the first term. Note this carefully. Let me define a strain tensor. Let me define a strain tensor which is given by, ds squared minus dS squared is equal to 2 E_{ij} dx_i dx_j and call this as a, what is the strain tensor? E and call this as Green strain tensor. We will come to the explanations of this and see how best you can relate it to our earlier lessons.

This term or this term is very accurate. I have not made any approximation, if you had noticed, when I wrote down this term. I have not made any approximation; as it is, by simple calculus, for simple calculus I wrote this; I have not made any approximation there. So, at this stage let me make an approximation.

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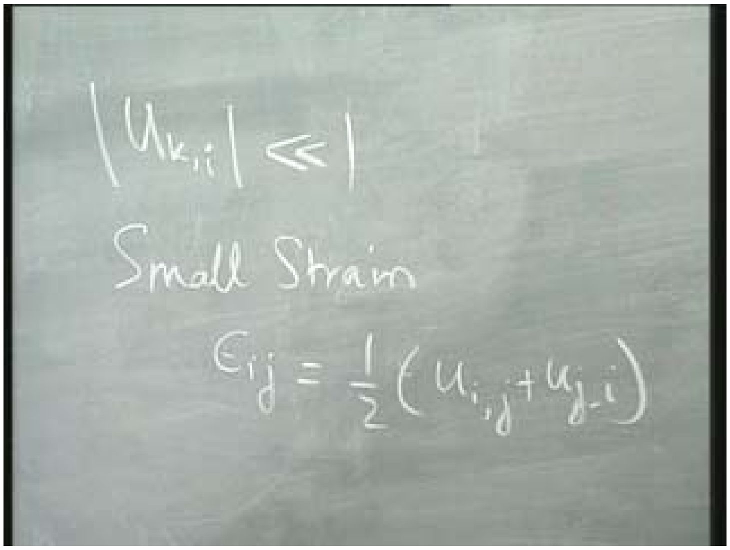

Let me say that the terms u_k comma i, or the gradients of u is very small when compared to, say, 1. The gradients of strains are very small. What happens now, when I say like that? What happens now? This last term here, u_k comma i, k comma j, this last term will vanish; will vanish which means that, that will go to zero. Look at this term and this definition. If you look at this term and this definition, it is obvious that E_{ij} is half of u_i comma j plus u_j comma i plus this term; that is what the definition is, because ds squared is this minus d capital S squared; minus d capital S squared, I said that it is minus $delta_{ij}$ dx_i dx_j and hence this chap and this chap go off and hence E_{ij} is equal to half of u_i comma j u_j comma i and plus u_k comma j.

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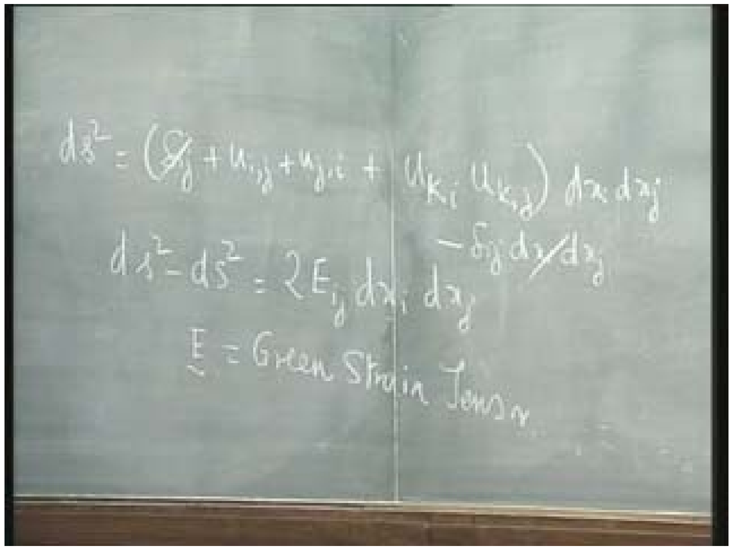

On top of it, I have put a restriction that the strains or the gradients of u are small. We call such a situation as small strain situation. We call this situation as small strain situation and we symbolize the strain in small strain case as epsilon_{ij} and epsilon_{ij} happens to be now what? Half into u_i comma j plus u_j comma i. But note one thing carefully that when I said small strain, I did not say u is small. It is a gradient of u that is small; u can still be large, but the gradient of u can be small and hence in which case, I can use epsilon_{ij} is equal to half u_i comma j plus u_j comma i.

Is this symmetric? This is a tensor. Why is that I have written half? You know, one of the questions which students ask is why they want to suddenly come and give one 2 there? Why do you want give it as half? We will come to that in a minute because if you want to give tensorial properties to epsilon, then epsilon_{ij} has to be defined as half u_i comma j plus u_j comma i because once I give tensorial properties you know what are the effects? I have lot of things to say about it, but before that let us see whether this is symmetric, epsilon_{ij} is symmetric?

From the definition itself, straight away, definition itself epsilonj j i is j comma i plus i comma j. So, strain tensor is symmetric. You would think by the way we define, though I have been harping on this issue of the change in length quite often, you would still be thinking that what is the relationship between this and what I know in from my earlier classes?

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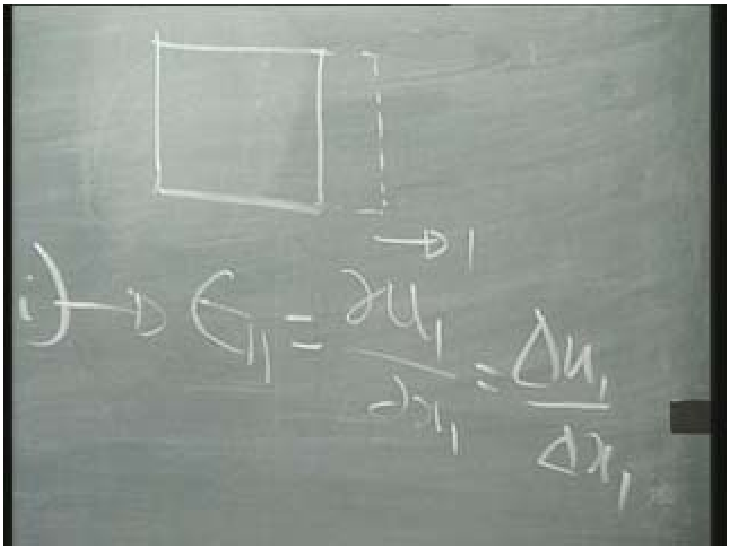

In order to just explain that let me take a small square; I hope it does not a small square. Let me say that the square becomes like this, where the change is say delta u_1. Now, I am interested in say epsilon$_{11}$, because I have elongated the square in the 1 1 direction. This is the 1 direction. What is epsilon$_{11}$ from this? What is epsilon$_{11}$ from this? Look at that; half of u_1 comma 1 plus u_1 comma 1. That means u_1 comma 1 that is dow u_1 by dow x_1, which is approximately from my delta ….. this equal to that is equal to dow u_1 by dow x_1 and this being a very infinitesimal thing so that is delta x_1 and hence epsilon$_{11}$ becomes what? Not 1; Epsilon$_{11}$ becomes this is elongated to delta u_1. This is delta u_1. What is that? This delta u_1 this delta u dow u_1 by dow x_1.

So, this is u delta u. This is delta u_1 by delta x_1. What is delta u_1? The difference between elongation of say 0.1 and 0.2; that is what we have been telling. (00:34:37) This moved by u, this moved by u plus du, so I have just put it together so that I said this moved by u where u is equal to zero; this moved by delta u. That is what I said; so, delta u_1 by delta x_1. What does it signify? Change in length by original length; so, this small strain definition still retains my first philosophy of change in length by original length. That is what comes out in this case, small strain case.

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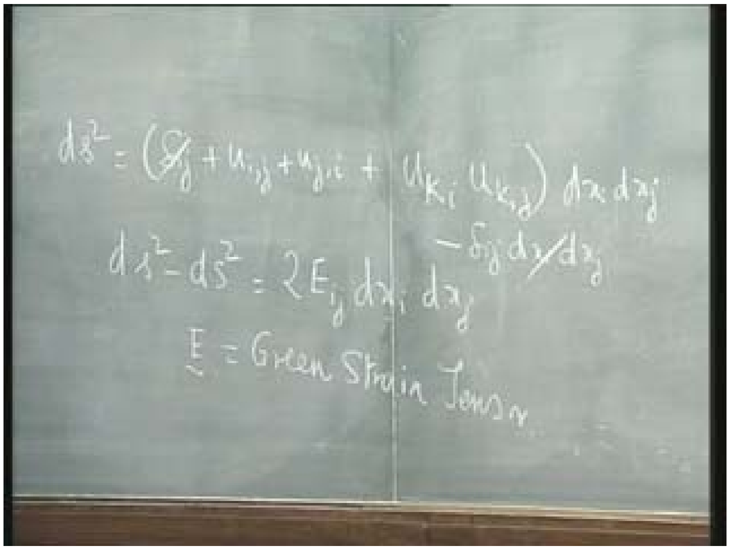

Come here for moment; look at this again. Physically, what is this situation? What happens during this situation? Look at this one. Physically, when will you say that it is no more small strain? What is opposite of it? Large strain; sometimes people call this as finite strain or what is the difficulty now when my strains become large? Look at this definition and look at my problem there. What I had put? No, no, no, no; that is not, I mean, we are talking about the issue of applying small strain or large strain. No; when this fellow becomes large, then original length and the final lengths are quite different. So, change in length by original length, simple definition may not be there. Then one person can come and ask me, why do you want to say that change in length by original length?

You guys may be smart; you may ask me why not I say change in length by new length? You can immediately look at say nominal stress and engineering stress and so on. So, that kind of confusion can be there and that is not a confusion; it is a very valid point and in order to take into account all these kind of valid points, the strain in the large deformation case is not measured by one quantity only. What I am trying to say is one of the quantities by which we can define strain is called Green strain. Another quantity is Almansi strain and so on. I am not going to be worried about all that right now, but nevertheless, you have to understand that the difference comes about because of the gradient of displacement.

When the gradient of displacement is small, we are not away from things what we know and we are right there, that the change in length by original length is still there. You can similarly look at a situation where delta x_2 goes up. Let us say that this is delta x_1 and this is delta x_2 and I am going to give a small problem for you to think for a minute. Suppose, there is shear strain, you know.

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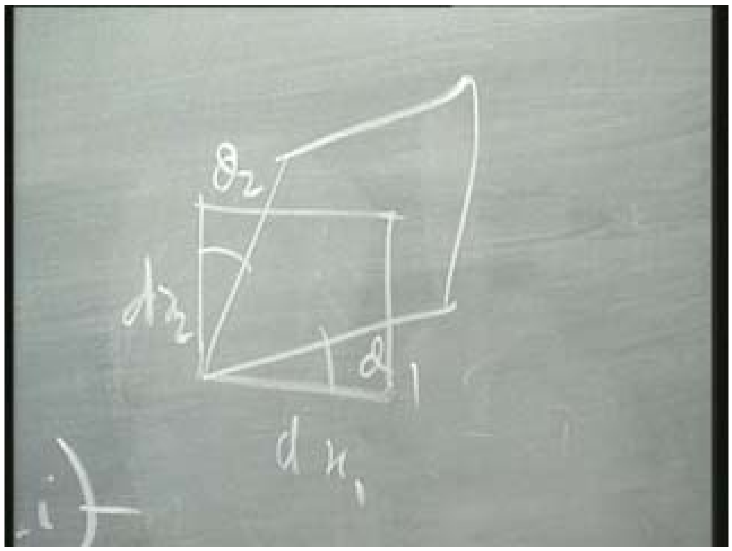

I want you to remember what shear strain is and suppose the body becomes like this. What is shear strain? Gamma, this is say, dx_1, this is dx_2. How do you define this shear strain, how do you define shear strain here? First thing is you have to note down what is $theta_1$ and what is $theta_2$ and then write down $theta_1$ and $theta_2$ and you would have defined what is called as shear strain, gamma to be, yes, $theta_1$ plus $theta_2$. What is that? What is that here? No, what is $theta_1$? dow u_2 because this is dow u_2 by approximately dow x_1 plus dow u_1 by dow x_2. Dow x_1, the original and the final lengths do not change much; that has been our small strain assumption. Though I have just put it for clarity like that it does not mean that there is so much of a change in length.

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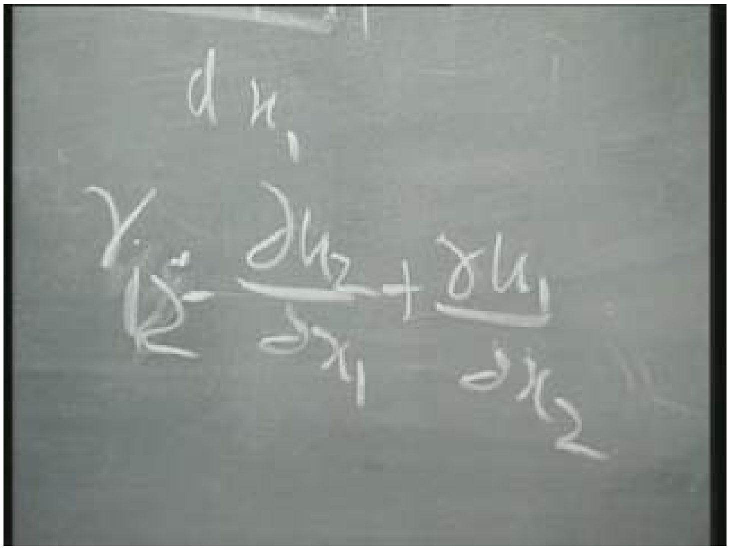

So, gamma, which you would have defined as dow u_2 by dow x_1 plus dow u_1 by dow x_2; this is how you would have defined. Let me call this is $gamma_{ij}$ or $gamma_{12}$. Let us say $gamma_{12}$ to signify that we are just in the plane 1 2. Compare this with my $epsilon_{12}$. What is the difference there? Half; so, this you have to be very careful. In fact we have introduced deliberately a half in our definition so that epsilon becomes a strain tensor; epsilon becomes a strain tensor, if not we could have retained this. In fact many books retained this or many course retained this and it is important to find out whether you are working with $gamma_{12}$ or $epsilon_{12}$. The strain what you are looking at is whether gamma or epsilon; the difference is half.

Here, though we defined it very, very, I would say tortuously if you want to call it, I know that you have to follow really every step. Thing is we are not very far off from what we know; we have put that nicely down in mathematical signs. This equation is very, very crucial for finite element analysis. Why? Basically because, this gives me a relationship between strain on one hand and displacement on the other hand. Remember, go back to our finite element days. Now, what we did about 4 or 5 classes back? ku is equal to f. So, what is that you are going to solve? You are going to solve for displacement. Afterwards, I have to get strain. How do you get strain? Strain displacement relationship and what is our next step? Stress strain relationship; let us be as rigorous as this so that all our definitions can be very nicely written down.

You know, $sigma_{ij}$ $epsilon_{ij}$ and once we complete stress strain relationship, we can get back into finite element analysis with much more confidence. We also know how to interpret results on stress. So, we will enter finite element with much more confidence after we finish what we call as stress strain relationship. Again, we will start always from what we know and extend it in a much larger setting. Let us see what you know about stress strain relationship? What is that you know about stress strain relationship, what is that you know?

Correct; so, what we know is sigma is equal to E epsilon. Then, what else you know? There is one more term which we know, which we should use. Poisson's ratio, Poisson's ratio. So, the two things that we know is E and mu.

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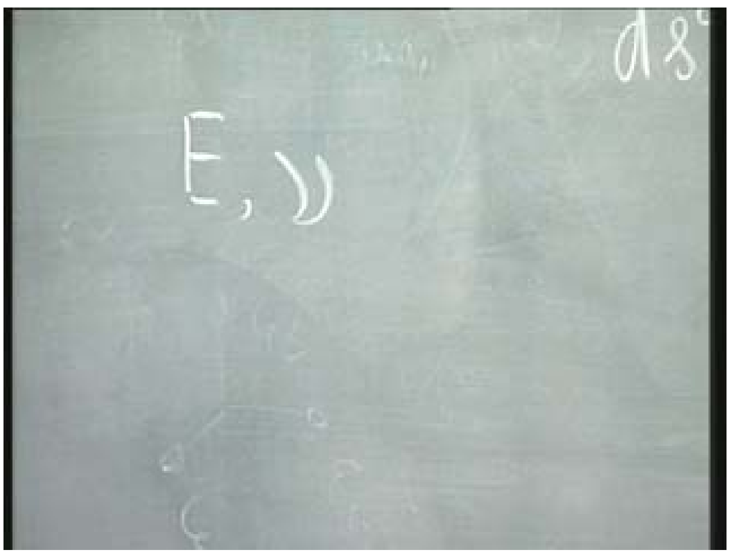

Now, the question is that is this is the way we define an elastic material? Do we say that when E is defined, we define an elastic material. because we have to set up this whole thing in a much larger canvas, then only it will be easier for us to work out the formulations, which comes to a very fundamental question. This whole thing comes to a very fundamental question as to what is an elastic material. We will worry first about elastic material and then much larger class of materials later. So, this brings us to a very important question as to what is an elastic material.

What do you think is an elastic material? Very good; there are usually two definitions. Initially also you said and now what you said; the first definition which students give is that stress is proportional to strain. This is first definition. The other definition which I would say much more accurate is that when I release the load, the shape is retained. This is a better definition. Why is that this is a better definition, because stress need not be proportional to strain. For example if you look at rubber, stress and strain are not proportional.

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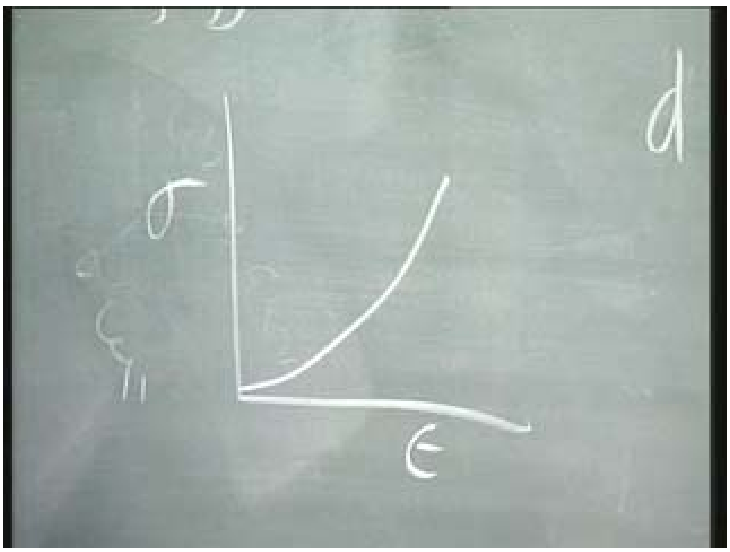

I can get a very complex stress strain curve where I cannot define E that easily; sigma versus epsilon. Fortunately or unfortunately this is not the way we define the elastic materials in solid mechanics. We have a slightly more rigorous definition. The definition comes from what is called as strain energy, strain energy density.

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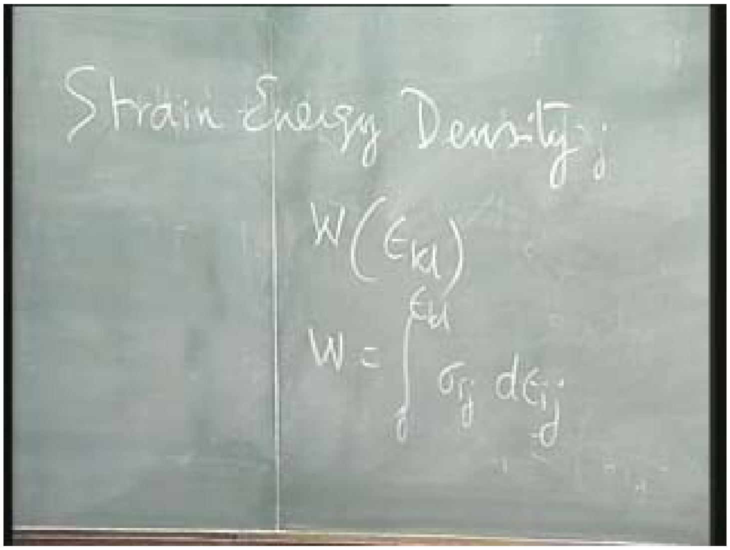

Again I am sure strain energy is not a very new term. You would have gone the other way; from stress strain curve, you would have said that strain energy is nothing but area under the curve. It is a nice way of looking at it in your earlier classes, but you would find that at this level, strain energy is much more fundamental, much more fundamental than what we call as the area under the curve. We are going to define the elastic material itself from strain energy or strain energy density function. When we say a material is an elastic material we postulate or we say the existence of a function called strain energy density function, W, which is a function of strain. We postulate, we say that if we call a material as strain, sorry elastic material, then we say that the material has a strain energy density function.

We will get back to what you know slowly, then, you will understand what these things are. So, we postulate the existence of a strain energy density function, which is a function of strains, which is a single valued function. There is only one value, single valued function. It is a positive definite function. Single value that means it will have only one value, positive definite, which means that it will have or whatever be the epsilon, it will have a positive value, positive definite function; single valued positive definite function.

So, an elastic material is one for which I can define a strain energy density function. I can define a strain energy density function. That is what is very important. Now, look at that. This is W is a function of epsilon k_i. How am I going to define W? We will get back to our original thing, integral zero to epsilon kl sigma$_{ij}$ d epsilon$_{ij}$.

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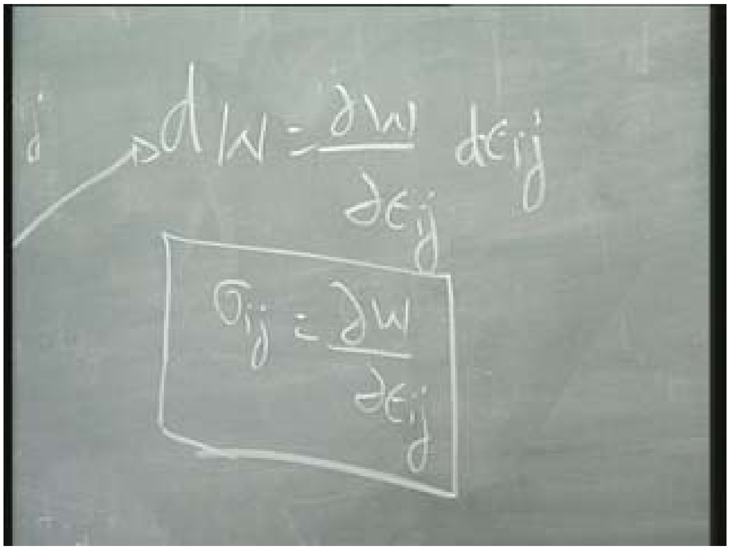

This function which I postulate that such a function exits for elastic material is defined as W is equal to integral zero to epsilon$_{kl}$ sigma$_{ij}$ d epsilon$_{ij}$, from which I can write dW to be dow W by dow epsilon$_{ij}$ into d epsilon$_{ij}$ and not from this, sorry; from this I can write like this. Now, these are the definition of W. So from this definition and from my dW it is very obvious that I can write sigma$_{ij}$ as what? dow W by dow epsilon$_{ij}$ and this forms the basis of definition of an elastic material.

We will continue this in the next class.