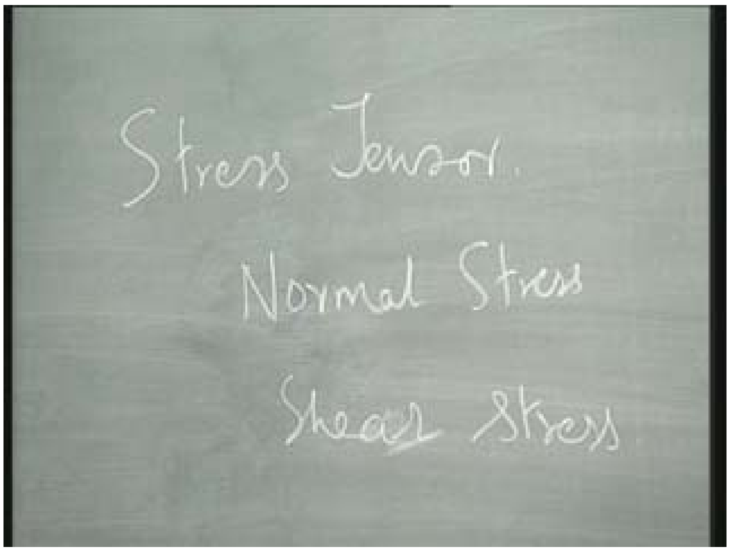
Introduction to Finite Element Method
Dr. R. Krishnakumar
Department of Mechanical Engineering
Indian Institute of Technology, Madras

Lecture - 8

In the last class we talked about what is called as stress tensor and we had completed our discussion on tensor. We saw that it is symmetric and so on.

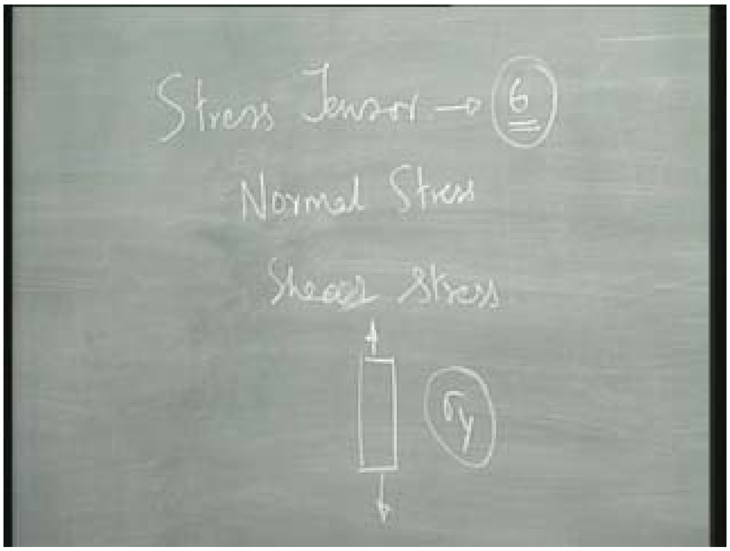
(Refer Slide Time: 1:06)



The question that can come to your mind is that in your earlier classes you would have probably studied about shear stress and normal stress. In fact what probably you would have studied is, in any plane you can find out what is called as the normal stress and shear stress. How is that related to what we have studied? There can be a question in your mind or you have studied about principal stress and so on. We will come to the principal stress in a minute, but before that let us see what this normal stress and shear stresses are?

Please note that we started our discussion by defining what is called as a stress vector. The normal stress and shear stress are nothing, but are obtained by resolving my stress vector, whatever I obtained, from the formula which I derived for it from stress tensor, either perpendicular in which case I get normal stress or along the plane in which case I get the shear stress. We are right in the same path as what you would have studied. It is not that whatever you have studied, it is not right or something like that. We are on the same path, but we are putting it in a more rigorous fashion, that is all. So that is called as normal stress and shear stress.

(Refer Slide Time: 2:57)

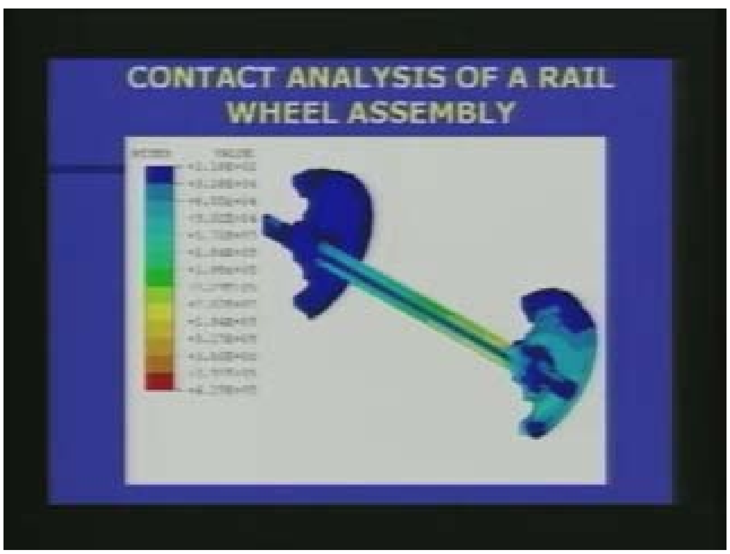


The other question which we just started is if I have a bar and if I do a tensile test, I get what is called as yield strength of the material sigma$_{y}$. I have just now shown that stress tensor has 6 quantities; 9, but because of my relationship that sigma$_{ij}$ is equal sigma$_{ji}$, I have 6 quantities. The question that may come to your mind is I have only one value as a material property, I have 6 values as stresses. Now, how am I going to tackle this situation or in other words the question is can I somehow combine these 6 chaps into one fellow? Some sort of an equivalent criteria for a uniaxial, please note that this is uniaxial situation. Can I get an equivalent uniaxial stress and compare that with the yield strength? Yes, it is possible.

(Refer Slide Time: 4:16)

The corresponding theory relies on or is called as Mises stress. Let us go back to our slide what we were looking at in our previous classes.

(Refer Slide Time: 4:32)

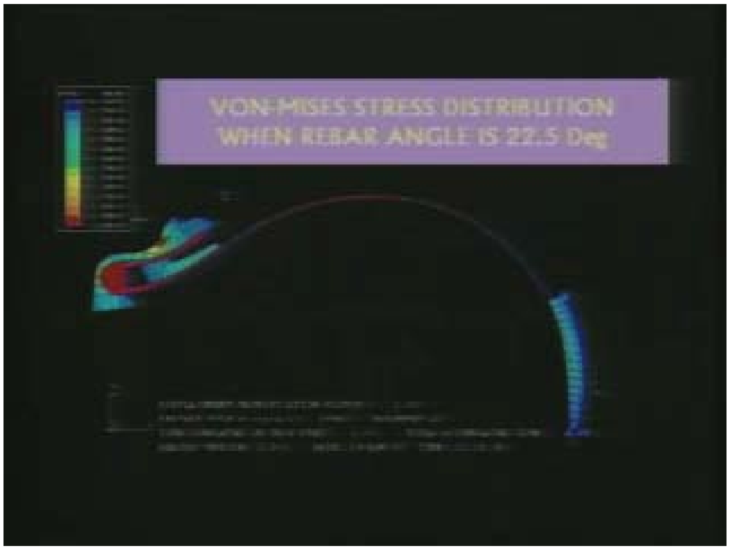


Look at that slide now. You see that, what we are seeing is Mises stress. We did a contact analysis of a wheel and axel together and what we were seeing is what is called as a Mises stress. Let us look at the next slide and see what we are looking at.

(Refer Slide Time: 4:55)

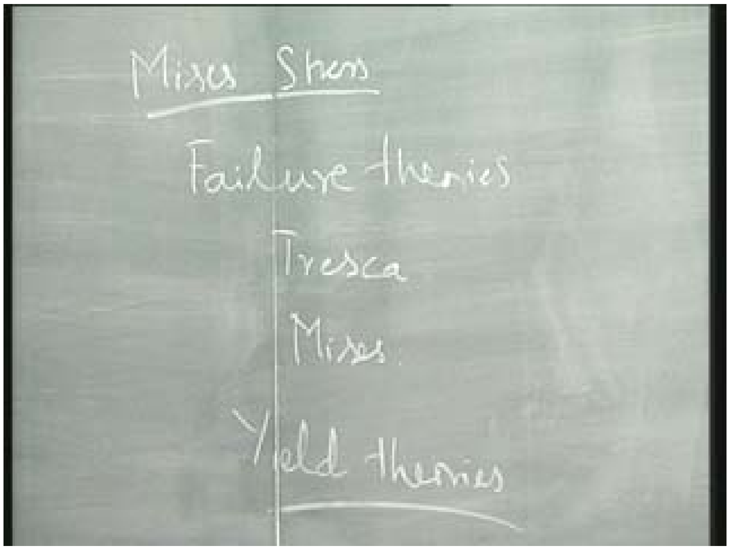
Again see clearly that it is Mises stress distribution, It is possible the theory of which I will explain. It is possible to get one equivalent stress called the Mises stress and then compare it with the yield strength. For example whatever stress value I have in this particular transparence or slide rather, I will straight away go and compare it with the yield strength of the material. It is about 37 Kgf per mm square or 370 Newton per mm square. I will straight away compare whatever value I am getting there to the yield strength of the material, I have obtained from my uniaxial test. Let us look at next transparency or next slide.

(Refer Slide Time: 5:51)



Here again you see that we are looking at Mises or Von-Mises stress, again from Mises theory. From all these examples, we noticed that we look at what is called as the Mises theory or we look at what is called as the Mises stress. 612

(Refer Slide Time: 6:12)

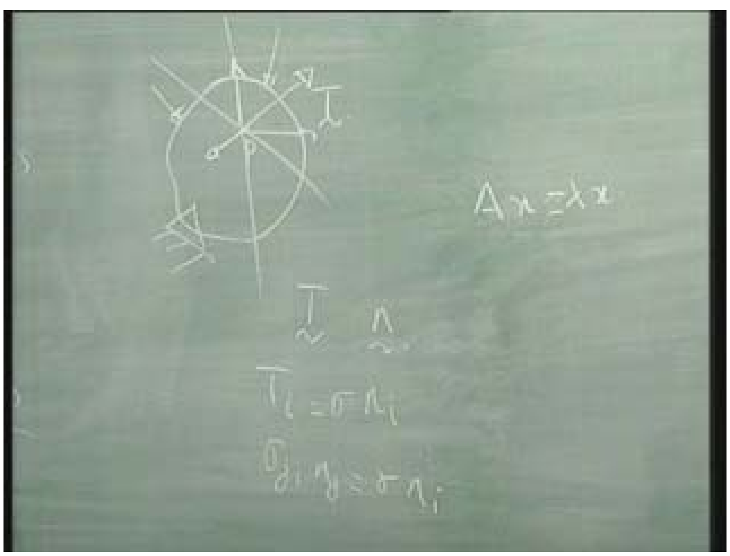


In your earlier classes, you would have called this theory as failure theories and probably you would have called this as distortion energy theory. You would have studied a number of failure theories. You would have studied what is called as maximum shear stress theory called Tresca's theory. This is due to a person called Tresca and you would have studied distortion energy theory and which we will now replace it with or call this as Mises theory. This term failure theory is a misnomer and unfortunately students do not pick that up very clearly. Failure does not mean fracture of materials. Lots of people make this mistake. Failure theories do not mean that when Tresca's criteria or Mises criteria is satisfied, the material is going to break into two, no.

For design purposes it has long been assumed, for 100 years now, that if the material starts yielding, it will not serve the intended purpose. This is what was first thought about and hence they say that failure means, this failure means not fracture but failure to do the intended purpose. That is what is meant by failure theories. It is not fractured; it is not failed means it is not fracture, but it is failed to do the intended purpose for which you have designed the component and hence people started calling this as failure theories. But the current terminology today is not failure theory but they are called as yield theories or these theories since they signify the yielding of the material, yield theories are nothing but yield criteria, criteria to signify that the material is going to yield. In much more simpler terms, failure theories signify the criteria for the material to yield. That is what we usually call this as; I mean that is what we mean when we call things as failure theories.

In order to go into the details of this, we need two things. One is not yet defined or one is not yet explained, Tresca. One is that we have to define principal stress and the other we will do is to decompose the stress tensor in order to arrive at the theory due to Mises. In most of the finite element packages, people look at what are called as Mises stresses and hence it is very important we understand what is the basis of defining Mises stresses and what exactly it means or else we will not be able to interpret the results. Before we go to Mises, let us look at what are the principal stresses? All of you know, I am sure, what principal stresses are, but we are going to do the same thing in slightly different setting. What is this setting?

(Refer Slide Time: 10:06)



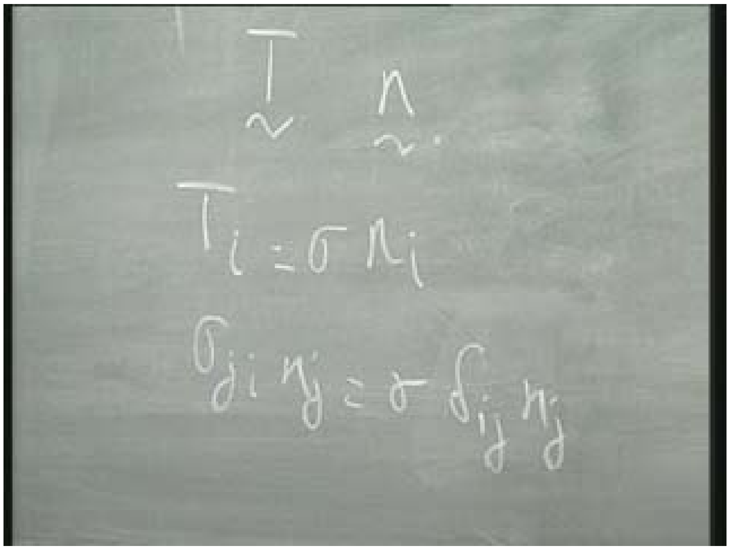
We have been looking at a complicated component with certain boundary conditions and certain loading conditions and so on and we were always looking at a point inside this. Now, the question is that if I consider all the planes passing through this point, can I get a plane or more than a plane where or on which the stresses or stress vector is wholly normal, completely normal. In other words can I get a plane or more than one plane where shear stresses are absent or to put it more clearly, can I get planes where my stress vector, please note this stress vector, is in the same direction as that of n. So, that is the question. Yes, that is possible and such planes are called as principal planes.

How do I write this mathematically? I write it mathematically by saying that T_i is equal to $sigma_{ni}$; T_i is equal to $sigma_{ni}$. This sigma is the magnitude of the stress vector which is totally normal to a plane whose normals are defined by these n's here. Please note when I resolve here T_1 T_2 T_3, they are along 1 2 and 3 direction. Suppose, I assume that this plane is the if if I assume that this plane is the principal plane, then if that is the normal then what I mean to say is the stress vector will be along that and what I mean by T's, please do not get confused, is how I am going to resolve T in three directions? That is what is meant by T_1 T_2 and T_3.

Let us go one step forward and write that equation with the result which we know for T. What is that result? $Sigma_{ji}$ n_j is equal to sigma n_i. Compare this with for example an equation of the form Ax is equal to lamda x; say, A is a matrix, x is a vector and Ax is equal to lamda x. What kind of problem is this? Correct; so it is an eigen value problem. Compare this equation $sigma_{ji}$ n_j is equal to sigma n_i with this equation. Do you see that there is a similarity between the two? Yes; so what does sigma actually signify? You can treat this is as actually a simple matrix, in order to understand it. Though the concept may be slightly deeper than that but nevertheless, right now it is easy to just treat this n as a vector, two vectors and you can look at this as a matrix. So, sigma n is equal to some sigma into, please note there is a difference between these two sigma's, into n.

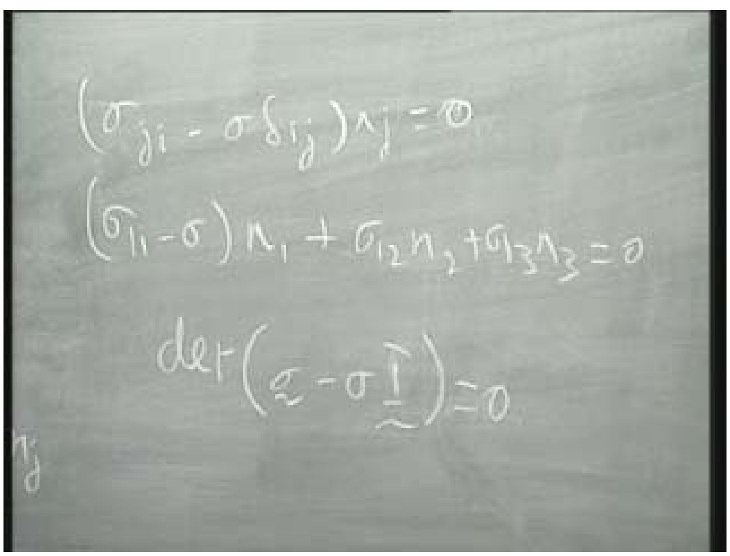
What is this? This is nothing but an eigen value problem of this sigma or principal stresses are nothing but the eigen values of this stress tensor, sigma. They are nothing but the eigen values of the stress tensor, sigma. We have shown lot of properties for sigma. It is symmetric, many more properties we will define later and hence it is possible to say that this sigma has eigen values and hence say that they are principal stresses or principal stresses come about because there is a property called eigen value for this matrix sigma or a tensor sigma. Though we started from here and came here we can always go from here, back to here and say that yes, if I say I get eigen values what does it mean? It means that they are nothing but the principal stresses.

(Refer Slide Time: 15:30)



Now, let us write down n_i in terms of $delta_{ij}$ sigma sorry n_j; $delta_{ij}$ n_j. n_i becomes $delta_{ij}$ n_j, so that, that equation can now be written as $sigma_{ji}$ minus, I am bringing that to the left hand side, minus sigma $delta_{ij}$ n_j is equal to zero.

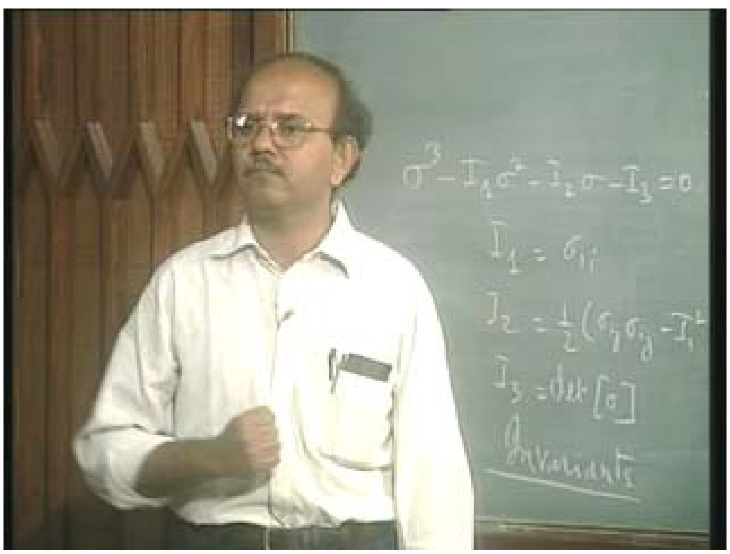
(Refer Slide Time: 16:20)



What else do you get from that equation or how many equations are there by the way? 3 equations. If you want you can write it down as $sigma_{11}$ minus sigma n_1. Let me write down the first equation; $sigma_{11}$ minus sigma into n_1 plus what is the second part? Sigma; you know now, $sigma_{12}$ is equal to $sigma_{21}$ and so on and so it does not matter because that makes it easier. You can say $sigma_{12}$ n_2 plus $sigma_{13}$ n_3 is equal to zero and you can write down two more equations. In order that we have by the way Before we forget, what is this $delta_{ij}$ by the way? What is this remember?

Kroneker delta and what is that? It is nothing but an identity matrix. Please remember that and now what does this tell us, from our very fundamental linearization? I can have two types of solutions. One is trivial solution for n_1, n_2, n_3, which means that all of them are zero and hence this equation is equal to zero. But if I want to have non-trivial solution, what is the condition I should have? If I have to have a non-trivial solution then, exactly; so, the determinant of the coefficients should be equal to zero. So, I can say determinant of sigma minus sigma I is equal to zero. So, sigma minus, please note the difference between this sigma and this sigma; always it is like that, but I am sure that the squiggle below will tell you which is a tensor and which is a scalar. What do I get from this equation? From this equation, I can get, a cubic equation. I mean from determinant sigma minus sigma I is equal to zero, I can get a cubic equation and the cubic equation can be written as sigma cube minus I_1 sigma squared minus I_2 sigma minus I_3 is equal to zero.

(Refer Slide Time: 19:17)



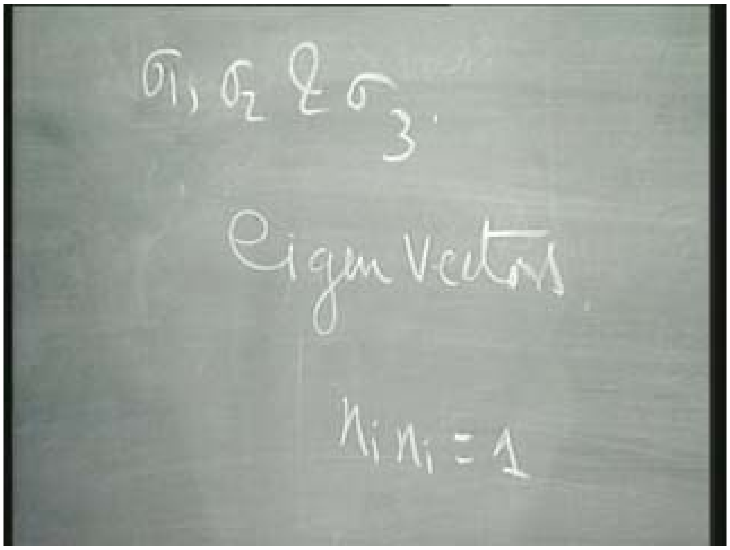
What I have done is, I have just left some of the steps; you can easily fill it up. That determinant is now expanded and written, I get a cubic equation, which is given by this particular form where I_1 is equal to $sigma_{ii}$. What is $sigma_{ii}$? Trace of sigma, correct; trace of this matrix, called sigma. I_2 is equal to half of $sigma_{ij}$ $sigma_{ij}$ minus I_1 squared and I_3 is equal to determinant of this matrix sigma. I hope you all understand how I got it? I just got it by taking the determinant of that determinant of sigma, minus sigma I. So, I have got this.

What are I_1, I_2 and I_3? I_1, I_2 and I_3 are called as invariants. In solid mechanics, we have nice names. The names signify or have lot of meaning to it. So, when I say invariants, it has a lot of meaning to it. So, it means it does not change; invariant, it does not change. It does not change with respect to what? Why do you call it as invariant? Is it with respect to plane? Look at this equation and tell me? Let us just slightly go back because it is a very important concept which people usually do not understand. Look at this equation here. I have got this equation and I want to find out sigma and sigma which is the principal stress I would say, I will not call it a sigma; let me say that I want to find out the principal stress and what is it that I have been given? I have been given the stress tensor, $sigma_{ij}$.

The stress tensor is a function of how I have chosen x_1 x_2 and x_3 or coordinates. So, the final equation which I should get and I should solve should be such that the principal stresses should not be a function of coordinates; correct. So, it should be independent of the coordinates which I have selected or else I can get whatever I want because principal stresses have to be determined without the interference of this coordinates. What does it mean? It means that this equation which I am going to solve in order to get my sigmas should be invariant to the coordinate system. Hence the invariants signify or say that they are independent of the coordinate system which you have used in order to get the cubic equation. Hence they are called as invariants.

Once I have this, it is not very difficult. It is very straight forward to find out I_1, I_2 and I_3 and once I know what are I_1, I_2 and I_3, I can very easily substitute them in this equation, solve this cubic equation. You know so many methods to solve them. What is the simplest method you can use? Trial and error will be too difficult. Newton graphs and Newton’s methods can be used in order to solve them. There are very well programme, I am not going to the details of it. If you have difficulty, let us see it. But I think it is a straight forward method to solve the cubic equation. Once I solve the cubic equation, I can get n's. So, first of all when I solve the cubic equation what is that I am going to get? I am going to get three values of sigma.

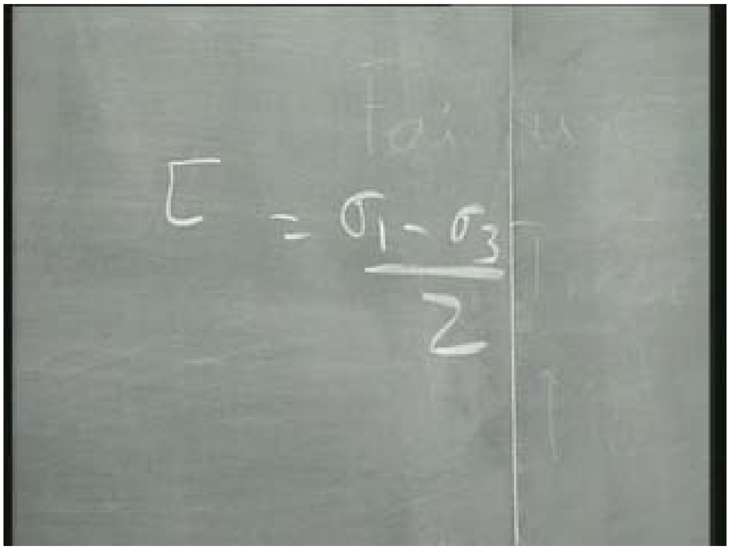
(Refer Slide Time: 24:11)



I am going to get $sigma_1$, $sigma_2$ and $sigma_3$. What are these things $sigma_1$, $sigma_2$ and $sigma_3$? They are the eigen values of the matrix sigma. Corresponding to this, there should be eigen vectors; there should be eigenvectors. In this case what are these eigen vectors? They are the principal planes. They are the n's. So, for each of these eigen value, I will get one set of n's, which I can get by solving the equation which we have written there, just now we saw that equation, along with the fact that n_i n_i or n_1 squared plus n_2 squared plus n_3 squared is equal to 1. By solving these two, we are going to have or we will get an equation or we will get set of equations to get n_i's. So, n's are eigen vectors and please go back to your theory on principal planes.

What is another important theory which you would have studied? Principal planes; do they have a relationship, principal planes? Perpendicular to each other - this is what you would have used. Principal planes are perpendicular to each other. In more sophisticated way, you say that they are orthogonal. Why because eigen vectors are orthogonal. So, you say that whatever you have studied before comes beautifully from our theory that we can define stress as a tensor and so on. Hence these principal planes are orthogonal. Once I define principal planes, I can get the maximum shear stress. I am not going into the details, you know it already, that the maximum shear stress can be defined as how do we define maximum shear stress?

(Refer Slide Time: 26:27)

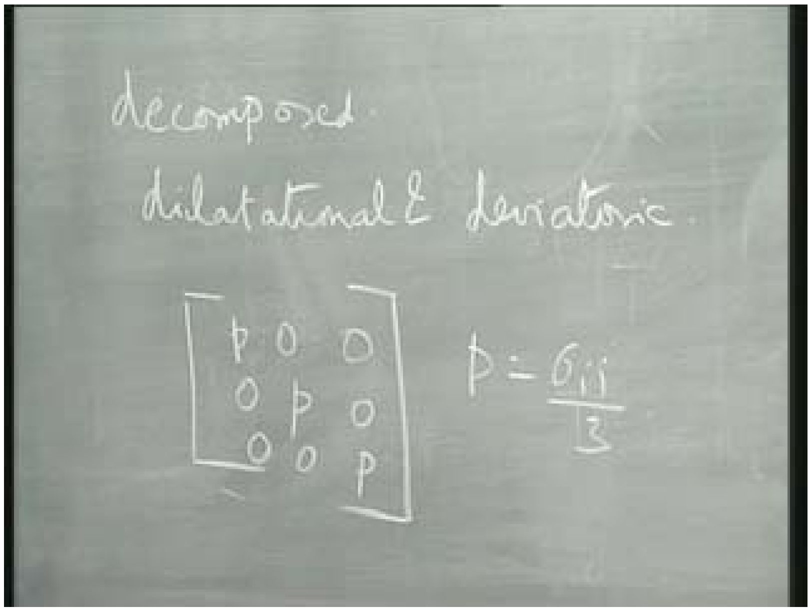


Let us say that the maximum shear stress is tau; correct. So, that is equal to sigma_1 minus sigma_3 by 2, after arranging sigma in a descending order; sigma_1 being more than sigma_2 and sigma_2 being more than sigma_3. The shear stress, maximum shear stress is given by sigma_1 minus sigma_3 by 2 and we also know that in the planes where maximum shear stress exists, there are normal stresses and so on. Again this is the fundamental strength of materials. Tresca’s theory simply states that when maximum shear stress reaches a value, which is obtained by an experiment where the shear stress reaches a critical value or when tau reaches a critical value, then we have yielding.

In a uniaxial situation, what is that? sigma_y by 2; so, a critical stress in the uniaxial situation is sigma_y by 2. When tau reaches sigma_y by 2, we say that yielding has occurred according to Tresca’s theory. But Tresca’s theory is valid and it is quite good. But there are certain difficulties, which we will not discuss right now, but it can also be used. More important theory that is used for metals is what is called as the Mises theory. Whatever we saw just now, we saw that principal stresses can be arrived at very naturally from our definition of stress and is quite useful as well if you are going to use Tresca's theory or certain brittle materials where you can use the maximum stress theory as well. You can say that the maximum stress is one which will signify my failure and so on. I am sure all of you have studied this in your design courses. So, now I am not going to repeat that.

On the other hand, let us go to what is called as Mises theory.

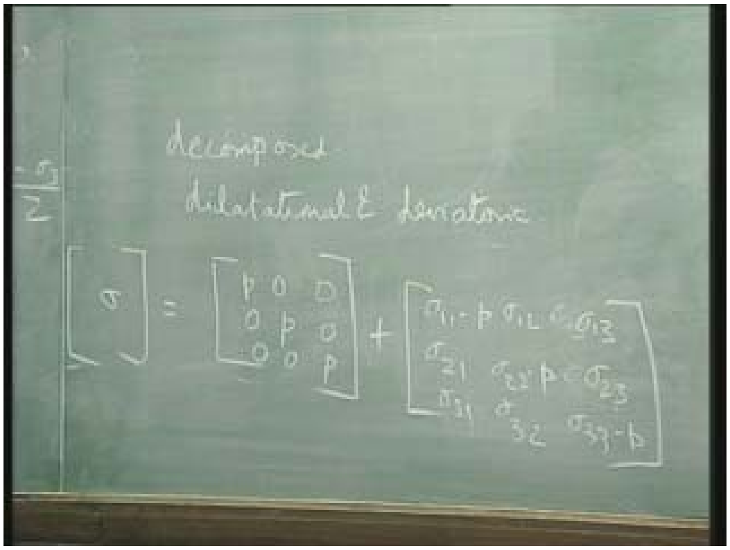
(Refer Slide Time: 28:54)



Mises said that the stress tensor can be decomposed, more importantly, addictively decomposed into two parts which he called as dilatational and deviatoric. He said that the stress tensor can be addictively decomposed into a dilatational part and a deviatoric part. A dilatational part is also called as the hydrostatic part and a deviatoric part. What he physically said is that the stress tensor can be split into two parts, one responsible for volume changes, the other for distortion. So, deviatoric is also called as distortion; so, one part responsible for volume changes and the other part responsible for distortion and that part is called as deviatoric part.

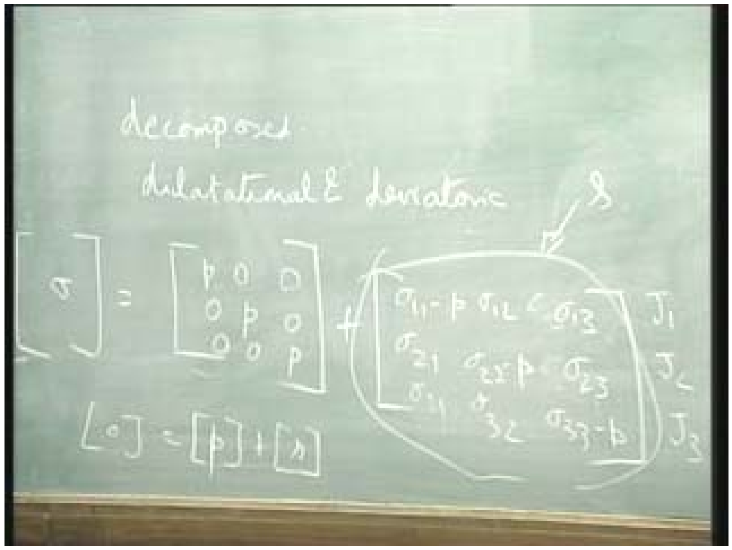
He defined the dilatational part, because note the word hydrostatic part or the dilatational part, by a tensor which he wrote like this where, what is p? p is equal to, correct, $sigma_{ii}$ by 3.

(Refer Slide Time: 30:40)



He said that the whole of this sigma tensor can be split into a dilatational part and a deviatoric part, which comes out, as I said, it is additive decomposition. So, what should be the other fellow here? He should be sigma_{11} minus p sigma_{12} sigma_{13} sigma_{21} sigma_{22} minus p sigma_{23} sigma_{31} sigma_{32} sigma_{33} minus p. I hope you are Is it clear May be I will write it further away. When I add these two, I will get my original sigma matrix. Why did he do this? He did this because he found out, he and many others found out, that of these two parts, it is the deviatoric part which is responsible for yielding and the dilatational part does not play a role in yielding. Yielding is not signified by the hydrostatic pressure which is superimposed on the deviatoric part. He said that that is the by number of experiments, empirical; that is how he arrived at it.

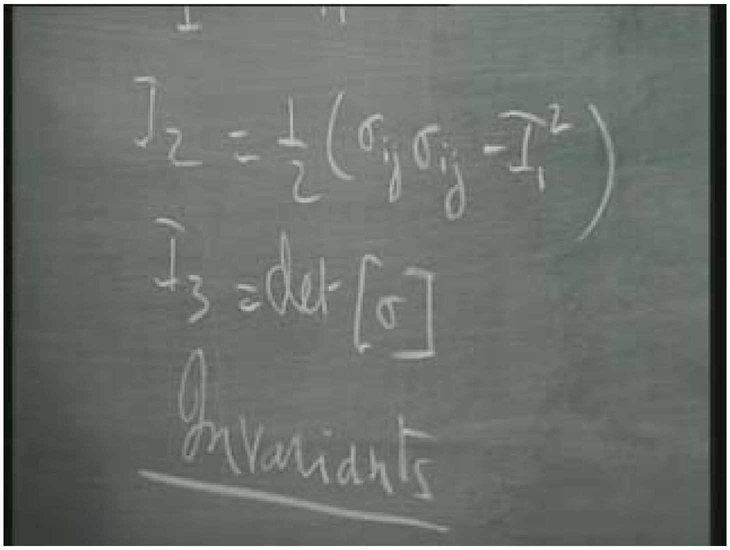
(Refer Slide Time: 32:25)



He looked at this tensor. This is the tensor again and said that this should have three invariance. He called this invariance as J_1, J_2 and J_3. He called this invariance as J_1, J_2 and J_3. What is J_1? Look at that; zero. J_1, J_2, J_3 are nothing but obtained from my equation here, this invariance; same thing because the three invariance are always present. Please note this carefully. These invariance are always present when I look at a second order tensor. When I say that tensor, second order tensor and I want you to write down three invariance, these are the invariance.

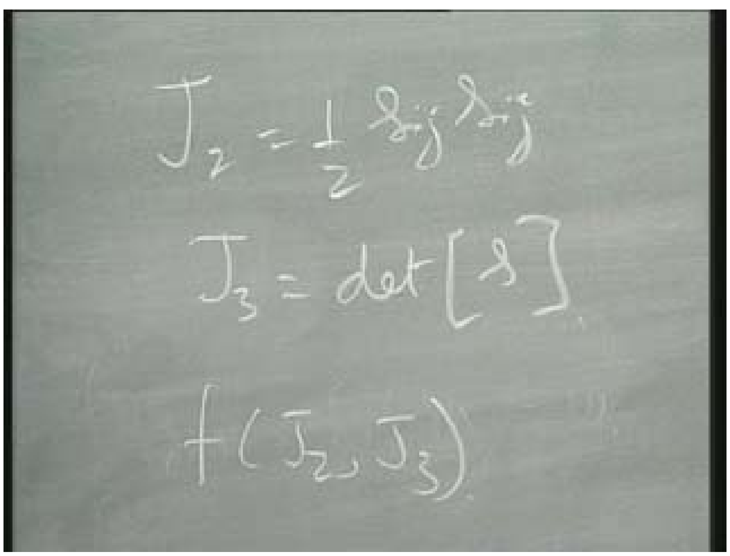
In the same fashion, the deviatoric part also has invariance defined and these invariance are called as J_1, J_2 and J_3. What happens to J_1, trace of sigma, here? Let me call this matrix as s, in order to distinguish this from sigma. Yes, any question? Any question? Yeah! So, we will call this matrix, what I am saying is this matrix is called s, so that I need not write this down every time. I can say that this matrix sigma is equal to p plus s. That is all; sigma is equal to p plus s. The first invariant of my s matrix, zero, that is called J. I do not want to again write as I, because I already have I defined for sigma and hence I write this as J. So, J_1 is equal to zero. What is my J_2? Come back here; come back to this equation here. What happens to this?

(Refer Slide Time: 35:07)



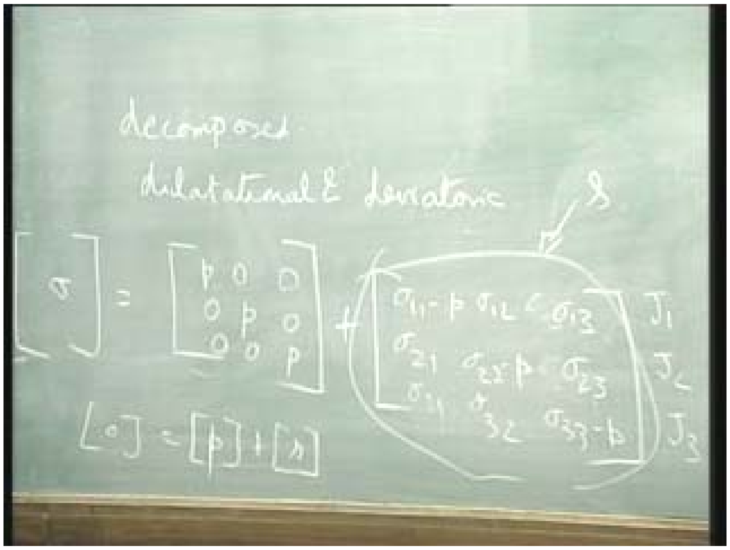
That goes off; zero, so, half, instead of sigma, what do I write? Correct; half s_{ij} s_{ij}. So, J_2 is equal to half s_{ij} s_{ij}. What is s_{ij} s_{ij}?

(Refer Slide Time: 35:31)



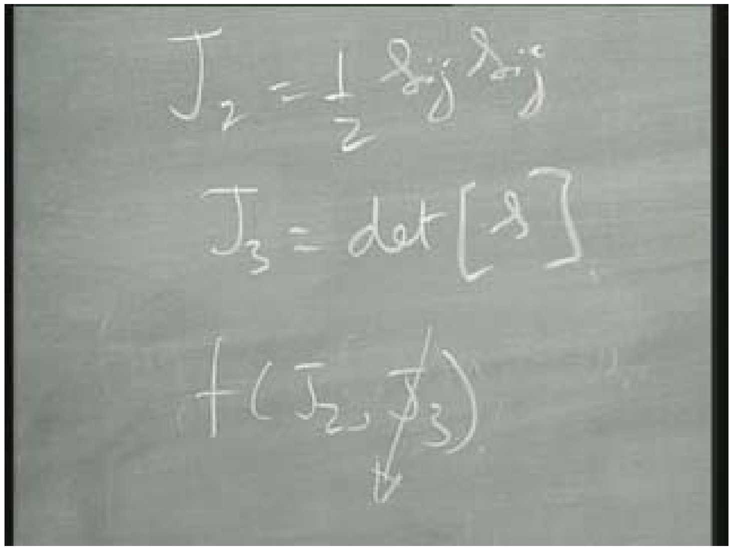
Just recapture what we did? Summation about i and j. That is the doubt? Clear and J_3, what is J_3? Determinant of s. J_3 is equal to determinant of s. The logic is very simple. If at all there has to be a failure or yielding in this case, sorry we will not use the term failure; if at all there is going to be yielding, then the yielding should be independent of my coordinate system. I cannot have a coordinate system and say that in this coordinate system there is yielding another coordinate system it will not yield and so on. My yielding theory logically should be a function of J_2 and J_3. My yielding theory should be function of J_2 and J_3. Just to recapitulate, this whole thing here, what we did was to split this sigma into two parts. By experiments, it was shown that dilatational part does not have much effect and deviatoric part has an effect, only has an effect on yielding.

(Refer Slide Time: 36:51)



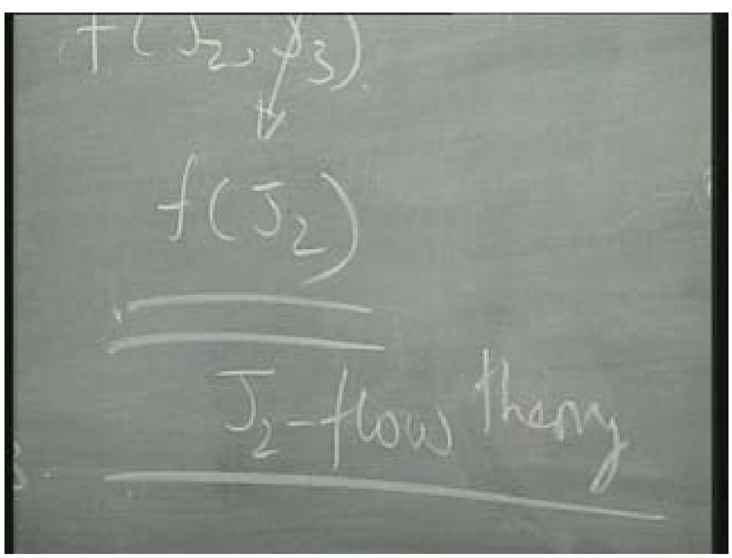
We took this up for further study. Then we found out that there are three invariance J_1, J_2 and J_3, three invariance and out of which J_1 goes off by our definition of the deviatoric stress and hence we have J_2 and J_3. The yielding is a function of J_2 and J_3. That is what is important, it is a function of J_2 and J_3. this is where Up till this point both Mises and Tresca are on the same line.

(Refer Slide Time: 37:31)



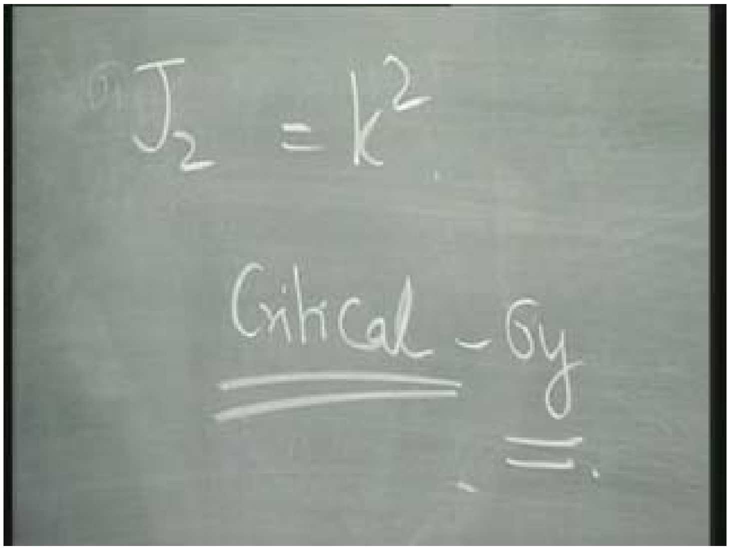
Mises said that he found, rather from experiments, that look that is not very important and for metals, note this carefully for metals, he found that if I write yielding to be only function of J_2 that is sufficient for me to define yielding. What he said is, to put it simply, when J_2 reaches a critical value, when J_2 reaches a critical value, material starts flowing; flowing means yielding. His whole emphasis was in the fact that J_2 reaches a critical value. In fact he defined the function of J_2 to be just J_2 and then said that J_2 has to reach a critical value. This theory, ensuing theory from Mises is also called by the name J_2 flow theory.

(Refer Slide Time: 38:58)



Someone says I am following J_2, in fact many manuals say that what we follow is a J_2 flow theory. What it really means is that they are following what is called as Mises theory. What is this critical value? That is our next question. If at all I know what is that I know from experiments. Usually there are two sides to the equation, one side the left hand side which is J_2. We will keep this like that so that if there is a future reference, we will look at it. On one side I have J_2.

(Refer Slide Time: 39:48)



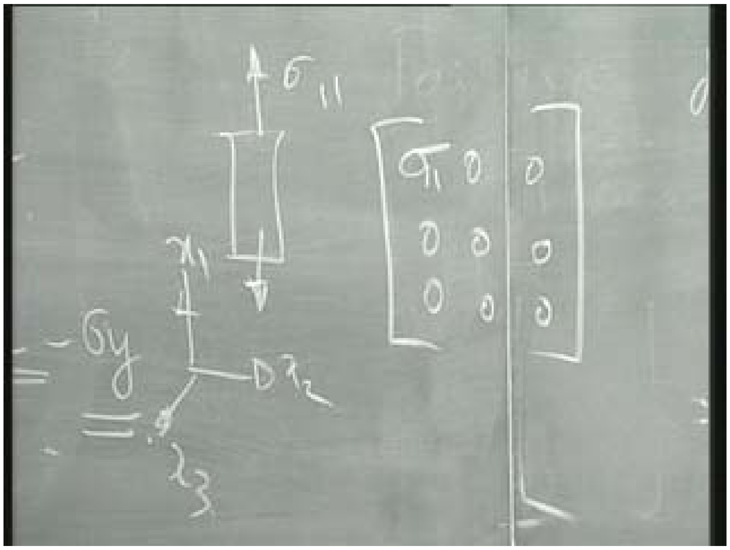
J_2 has to reach a critical value which means that I have two sides of the equation. Right hand side, I should have something to do with material. J_2 depends upon what? J_2 depends upon the loading, geometry, boundary conditions and all that because J_2 ultimately depends upon sigma and sigma depends upon my loading conditions and so on. The left hand side of this equation depends upon my loading conditions, boundary conditions and what not, geometry and all that. The right hand side should depend upon the material property. Let us look at J_2. J_2 is square of stress. Let me call this as say k squared; it reaches the critical value called k squared.

What is this critical value? You know people We talk about all the time critical value. What is this critical value? This can also be looked at some criteria or people call this also as a criteria function. What is critical value? What is it in my hands? I can take a specimen and I can load it, look at yielding. That is in my hands. The critical value as far as I am concerned should point out the yielding of the material or should be related to my $sigma_y$, which I can see, monitor, measure or whatever it is. So, the critical value should depend upon $sigma_y$. What is that I am going to do? I am going to apply whatever theory I have developed so far for yielding for a uniaxial case and see whether I can get the value of k which should ultimately depend upon $sigma_y$. That is what we are going to do now.

Any questions so far? The logic is clear? This is a very, very important part not only in design but also in finite element because ultimately when you interpret the result, you are going to interpret only in terms of Mises stress. Usually people interpret finite element results without understanding what Mises stress is. They will just look at the Mises stress and then start interpreting. But actually you should understand that we are applying our well known, within quotes, failure theories or yield theories and that is how we are doing the whole thing. May be this is not the way you would have studied it in your earlier classes. You would have just got one expression for distortion energy or whatever it is, but this is the logical way in which, solid mechanics or continue mechanics people develop it. It has lot of implications later.

We will stop with just one sentence as to what is a stress tensor for a uniaxial case. This is the uniaxial case.

(Refer Slide Time: 43:04)



What is the stress tensor? Let me have say x_1 in this direction, x_2 and x_3, in which case how do I write the uniaxial stress tensor? Let me call this as, say, $sigma_1$ or $sigma_{11}$ if you want. So, it will just be $sigma_{11}$ 00 000 000 period. This will be $sigma_{11}$ 00 000 000. Remember that this is only the stress tensor, not the deviatoric stress tensor. What is p in this case? $Sigma_{11}$ by 3 and hence I have to subtract that from this in order to get the deviatoric part.

We will stop here and proceed to the next part in the next class.