Introduction to Finite Element Method
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Lecture - 7

In the last class we stopped at deriving the equilibrium equation. We said that the equilibrium equations are similar to what you have studied in your earlier classes on say engineering mechanics and sigma F is equal to zero was one of the equations. We said that there are two equations, sigma F is equal to zero and sigma M is equal to zero and we said we will concentrate first on sigma F is equal to zero. Before that there were one or two questions which I think we should get to know the answer, because quite an interesting question was their initially, before we just ended the class. What was the question?

I have two tensors, say sigma and delta; the two tensors. I actually operated on these two tensors and said that I get a scalar.

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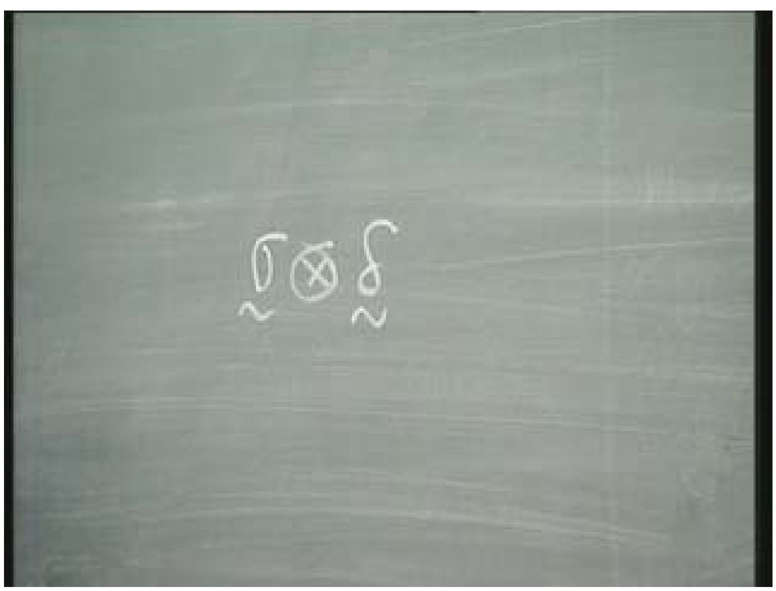

If you remember I said that $sigma_{ij}$ $delta_{ij}$ is equal to $sigma_{ii}$. I operated on these two tensors. The question was what is this operation and what are the types of operations that are possible and what is the result? Is it that when I operate on these two tensors, on two tensors I will get only a scalar? The question was very much valid because I said that these can be also looked at as a matrix. If I just say multiplication of matrix, I should in fact get another matrix or another second order tensor. If this is the understanding, yes, you are correct, but the only difference is that there are number of operations that are possible with two tensors. This operation for example is called as a dyadic operation.

What is this operation? Let us understand this, let us not get confused about it. Suppose I have two vectors; let us go from what we know.

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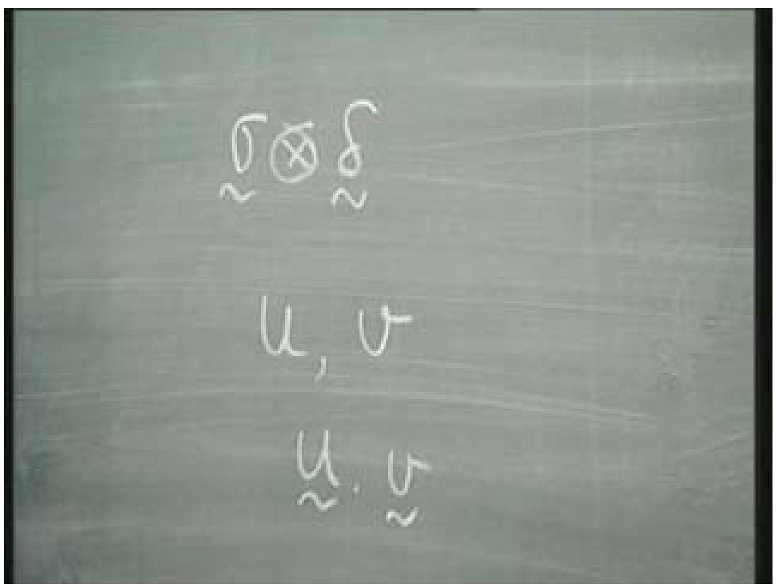

Suppose I have two vectors say u and v. All of you know that I can do two operations with u and v. A dot product can be obtained which is say u dot v and what is this? What is the result of this? Scalar; what is that you do here? Actually it is u_1 v_1 plus u_2 v_2 plus u_3 v_3 and how do I express this in indical notation? u_i v_i. So, one operation with two vectors gives me a scalar.

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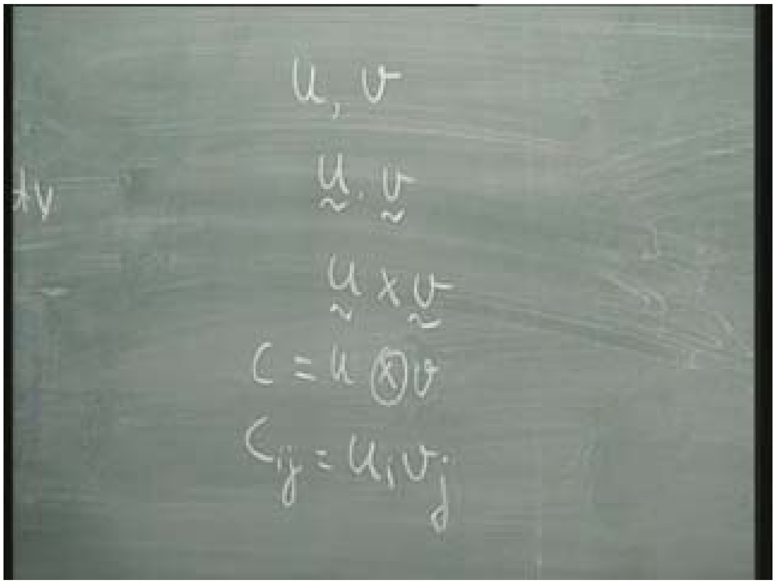

Let us look at another operation. Let us look at this operation? What is this operation? Cross product; what is the result of this operation? What is the result of this operation? Another vector; so, by operating with the same vectors on one hand I can get a scalar, on another hand, I can get a vector. We saw how to express this also in terms of indical notation yesterday. Similarly you can look at a number of operations with tensor also. What did we do here? Actually you have u_i v_j. I made j is equal to i in order to get this scalar product. In fact it is not that there are only two operations with vectors. I can have another operation called u dyadic v. The result is a second order tensor; the result is second order tensor. The second order tensor, say for example, let me call this as some tensor C, where this operation indicates that c_{ij} is equal to u_iv_j; c_{ij} is equal to u_iv_j. You can see that Before we proceed to this first equation, you can see that with the same two vectors, I can get a scalar by operating it; by operating it with another symbol and another rule we get a vector and yet another operation gives me a second order tensor. Now the question is what are these operations? what are we trying to do here?

These operations, in later classes will mimic certain physical phenomena. In order to understand what this means, this particular operation means, you can look at for example force and displacement. If you want to get say work, done what will you do? Yeah! So, you will look at dot product. You would see that when I have these vectors, especially in this course with the physical meaning, particular operations will give rise to certain physical quantity or phenomena. This is independent of the tensor analysis and so on, but for engineers it is easier to say that this dot product between these two vectors gives me this quantity; cross product gives me this product and so on.

Now coming back to this equation in the same sense that I can have different operations; I can have different operations defined on a second order tensor as well. This has sigma_{ij} delta_{kl}.

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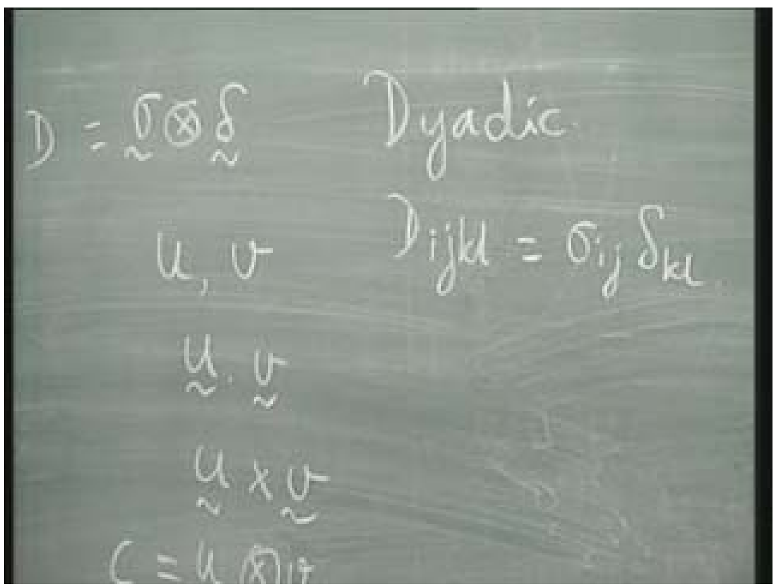

So I can get a fourth order tensor by an operation which is called as a dyadic operation; by a dyadic operation, such that ijkl, let me say this results in say D. Let me say that this results in D such that D_{ijkl} is equal to sigma_{ij} delta_{kl}. Whenever I put this operation due to some physical phenomena that is taking place, the result will be a fourth order tensor whose components will be given by this operation. After all vectors, tensors and other things that we are going to use has some physical meaning. Stress as We understand stress; even though I defined it very rigorously, we know what is stress and what it means and so on. This kind of intuitive feeling you would have developed already in the previous classes.

The question is why are you defining so very rigorously all this kind of things? Why are you now taking us to a tensor? Why are you defining stress as a second order tensor? This is another question, why do you want to define? Why not we stick to what we know as a matrix. Now this takes you to a slightly higher mathematical plane. What is that and that is going to form, that kind of plane is going to form the bases of your further understanding of subjects like this. What is that plane and what is that understanding? When I prove or show that stress, I have not yet done it; that requires some more rigour but nevertheless, when I show that this stress is a second order tensor, I am bestowing on that certain qualities of a second order tensor.

What does it mean? If I keep defining, sigma is a second order tensor, epsilon is a second order tensor, epsilon dot is a second order tensor and so on, a number of quantities, that means that these guys will take the qualities of a mathematical, say, understanding called tensor or mathematical definition called tensor. A second order tensor has certain properties, independent whether it is stress is a second order tensor or whatever, it has certain properties. It has certain Eigen values, Eigen vectors and so on. So, all these things will come to the second order tensor. We are going to see in this class or maybe I hope it is in this class that we are going to see principle stresses; how are they defined?

You would have studied it in one fashion. But once I define second order tensor is the one which is going to be characterized by stress, then I can say that these are nothing but Eigen values of this tensor. I need not go and study principle stress, principle strain, principle so and so, as long as I say these guys are second order tensors. So, they get, they inherit all the properties of a second order tensor, once I define a physical quantity to be a second order tensor. In the same sense, when I say vector immediately you would go and tell me or you would tell me immediately that a vector has a magnitude and direction. What is that you have done? What is that we are looking at if, when I say displacement? It is a vector or velocity is a vector or force is a vector. What is that I am doing? I am saying that these quantities will follow the rules of a vector. These quantities will follow dot product; these quantities will follow cross product and so on.

Once I say these things, I need not worry about saying or looking at rules for these operations. One of the operations will give me what is called as the fourth order tensor.

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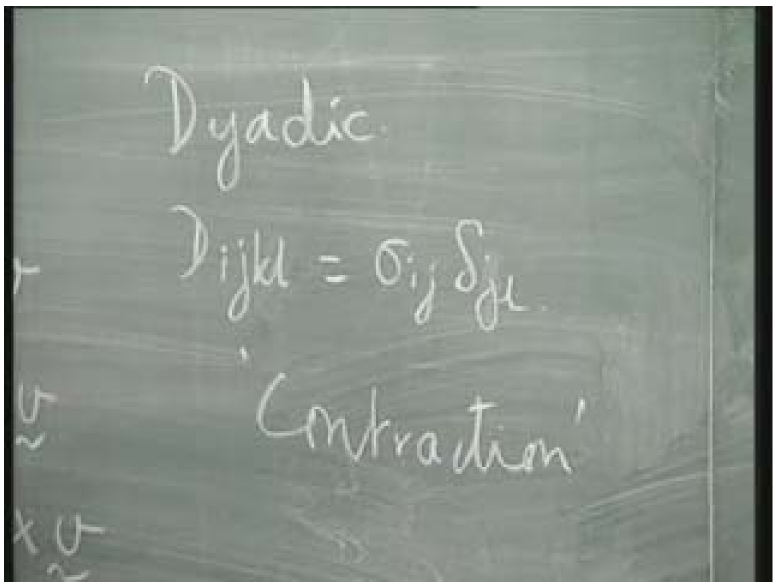

Another operation called contraction operation, where I will make, say two quantities to be the same say i and k to be the same or j and l to be the same and so on, will give me a second order tensor. Sigma$_{ij}$ delta$_{jl}$ which means i imply summing up about j. Let us not go into details because you can very systematically define what is called as transpose and so on. Let us not worry about that right now, so that we do not deviate too much away from what we are doing. But nevertheless since this question was asked we can see that I can now define a contraction called contraction operation on a fourth order tensor. What did I do? I simply said let me give a rule to you by making two of these indices to be the same. As soon as I define this rule, I also follow all the summation notation which means that the resulting operation or the resulting quantity is a, fantastic; it is a second order tensor.

I can give you one more rule. Next rule is that take the trace of this second order tensor. That means that second order tensor is a matrix; sum of the diagonal terms. What that does mean here? That means that make the other two quantities, other two indices also to be the same.

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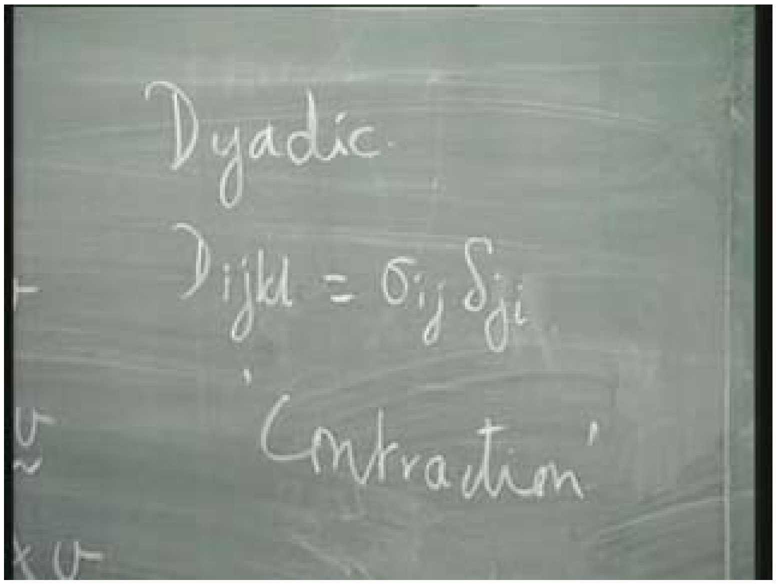

So, I can sigma$_{ij}$ delta$_{ji}$, that is another operation; another operation. Now you will see that why am I doing all these? Why is that this becomes important? You will see that like we had F, what is this? Force and a displacement define what we called as a physical quantity, work, I am going to say that I will operate with sigma and epsilon or epsilon dot rate of strain.

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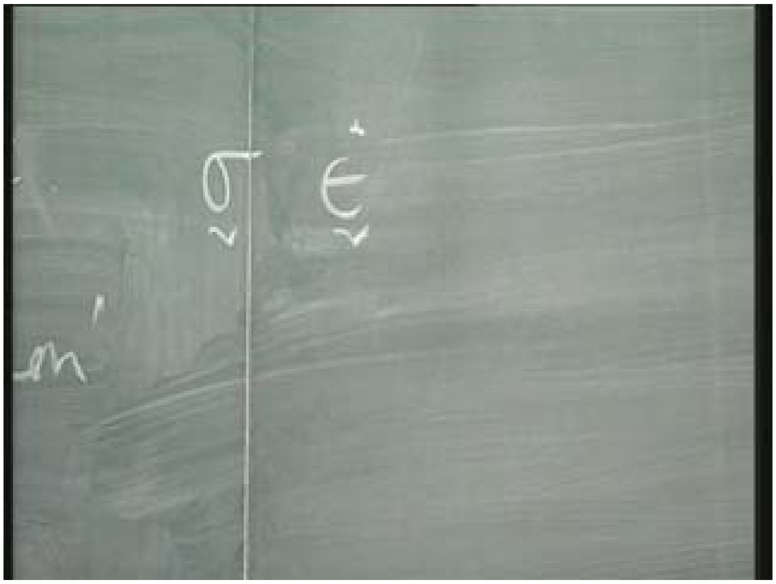

I am going to define strain may be in the next class. with if I going to If I can operate with these quantities, I will get some other physical quantities, some other physical quantities like for example stress power. These operations we will carry out later, we will make use of such operations with the tensors. There are number of operation with tensors. In the same fashion, they are nothing but an extension of what you already know as operations with vectors. We will come back to this topic after some time. But we will get back to the problem which we were dealing with in the last class.

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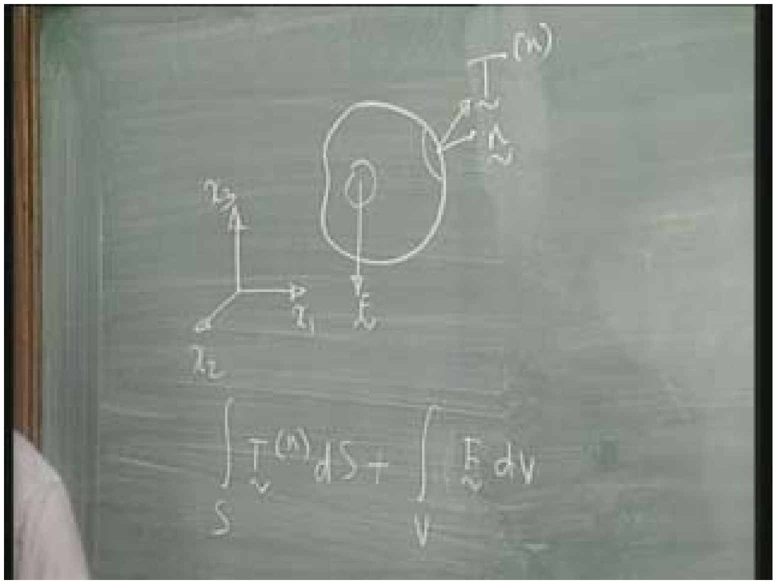

Let me recapitulate this problem. We were trying to get at the equilibrium equation. We had said that there are two types of forces. The surface force has resulted in a stress, what is called as surface traction and body forces. Body force is defined as F per unit volume of the body. If it is going to be defined per unit mass, then you have to correspondingly bring into account the density of the body and so on; rho of F has to be brought.

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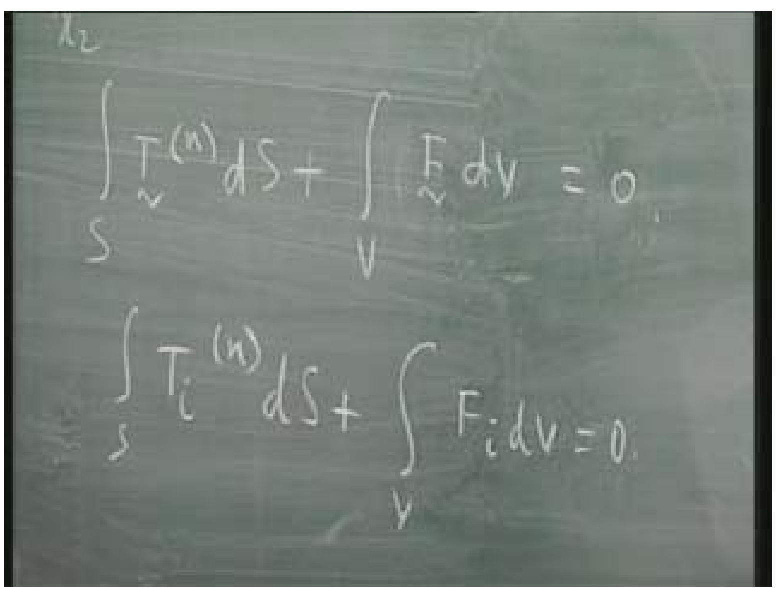

This is the equation and that is going to be equal to zero. We had defined this in terms of an indical notation as well and said that, that is how we had written it in terms of indical notation.

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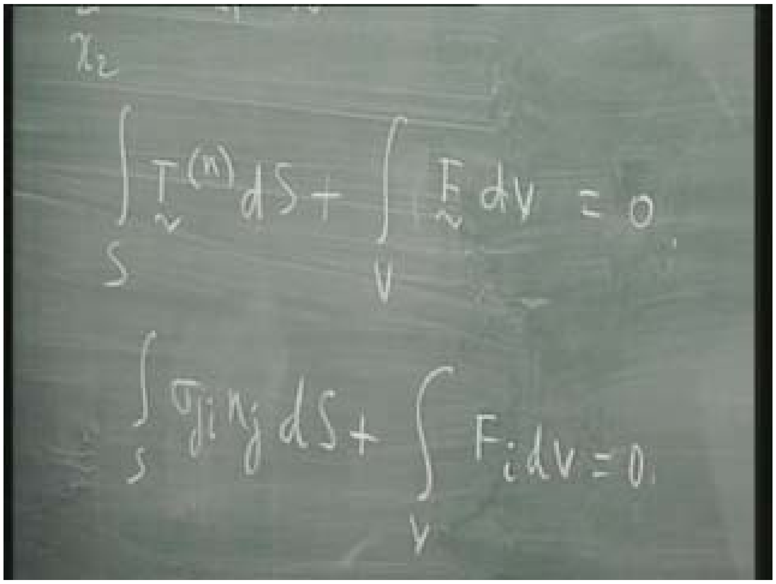

We also said that we can replace F, sorry T in terms of sigma so that this term can be written as sigma$_{ji}$ n$_j$. The next thing That is where we stopped in the last class, just summarized it and next thing we said that we want to replace the surface integral by means of a volume integral and we want to use divergence theorem. That is where we stopped and what is divergence theorem? We had already defined what is divergence theorem?

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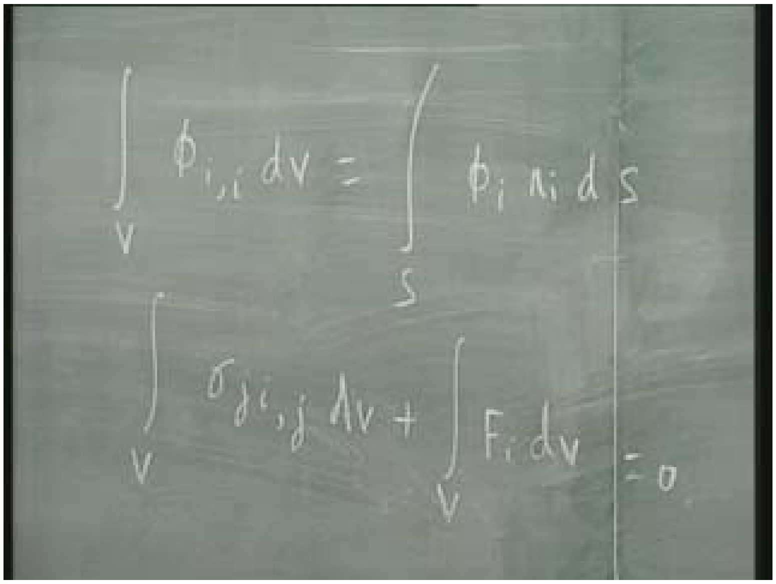

Say, take a quantity phi; phi i comma i dv is equal to surface integral phi_i n_i dS. This is what we had talked about in the last class. I had already said that this coma means it is differentiation with respect to x_i and so. Now, applying this equation to my last step here in this equation, the last step of the previous equation for this place at this place (Refer Slide Time: 18:42), what do I get? Sigma_{ji} n_j will become now, what is that it will become? Sigma_{ji} comma, that is important, j now volume integral dV plus integral V F_i dV is equal to zero.

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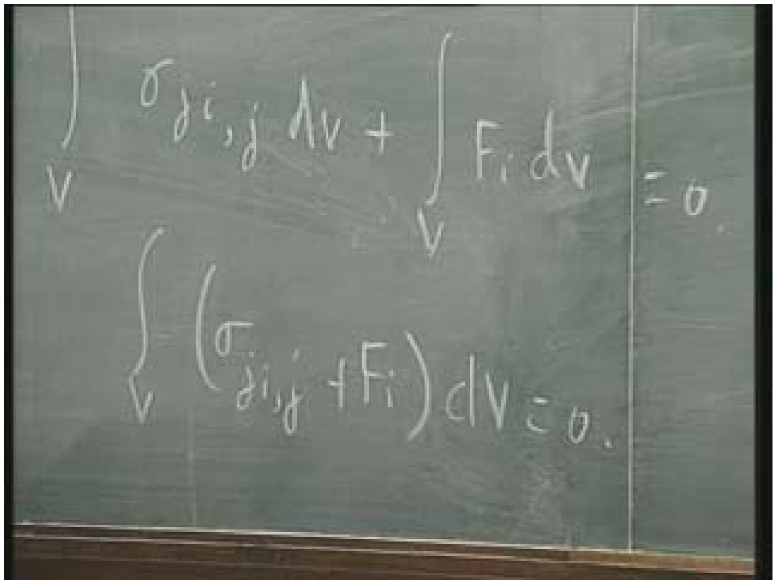

Bringing them under one case or one integral symbol, I can write this as sigma$_{ji}$ coma j plus F_i dV is equal to zero. What is that you learn from this? What is that you learn from this or what can you conclude from this? Look at this equation carefully. It means that the whole body is under equilibrium. When I say whole body is under equilibrium, if I take a small chunk from this body, a small chunk from this body and apply this equation, it has to be under equilibrium. This equation has to be valid for arbitrary volumes of the body. I take small volume, I substitute it in terms of traction forces, this has to be valid. So, this has to be valid for arbitrary volumes of the body or in other words this has to be valid at every point of the body.

I can consider infinitesimal volume around a point and say that I take this out, apply the same procedure and I will get the same equation. For every infinitesimal volume, at every point that you can define, this equation has to be valid. What does that mean? That means that when this total guy is going to be zero, it is not zero on an average sense. It is not zero in an average sense but it is zero at every point in other words zero because this chap here is going to be zero at every point.

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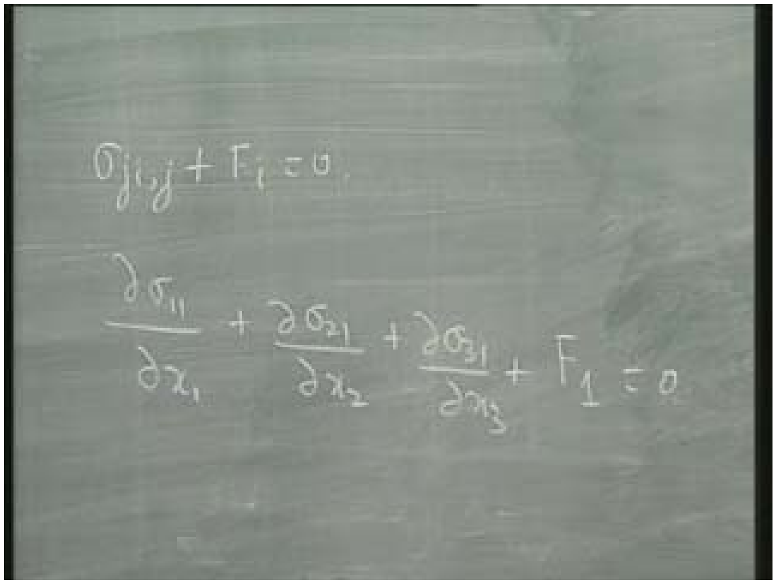

I get sigma$_{ji}$ coma j plus F$_i$ is equal to zero. So, that is the equation which actually I get by considering equilibrium equation or by considering the equilibrium is called as an equilibrium equation. Please note that I started with sigma F is equal to zero. How many equations are there by the way? There are three equations. Correct; there are three equations. Please note this carefully, students tend to make a mistake that this i is not a repeated index. This i is not a repeated index. This i sits in one part of the term, this i is another part of the term. The repeated indices are valid only if it is grouped like this.

So, you should not say that this is one equation because both j is repeated as well as i is repeated. Please note ultimately the summation operation signifies a tensor operation; summation operation signifies a tensor operation. Hence this means that I have to apply i is equal to 1, i is equal to 2, i is equal to 3 and the summing up is valid for j. So, you can say that then, i is equal to 1, I get, what is the type of equation I will get? dow sigma$_{11}$ by dow x$_1$ plus dow sigma$_{21}$ by dow x$_2$ plus dow sigma$_{31}$ by dow x$_3$ plus F$_1$, which is the body force in one direction is equal to zero. Like this you can write down three equations. This is the equilibrium equations; very, very fundamental equations. What all you do is to solve this or variant of this; you either solve this directly or variant of this. Through that procedure only we are going to get stiffness matrices and so on.

Let us now look at the next part of the equilibrium equation. Just to recapitulate so that you are not lost in these whole things, what all we have been doing, please note that we started, where did we start? We started with a small anomaly in the definition of stress which we know. We had In all our earlier classes or in the beginning classes of your engineering you would have learnt stress to be force per unit area. We thought or we saw that there is an anomaly between that definition and the result what you saw in the first class and hence we started defining stress. We defined it much more rigorously; we did not compromise on your understanding. We said, yes, your understanding is right, but I have to define that in a slightly more rigorous fashion.

Hence I defined stress taking a point and taking a plane. From that we define state of stress, from that we came to equilibrium equation. One small factor is left, but I am going to do this derivation, not because it is very difficult, though the result is extremely important. I can do that in a much less rigorous fashion, but I am going to do this in a very rigorous fashion basically for you to understand the use of indical notation. It is going to be a small jugglery here and there and hence that will enhance your understanding of the indical notation and hence I am going to do that only from that point of view. We will come back to this figure now and put a small say add some things which we have not done right now I hope. Any question on this?

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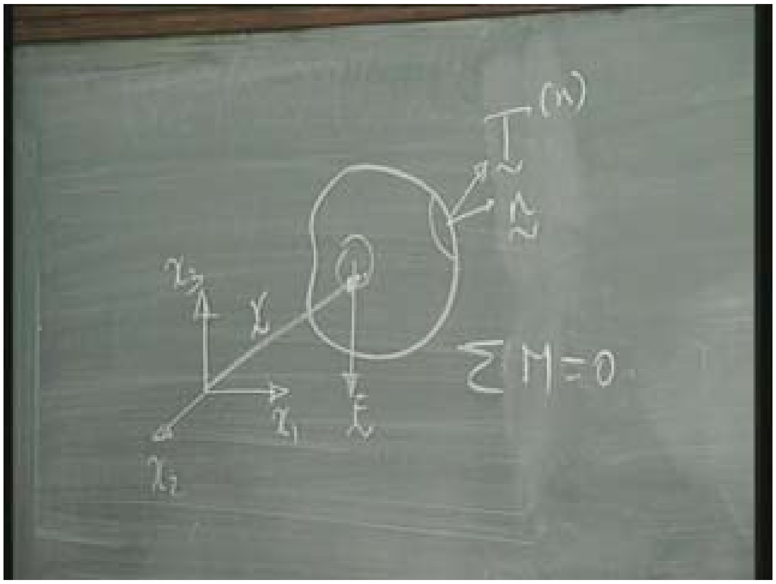

Now, let me add to this figure what is called a position vector. Let me define a position vector say to that point to be say r. What is r, a position vector, which means that it is defined by means of x_1 x_2 and x_3. They are nothing but the coordinates of that point. So, that is the position vector. What is that I am going to do now? I am going to say that sigma M is equal to sorry sigma M is going to be zero and what is the effect of sigma M? What is the effect of sigma M? That is what I am going to do now. How do I do that? What is that I am going to do? I am going to take a moment about, say, the O, origin. What is the result or how do I take the what is the moment or how do I get it? By defining the cross product.

You see that there is a vector operation called cross product which you are using because you have understood what moment is and when I do a cross product, I will get this moment. See, lots of difficult things are simple or made simple by taking a cross product. Once you understand what the cross product is, you can take moment without much problem. So this cross product is associated with this kind of activity, determining moment or physical phenomena.

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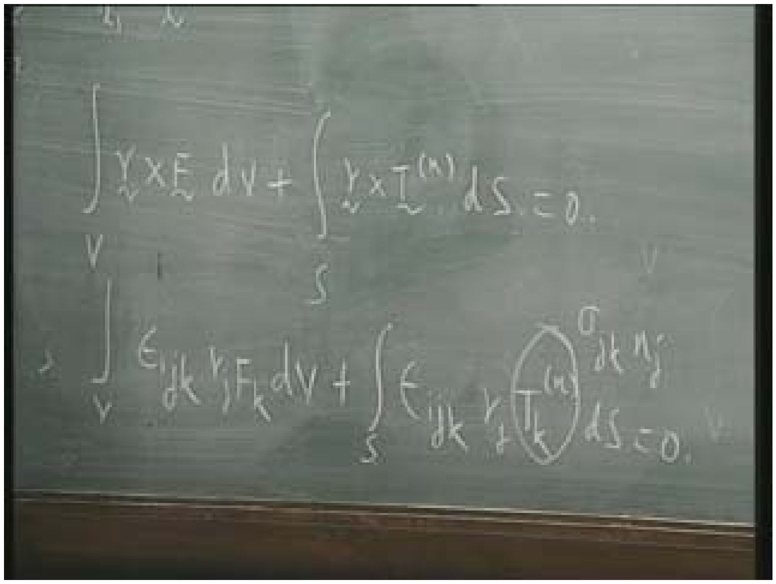

So, let me write that for the body forces and then for the surface forces; that is equal to zero. Yeah! Any question? Now let me express this equation in terms of indical notation. What is the indical notation we had used? Please note that carefully for cross product or please look at it what is the indical notation that we have used? Correct, fantastic! So epsilon$_{ijk}$, epsilon$_{ijk}$; here r_j F_k; r_j F_k d_V plus same thing epsilon$_{ijk}$ r_j T_k n dS is equal to zero. T_k is the stress vector. Now I want to replace this by our stress tensor. You know the operation I am going to do, same thing as what we did before. What is that I do? This quantity, I am going to replace this by sigma. How do I write that? How am I going to write this? That is the crux of the whole thing.

Let me make a, I would say mistake. Can I write this as sigma$_{jk}$ n_j? Can I write like this? Because, T_k Immediately you have a tendency because, I have been using j there, so, immediately or Remember we wrote T_a is equal to sigma$_{ji}$ n_j; that is how we wrote. So, when I say k here, immediately the tendency for students is to write jk n_j; that will be the tendency to write. Now you replace it with that. So, let me replace that with that. Let us see what happens?

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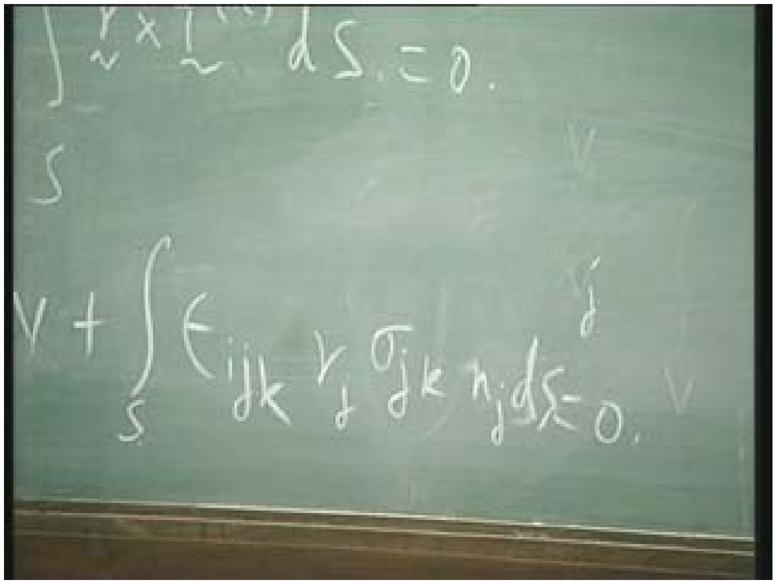

Sigma$_{jk}$ n$_j$ dS; this is n. Look at that equation. Does it appear nice to you? Does it appear nice? What is the problem? Fantastic; so, j is repeated. I am not very happy with that or it is not the right way to do things. I know that these two indices have to be repeated and I know that they are dummies sitting there and I also know that if I use any other index, I will mean the same thing. So, what is that I do? I am saying that this is wrong and so let me replace it with another index say l.

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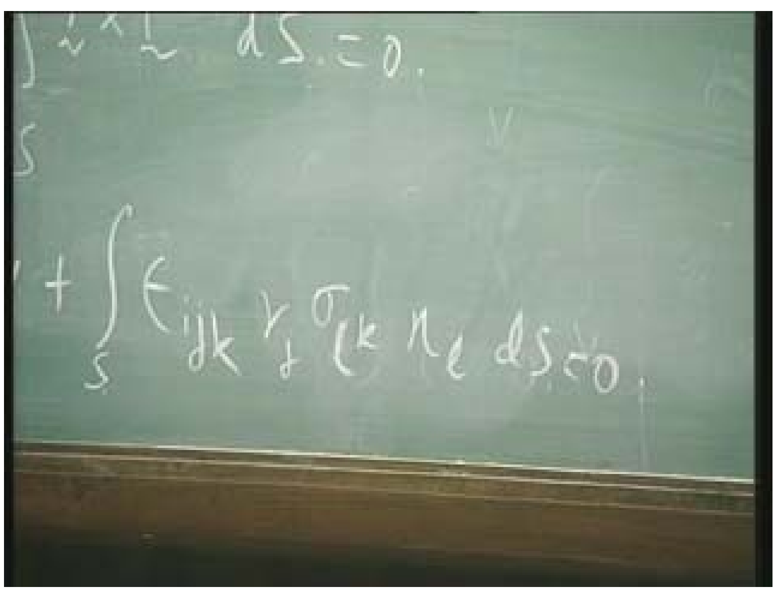

Hope it is clear; n_l dS is equal to zero; n_l dS is equal to zero. Any question? Yeah! Yeah! How is j repeated? See, when I put j here it does not matter, whether it is in a same variable or in an independent variable, as long as they are in the same conglomeration. For example we said $sigma_{ij}$ $delta_{ij}$ or ji, when they are in the same conglomeration or same assembly, to understand that, then I said that you have to add it because j gets repeated. That is $sigma_{ji}$ delta $sigma_{ij}$ $delta_{ji}$ or $delta_{ij}$ I said you have to add it. $Sigma_{ij}$ $epsilon_{ij}$, you have to add it or sum it up rather; summation notation is valid. It is not that the summation is valid, only if it gets repeated for the same variable or the same tensor; it is not that the $sigma_{ij}$ should become $sigma_{ii}$.

Say, for example u_i v_i; i gets repeated in u and v. The result is a scalar. Same fashion, here it does not matter where it gets repeated. So, it becomes $epsilon_{ijk}$ r_j $sigma_{lk}$ n_l; looks quite complex but once you understand, it becomes very easy to understand. Take a minute and look at this and see what the result of this kind of operation is or what results from this operation? What are the fellows who get repeated there? k gets repeated, j gets repeated, l gets repeated and what is that ultimately you have? i, which means that what is it? It is a vector. That is the result of a cross product.

Now, we go ahead and apply divergence theorem to this particular part of the equation because I want to convert this surface integral into volume integral. Let us write the first step here. Integral $epsilon_{ijk}$; I am going to again, you know be aware; I am going to ask you question.

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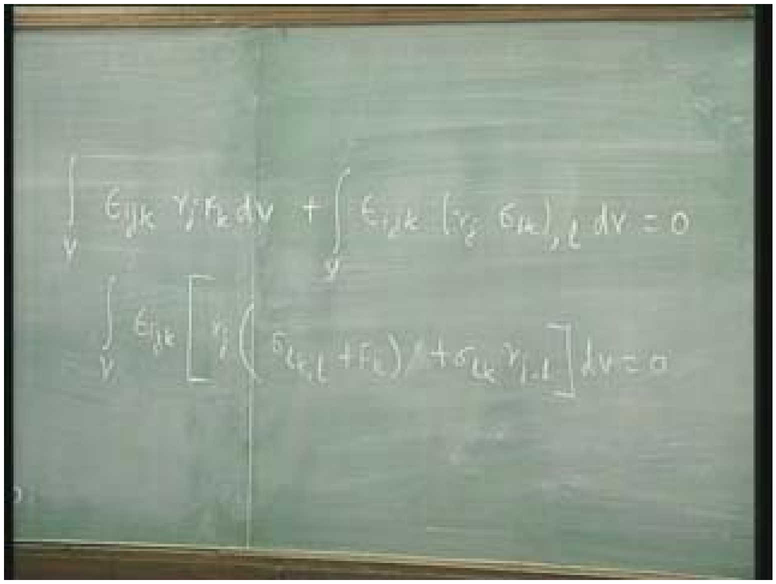

I am not going to write that so that you will become familiar with it; F_k d_V plus plus let us see what you have to offer to me. Where? How am I going to write this? How am I going to write that? Because of the fact that this differentiation, I mean differentiation with xl does not give me, it is zero so I can write this as epsilon_{ijkl} sorry ijk r_j. No; both of them are there know, no both of them are there. So, r_j, correct sigma_{lk} coma l dV is equal to zero; sorry. Let me bring them together and see what are the things that I am going to get with it? So, let me write this as V epsilon_{ijk} into r_j into I have to differentiate this r_j coma l sigma_{lk} plus r_j sigma_{lk} coma l. So, the first part I am taking; r_j into sigma_{lk} coma l plus F_k, dV we will put it at the end plus sigma_{lk}, correct, r_j coma l the whole thing dV is equal to zero. Is every one with me? What does this give me? Actually we have done quite complex operations. I think now it is so very logical and straight forward. I do not think there is any difference, I mean difficulty in this. Now, look at that and tell me how I can simplify this. How I can simplify this? What is the first term?

That is the first term is zero. Why? Because, it is the result of my equilibrium equation; very good. I want you to participate because, then you will understand whatever we have done; fantastic.

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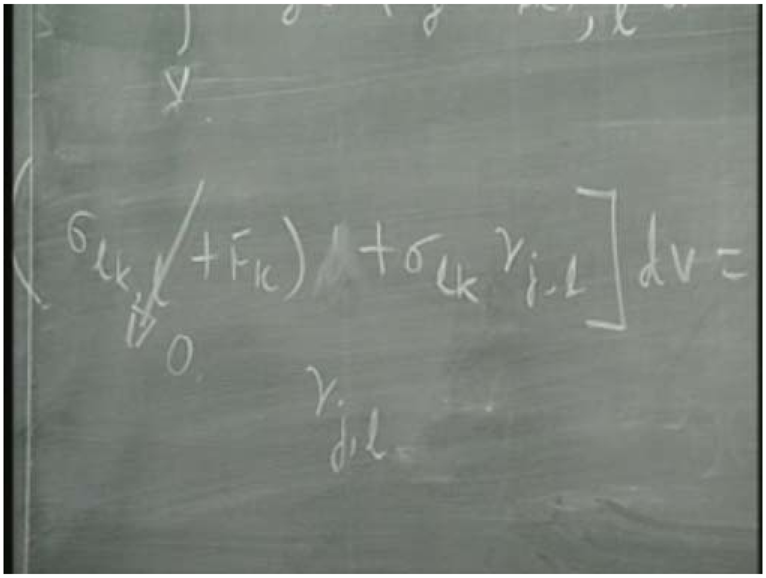

So, this term goes to zero. I have only this term. Now, let me look at that term r_j coma l. What is r_j coma l? No; dow r_1 by dow x_1 or dow r_1 by dow x_2 and so on. So, what would be the result of this? What will be the result of this? No, no; please note this carefully. What is r? r is nothing but the position vector; position vectors consists of r_1 r_2 and r_3. Note this; r_1, r_2 and r_3. r_1 corresponds to the position in the x_1 direction, independent of x_2 and x_3. There are in other words In this euclidean space we need three quantities to define, r_1 r_2 and r_3. So, r_1 They are all independent. So, r_1 is the quantity which is independent of 2 and 3; 2 independent of 1 and 3,3 independent of 1 and 2. What is that I will get out of this? What are the quantities that are involved here? They are say dow x_1 by dow$_1$ which means j and l is 1; dow x_1 by sorry dow r_1 by dow x_2 dow r_1 by dow x_3 and so on. Since the r's, it depends upon only the corresponding x, so what will be dow r_1 by dow x_1? That will be 1. But dow r_1 by dow x_2 is zero; correct, because it is independent. So, many people what they do is, which may be also easier to understand is, instead of r they will write x also; or in other words the position vector x will be written as x_i e_i.

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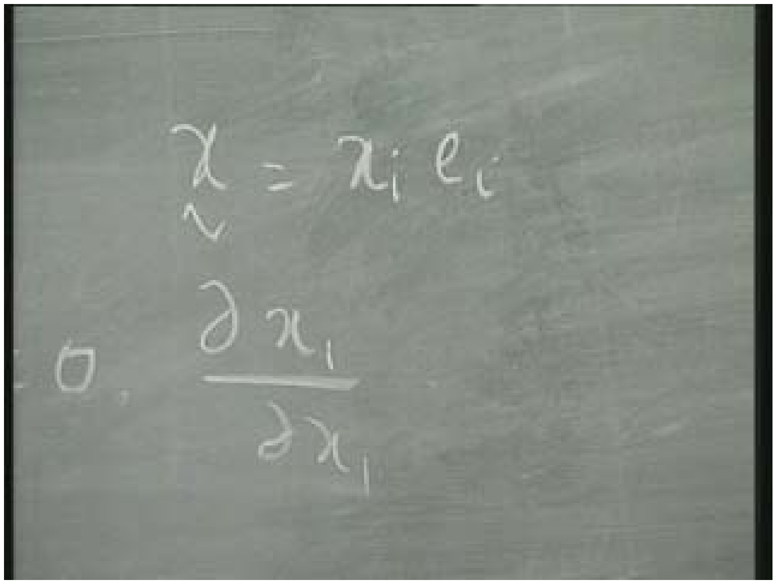

I did not do that in the beginning because, it may be slightly confusing. But I think now it will become clear so that when I now differentiate dow x_1 by dow x_1, it will be 1 and x_1 by x_2 will be zero and so on. It is also a nice way of looking at it. The whole thing comes from the fact that this coordinates are independent. So, if you want you can replace it, this r by x as well so that you do not get confused.

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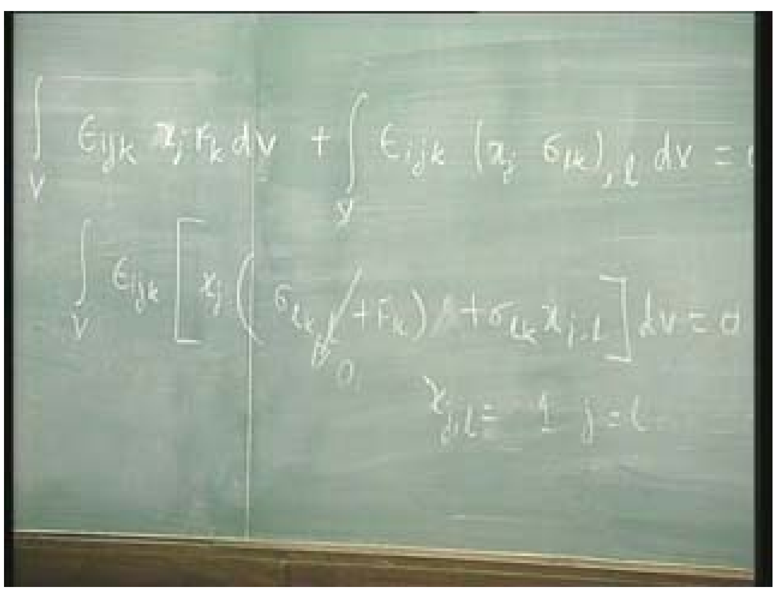

I hope you understand that and you can replace it by x as well without losing the understanding of what is position vector? So, this term now is equal to how do I write this now? So, x_{jl} is equal to 1 when j is equal to l and is equal to fantastic, one minute and equal to zero when j is not equal to l.

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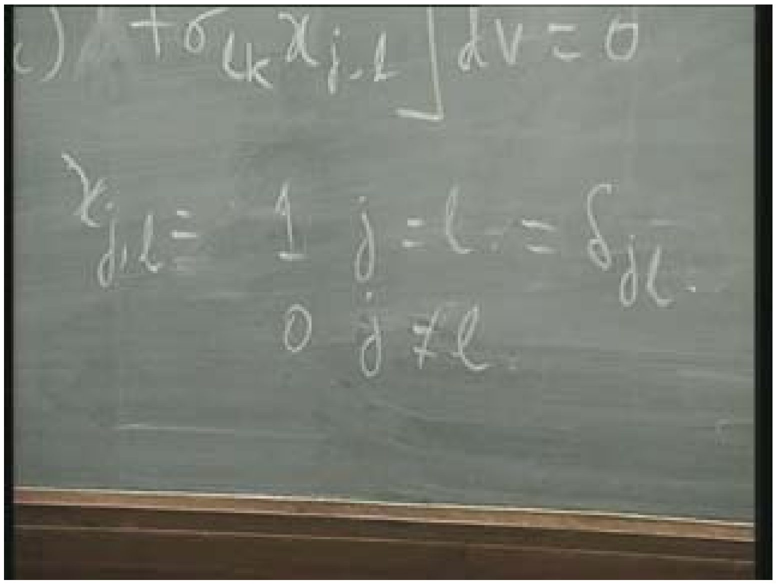

Fine; so this is equal to $delta_{jl}$. Is every one with me? So, this x_j coma l is equal to $delta_{jl}$ is one thing where you will repeatedly be using in mechanics; lot of places you will use this. It comes from the fact that x_1 x_2 and x_3 are independent quantities and x_1 depends upon only the corresponding x and so on. Extending a same argument like what I did in the last derivation that this chap, this fellow who is inside has to be independent of the volume which I am considering, this term inside the bracket has to be equal to zero. Whatever term that is inside the bracket (Refer Slide Time: 43:12) has to be equal to zero; in this bracket. What is that I will get out of this? I will say that $epsilon_{ijk}$ sigma; before we proceed what is $sigma_{lk}$?

Let us simplify this. $Sigma_{lk}$ x_j coma l, what will that become?

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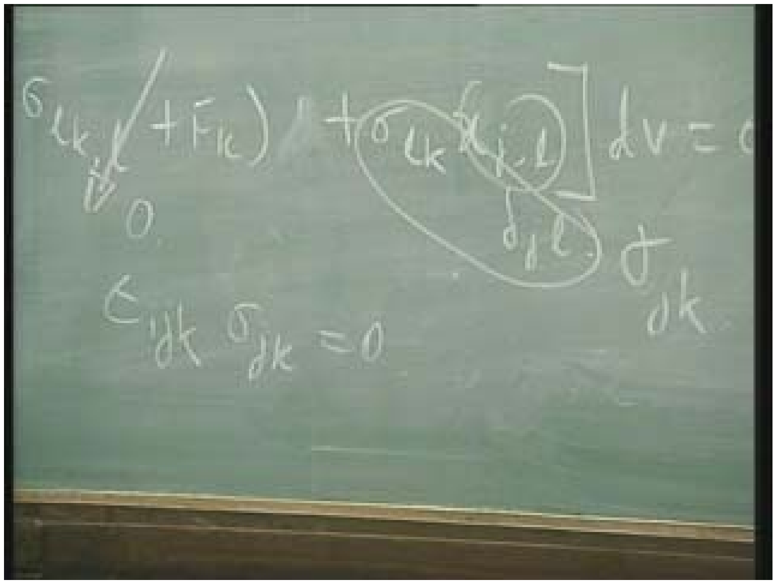

This is delta$_{jl}$; this is what we saw here. What will this become? What will this or what will sigma$_{lk}$ delta$_{jl}$ should become? What is this? What does this indicate? This indicates that this is equal to 1 only when j is equal to l. When j is equal to l it has, one minute, j is equal to l, this is equal to 1. When j is not equal to l this becomes zero. So, sigma$_{lk}$ delta$_{jl}$ can be written as sigma, no, no; when j is equal to l only this will have a value or else it will not have a value. So, when j is equal to l only, it has a value or else it will not have a value. So, l will be replaced by j. That is it. So, this whole term sigma$_{lk}$ delta$_{jl}$ will become sigma$_{jk}$.

If you have a doubt, you just write it down in your note book; expand this and you will understand because delta$_{jl}$, see, this is again one of the operations which you will quite often do. When you multiply some quantity with delta$_{jl}$ or delta quantity, since delta has a value only when these two are the same, immediately what you can do is go and replace; remove delta, go and replace this index with this index, so, sigma$_{jk}$. Extending my previous argument which I am not going to repeat epsilon$_{ijk}$ sigma$_{jk}$ is equal to zero. So, when I do this, I mean you expand it. I am not going to do that now; thing is quite simple now for you. When I do that I get to the fact that sigma is a symmetric tensor or sigma$_{ij}$ is equal to sigma$_{ji}$.

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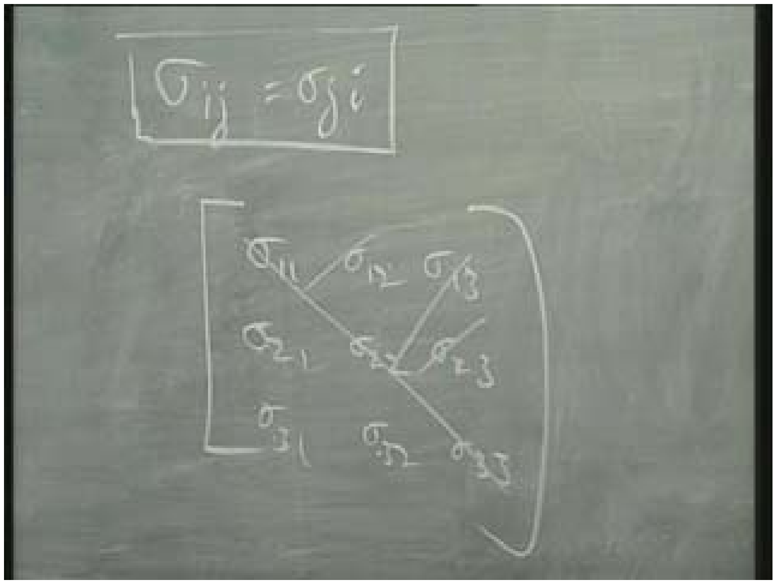

So, sigma_{ij} is equal to sigma_{ji}. What does it mean? It means that in the tensor sigma_{11} sigma_{12} sigma_{13} sigma_{21} sigma_{22} sigma_{23} sigma_{31} sigma_{32} and sigma_{33}, this is what you have to define and this is equal to this, this is equal to this and this is equal to this. As I told you, it had lot of simplifications using indical notation and once such a derivation you go through it again; may be you can go through it two or three times. Once you go through this derivation you will understand completely the operations you can do with indical notations. Patiently go through this and you will understand, then you will not have any doubts about indical notations. This is the reason why I did this.

So, with these two derivations we move over to another important property, two more properties. The ultimate aim, before we go there I mean, we will close this class by looking at the ultimate aim of what I am trying to do. The ultimate aim is to interpret results whether it is design or manufacture or whatever it is. What is my ultimate aim?

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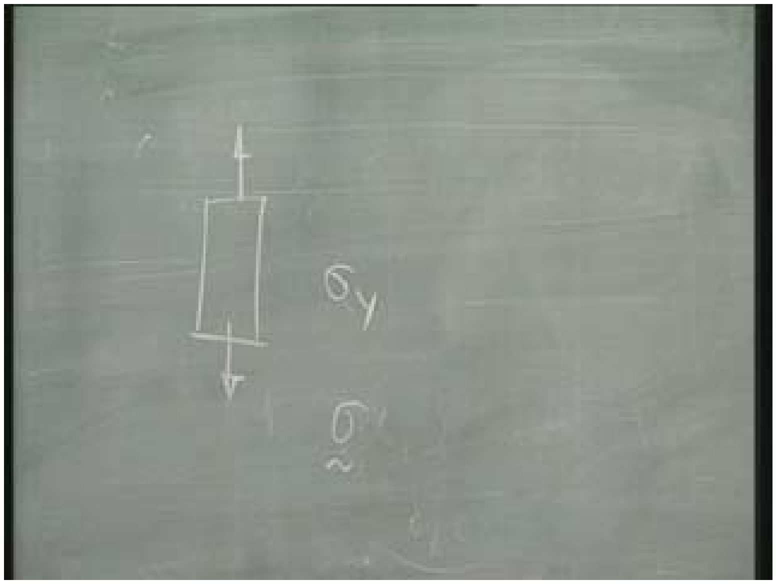

Now, all of us know that some tensile tests are done with different specimens; standard tensile tests are done and you have what is called as yield strength of the material, $sigma_y$ or in other words what we find from tests is given by one quantity or yield strength of the material. The question is that look, you have defined 9 quantities and called this by various names, tensors and all that. Now, as a designer my interest is to tell or to determine when my material will fail; that is my interest. You have given me 9 quantities here. But, from the material result or test result I have only one quantity. How am I going to link up this with this? Which are the 9 quantities that I have to take or can you now combine and tell me what that quantity is which I have to take? That is the question and that is what I am going to answer in the next class.