Introduction to Finite Element Method
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Lecture - 6

There are some questions on what we did in the last class. Let us clarify some of the doubts because I know that there has been a jump in the level of the subject which we have been dealing with till last class and lot more mathematics was involved in the last class. So, let us just recapitulate what we did and clarify certain of the doubts.

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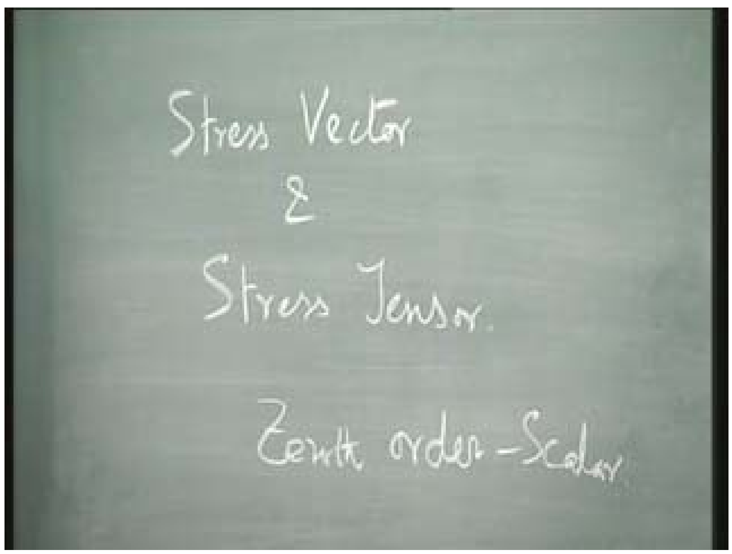

The first confusion that usually students have is what is the difference between stress vector and stress tensor? The first thing that we should understand is that tensor is a generic name, generic name. Tensor can be zeroth order. Zeroth order tensor is nothing but a scalar.

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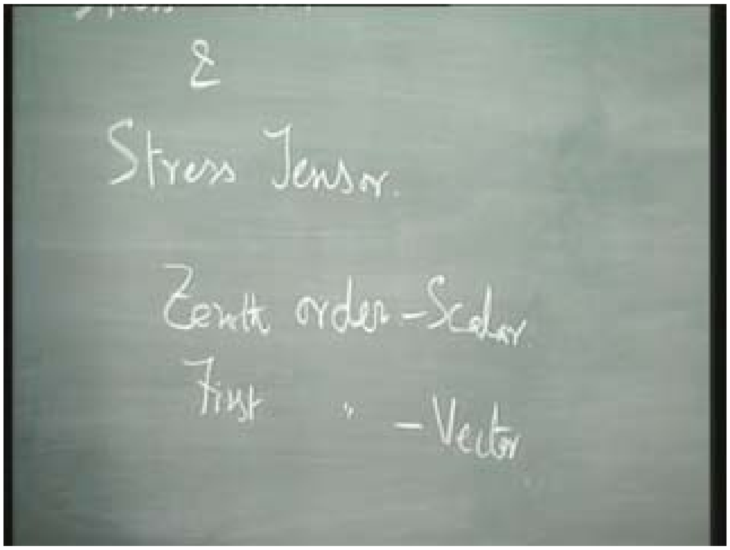

The first order tensor is called as the vector. Then the tensor keeps moving to second order, third order, fourth and so on. It is a generic name. Now, the question is what is a vector and what is the stress vector and what is the stress tensor? If you remember we were dealing initially for the definition of stress, a vector quantity; for the definition of the stress, we were dealing with the vector quantity. All of you know that vector has a magnitude and direction. It is at a very fundamental level or I would say in the earlier classes you would have defined vector to have something magnitude and direction.

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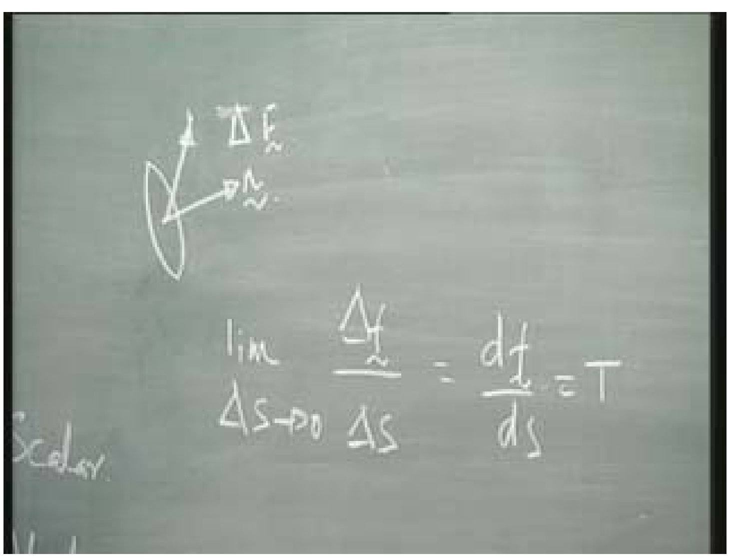

For example we took that small surface around the point P and then said that let this be the normal of the surface and let this be the force or the resulting stress vector. This is nothing but let me call this is as delta f or F which means it is the force and remember that we said limit as delta S tends to zero, we define stress as delta F by delta S or this is nothing but the definition of df orF by dS. I need not even specify that it is nothing but df or F by dS and this is what we called as a vector T. Because this is a vector, we divided this by an area and hence T is a vector.

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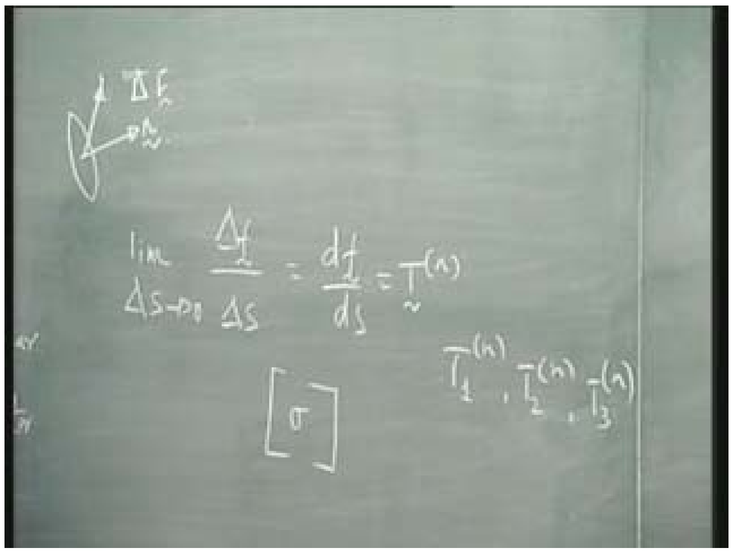

Now, we put one n there and a squiggle at the bottom. This is to indicate that this quantity is a vector. This is to indicate that the T depends upon the plane that you choose; we saw that already yesterday. Your question is what do you mean by T_1 n, T_2 n and T_3 n? That is the question. What do you mean by T_1 n, T_2 n and T_3 n? What are these things? Is that right? They are nothing but the quantities associated when I resolve this vector in the x_1, x_2, and x_3 directions. This you would have In your earlier classes you would have written T is equal to T_1 n_i plus T_2 n_j and T_3 n_k plus. In this course we are not going to use i j and k because of the indical notations that I am going to introduce shortly. Hence instead of i j and k I call them as e_1, e_2 and e_3. They are called as the basis vector. So, T is now defined as T_1 e_1 plus T_2 e_2 plus T_3 e_3. So, this is a vector. We went ahead and derived a matrix ultimately for sigma and called this as state of stress at a point. Remember that we called this as a state of stress at a point. We said that there are 9 quantities, 9 quantities with it.

We associated two indices with the sigma i and j. One index, the i index indicated the plane, j direction. So, i and j are two indices whereas here, how many index? Only one index. The 9 quantities cannot be expressed in terms of three baseses as e_1, e_2 and e_3. There have to be nine baseses to express this. Let us not worry about too much of theory in tensors right now, but remember that this has 9 and that has 3. So, this is called as a second order tensor. People deal with this in a matrix form. That is an easier way for an engineer to deal with a second order tensor. That is an easier way to deal with the second order tensor. But though mathematically there is a difference between viewing it as a matrix, viewing it as a tensor, but for all practical purposes when we do manipulations with it, the concept of matrix is extremely useful to us. Hence we look at it as a matrix.

Why did we define this? We had a long discussion and ultimately we came up with a relationship between the sigma and T.

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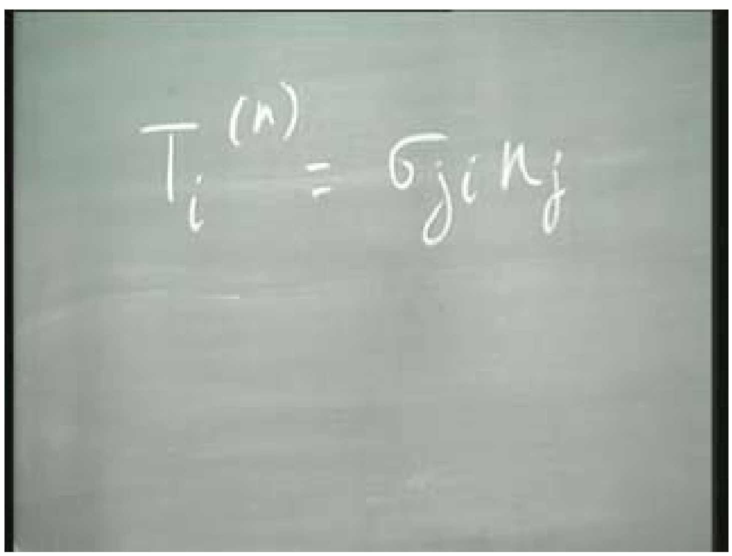

You remember what was their relationship? We said that say for example T_i n is equal to sigma$_{ji}$ n_j. So, we came up with the relationship of this kind. What does this relationship tell us? It tells us that if you give me a state of stress at a point P which means that you give me all the sigmas, sigma$_{11}$, sigma$_{22}$ and so on and if you ask me a question what is the stress vector at this point P at a plane n, then using this equation I will tell you or I can calculate and tell you what are the T_1's, T_2 and T_3 or what are the

T_i's rather? I will be able to tell you T_i's. That is what I mean by this expression. Now, let us expand this. Is that question clear? Is that clarifies your doubt? Please also note there is one more question that what is T and what is n? They are totally different quantities or what is delta f and what is n?

See, sometimes when we talk and when we use it we have a tendency to say force and then say corresponding stress; stress vector and the force vector. I think when we use it, it will become very clear the difference between the two and this definition here will tell us what is the difference? Please note that both of them are different. One is the normal, the other is the force that is acting. There is a force that is acting or the effect of the other body onto this body where P sits. If I consider a corresponding point in the other half of the body, remember I had cut it into two pieces; corresponding point in the other half of the body, then the force will be acting in the opposite direction, like bar we ….. I do not want to repeat what we did, but like that bar where we cut; one side there was a force like this, the other part will have a force in the opposite direction.

What does this equation now mean? There are indices. As I told you they are summation indices. There are rules for summation. These indices have in other words These rules are not based on lot of physics; you know it is based on certain rules to make things easier for us. Why do you sum up? You cannot ask a question because it is a convention; it is rather a convention like for example you write a vector with the bold letter, like that it is a convention. What is the convention? The convention says that any vector or a tensor or whatever quantity it is, can be expressed or even a scalar can be expressed in terms of a letter and an index associated with it; by means of a letter and an index associated with it.

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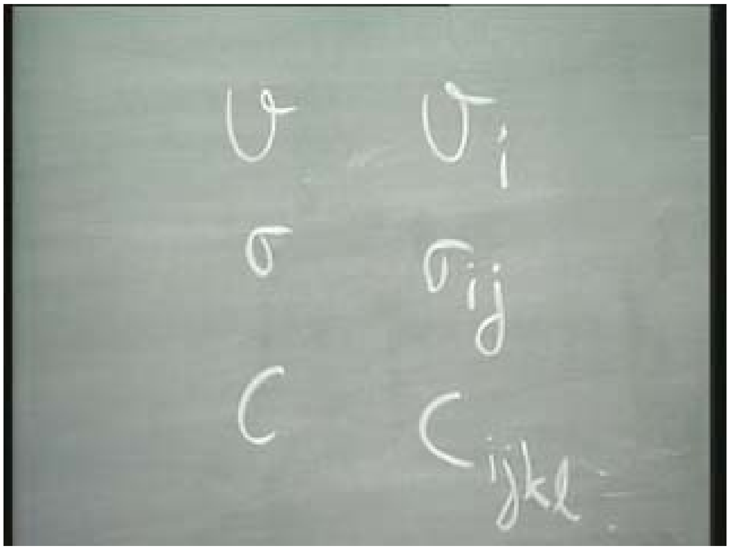

For example, v is a vector. All of us know that it can be resolved into v_1, v_2 and v_3 and we express this as v_i. Sigma, what we just now saw? We just now saw sigma and we associated with sigma two indices which we called as $sigma_{ij}$. I can associate, say for example, what is called as elasticity tensor, say C with the index C_{ijkl}. These indices are not repeated. i and j are different and so they are free indices. At one level, at a very simple level, this also indicates the order of the tensor; number of free indices indicates the order of the tensor.

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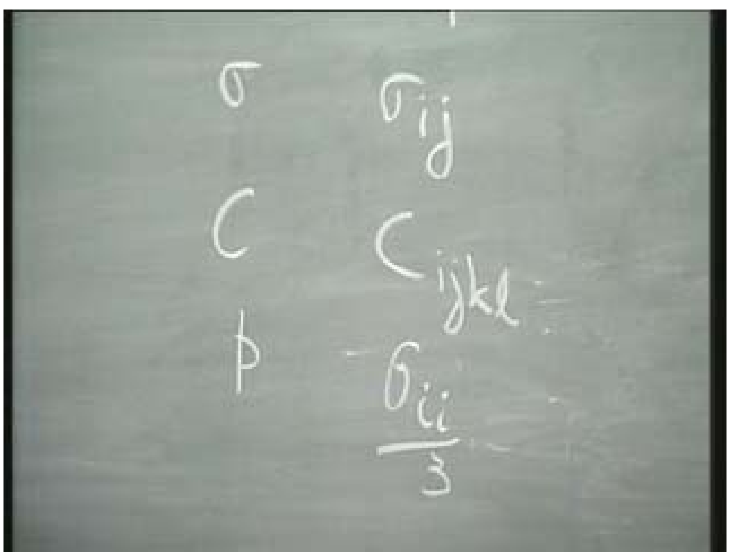

I can write say for example p as σ_{ii} by 3. There is a difference between this and this? What is the difference? What is the difference between the two? It gets repeated, i now gets repeated. Here it does not get repeated, here it gets repeated. The convention here is whenever an index gets repeated it means that you have to sum up over the range of that index which means that σ_{ii} is equal to, what is it? σ_{11} plus σ_{22} plus σ_{33}.

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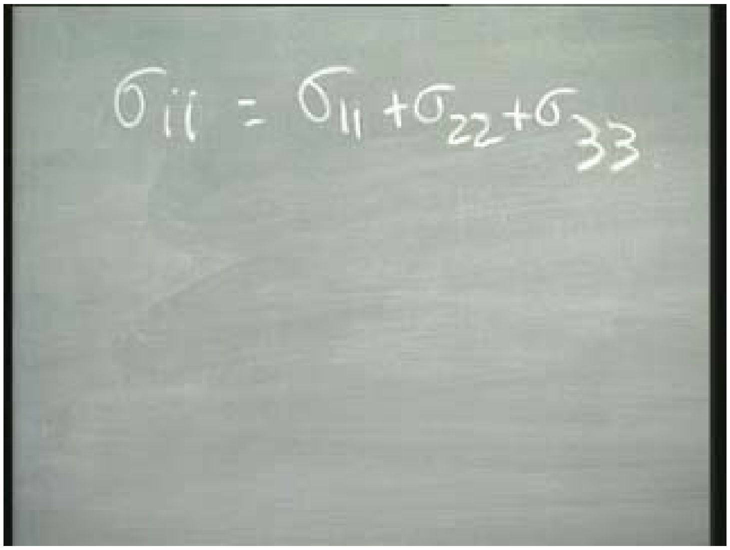

So, σ_{ii} is equal to σ_{11} plus σ_{22} plus σ_{33}. Here we have divided by three; it does not matter. σ_{ii} means that σ_{11} plus σ_{22} plus σ_{33} whereas, here in this case when I say about σ_{ij} there is no repetition, which means that i and j takes values ranging from 1 to 3. So, I have 9 different quantities. That is what it means. This means that I have σ_{ij} means that I have 9 different quantities whereas when I say σ_{ii}, it means that there is only one quantity. What is this σ_{ii} then? Whether it is a scalar or a vector or a tensor or a second order tensor or what is that? Yeah! That is true that is scalar because there is only one quantity. There is only one quantity and hence this is a scalar. So, this kind of index which does not get repeated, stands out free, tells me the order of the tensor. So, this is a fourth order tensor, this fellow is the second order and so on.

Tensors are also classified by transformation of coordinates. I will not deal with it in this class, I will take it up in a later class. The tensors also are dealt with from transformation. We will go step by step to understand it, let us not lose the thread. We will come to that later, but right now let us only worry about how to study or how to understand the indical notation; I know that they are difficult notations. It is not that always we put these indices as a subscript or below here.

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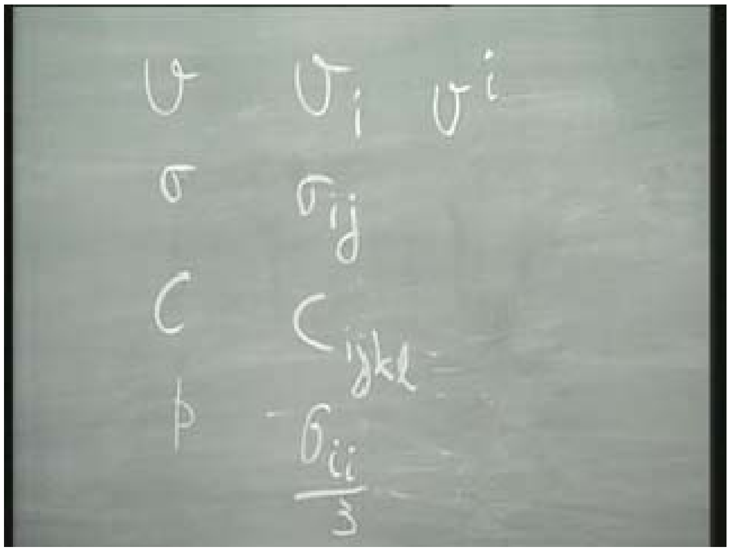

Under certain conditions, we can also or we will also put it as a superscript as v i. But, in this course fortunately we will not use indices as a superscript but it has its own meaning and so on. Now we will only use it as a subscript in this course. When I go to superscript, then I have to deal much more with tensors which I am not going to do, but we will deal only with this kind of indices. Summation index, as you right now would have understood and got a grip of it, is that whenever there is a repetition you have to sum up. Again, this mechanics people have a convention.

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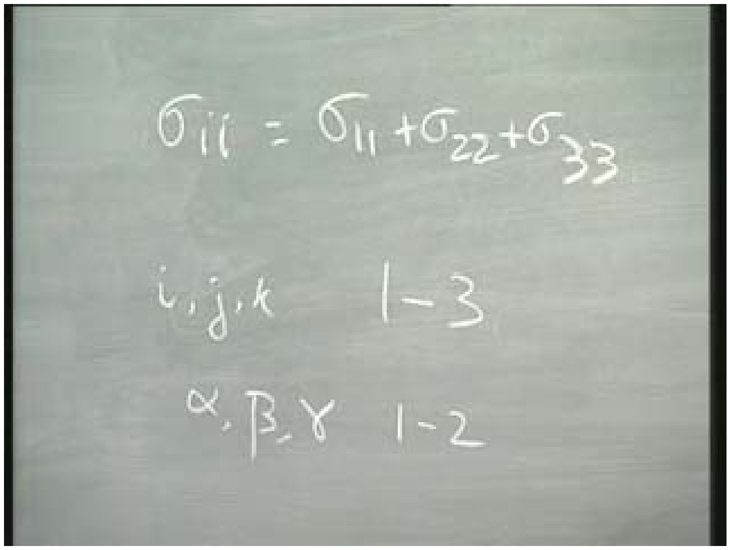

They say that as long as you use roman letters for subscripts say i, j, k, a, b, c, d, whatever it is, these roman letters then the range of the subscript is from 1 to 3. So, when I say sigma_{ii}, my range is from 1 to 3. You can also use for subscript what are called as Greek letters. So, say for example I can use alpha, beta, gamma and so on. I can use for example sigma alpha alpha, subscript; instead of i and j, I can use alpha, beta, gamma and so on. So, when I use alpha, beta, gamma, my range is restricted to 2 or in other words my range is 1 to 2. What does that mean? It means that sigma alpha alpha means that it is sigma_{11} plus sigma_{22}; that is all, sigma_{33} does not come into picture. Only when I want 3, the range to be 3, then I use i j and k. Any questions on it?

Now let us go back and look at this expression. Having clarified all these things, let us now write down this expression.

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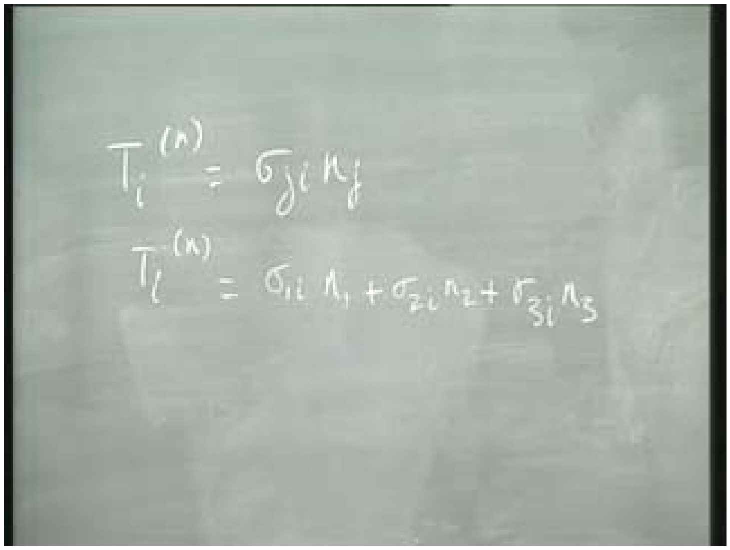

Look at the left hand side and look at the right hand side. What do you see in the left hand side? You see one i. On the right hand side you see i stand out or free and j is what is repeated. So, what do I do now? I sum up over j. This expression can be written as $sigma_{1i}$ n_1 plus $sigma_{2i}$ n_2 plus $sigma_{3i}$ n_3. This is for one i, where or 1 expression say i is equal to 1; rather I should say when i is equal to 1, I get one equation. Then, when i is equal to 2, I get another equation. When i is equal to 3, I get three equation. The three equations with each of the equation having three quantities are summed up into one simple expression.

Please note that this j is a dummy index. Because of the statement which we made just now, I can replace j by k, for example.

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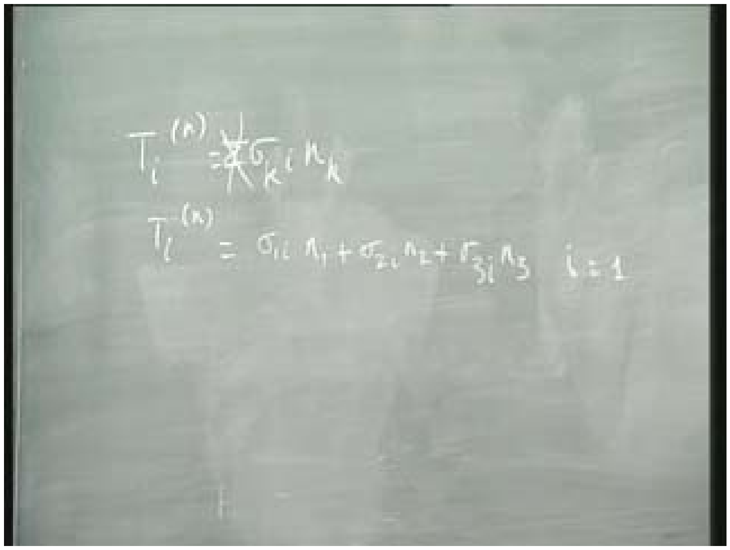

I can replace this by k. Do my expressions change? No; it does not matter. But when I replace it by alpha, beta it will change because I will lose one term. This is a simple summation convention. In your earlier classes what you would have done? You would have put one sigma before it and you would have said that by putting a sigma and i varying from 1 to 3 and j varying from 1 to 3 or k varying 1 to 3 this is what you would have done.

Now instead of doing that what is that I have done? I have said do not worry about the sigma, putting a sigma here; do not do that. But whenever you get a repeated index you sum it up. Note that your left hand side and right hand side should balance as far as these indices are concerned. What do I mean by that?

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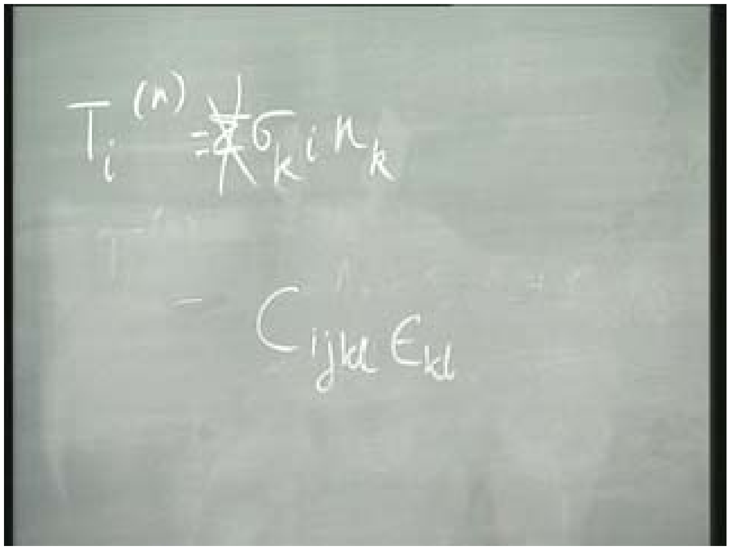

Let me write one more small expression which we will use later to understand what I am saying. Let us write C_{ijkl} epsilon$_{kl}$. Just take a second to understand what this means. Look at that. You have ijkl kl? What is this first one? It is a tensor, correct. But what is the order? Fourth order. What is this? Second order tensor; so, fourth order tensor now operates on a second order tensor. What it means? It means that this is something like a function which operates.

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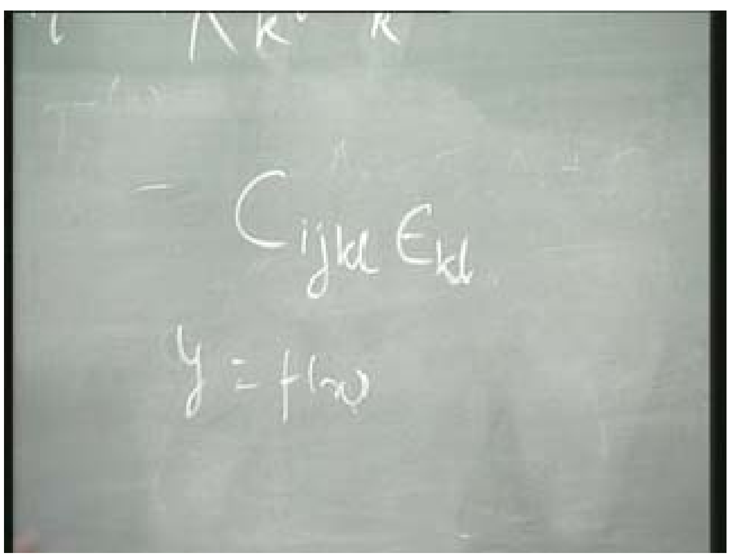

You would have probably seen and understood very well say y is equal to some function of x, more sophisticated fashion you would have called this function as a mapping, function maps x to y. You would have written it in a different fashion. As you grow and as you take higher and higher levels of mathematics, initially you would have called this is a function, then you would have called it as a mapping and so on. So, you can say that this is an operator. It operates on epsilon_{kl} to give one more quantity. Please note that this quantity has to be what? Correct; it has to be a second order tensor so that my indices balance each other. In fact this is the famous expression for the stress strain relationship.

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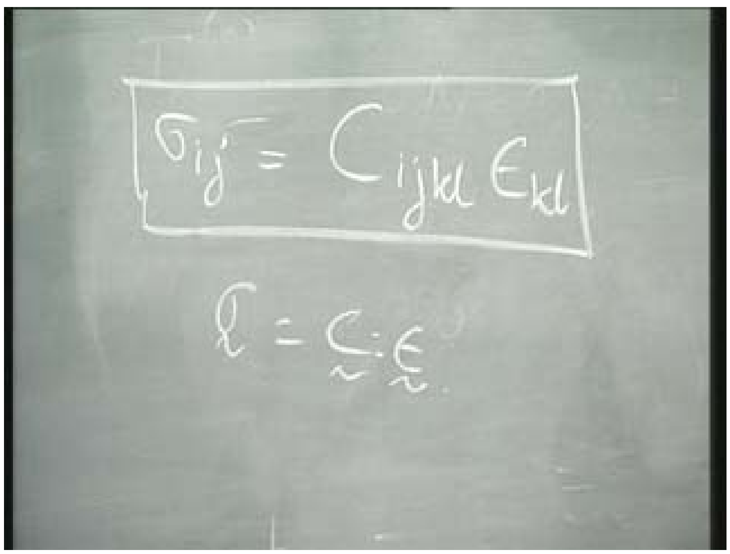

See, sigma_{ij}; that ij fellow is here, then kl is here and epsilon_{kl}. So, ijkl epsilon_{kl}; kl gets repeated. So I will have how many equations by the way? Totally how many equations? 9 equations, correct. i and j 11, 12, 13, 21, 22 and so on. So, I can have 9 equations and each equation will have how many terms? 9 terms, correct. Because kl will have values ranging from 1 to 3, so I will have 9 terms. Totally I have to write 81 terms; 81 terms. What is that I have done? I have summarized the whole thing in one simple single equation. In fact, as you go along many books, they will not even use this kind of indical notation. What they will do is to write this as sigma is equal to C epsilon. Some books put two colons or colon rather, two dots before it and so on. That is why, say, this convention how to write it may vary from one book to the other. So, you have to look for it. Actually these are tensor operations. We will not go into the details of it, but it is important to understand what these indical notations mean. Any questions?

Yeah! There can be one small doubt, ….. what happens when this kl gets repeated 3 times? 2 times is fine, why not 3 times? Can I have one more term and have again another kl? In tensor analysis usually it is a bad practice, again I am saying it; it is a bad practice to write thrice. It should not get repeated thrice. But many times or a few times, I should not say many times, sorry for that; but few times it may so happen that you may have to write it again. It may so happen, in which case the convention is that we will abandon summation; summation is not implied. As far as possible you should not write it, repeating it three times. You should write it such that it only repeats twice.

Now, having understood this, we will move to slightly more difficult conventions.

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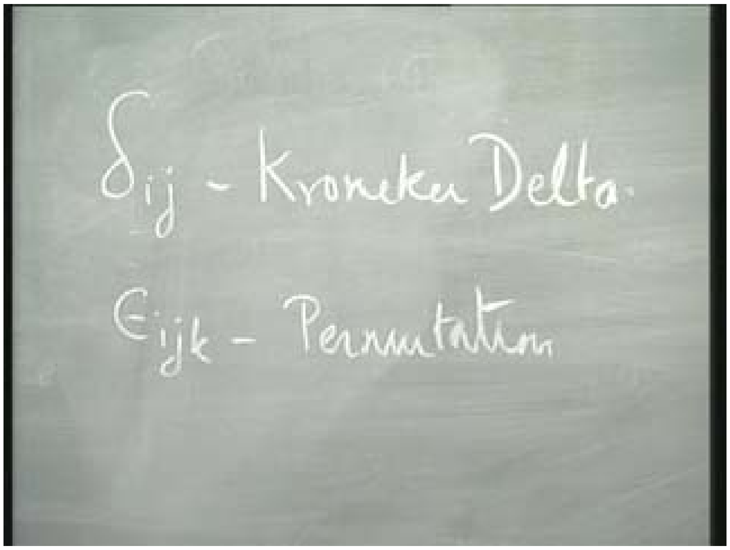

These conventions come from two letters called delta written as delta_{ij}, which is called as Kroneker delta and a permutation symbol, epsilon_{ijk}; this is called as permutation symbol. These two symbols have some special meaning to it; these two symbols have some special meaning to it. Let us understand what these things mean? Now, the first question that you may ask is what is so great about these symbols or why is that these symbols are used? Why do you want to introduce this? These symbols are used for easy manipulation, as you will see in a minute. First let us state this symbol; it is again a convention.

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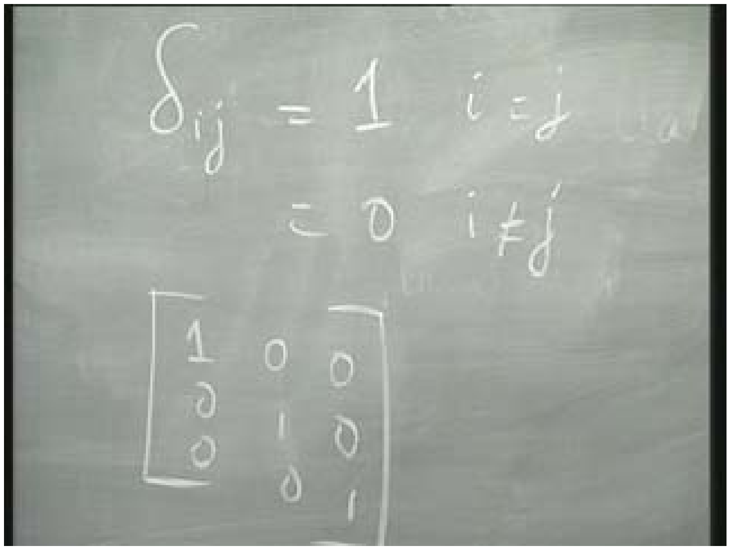

The convention states that when i is equal to j, how many quantities are there by the way for delta$_{ij}$? 9 quantities; so, again it is a matrix, delta11 and delta$_{12}$ and so on. When i and j are the same which means that it is the diagonal of the matrix delta$_{11}$ delta$_{22}$ and delta$_{33}$, then this Kroneker delta takes a value of 1; takes the value of 1, when i is equal to j. It doesn’t mean ii. Please note, it is not i i. Delta$_{ii}$ has a different meaning, that is summation. When i is equal to j, that means 11, 22 and 33, then delta$_{ij}$ takes a value equal to 1. When i is not equal to j, Kroneker delta takes a value of zero, when i not equal to j.

Can you recognize what this really means? Can you put down for a second this in a matrix form and tell me? Correct; unit matrix. So, when I look at it, it is 100 010 001. In the matrix notation this becomes a unit matrix.

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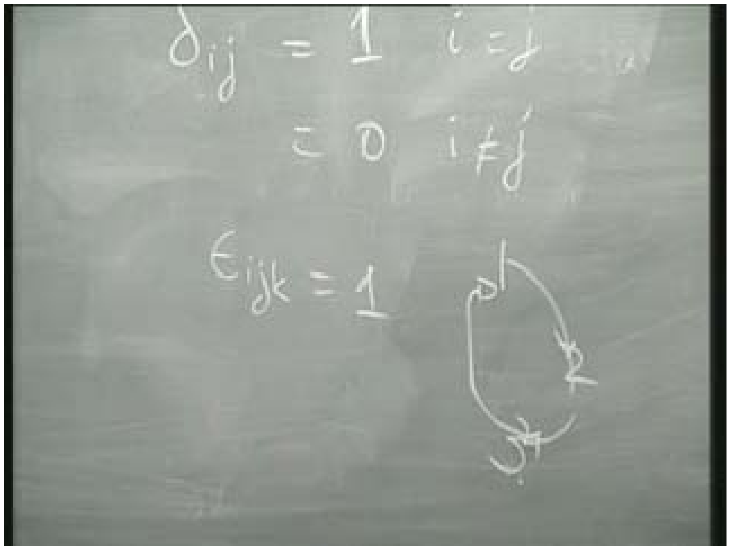

We will look at small operations with this in a minute. Before that let us understand what is $epsilon_{ijk}$? $Epsilon_{ijk}$ is a permutation symbol, can take on values with i j k ranging from 1 to 3. So, it is a third order tensor. This is a second order and that is a third order tensor. As the name indicates this is a permutation symbol and hence we can write first 1 2 3 in a particular order, say. Whenever ijk is such that they take a clockwise order that means $epsilon_{123}$, 123 312 or 231, we say that the corresponding value of epsilon is equal to 1. These are clockwise.

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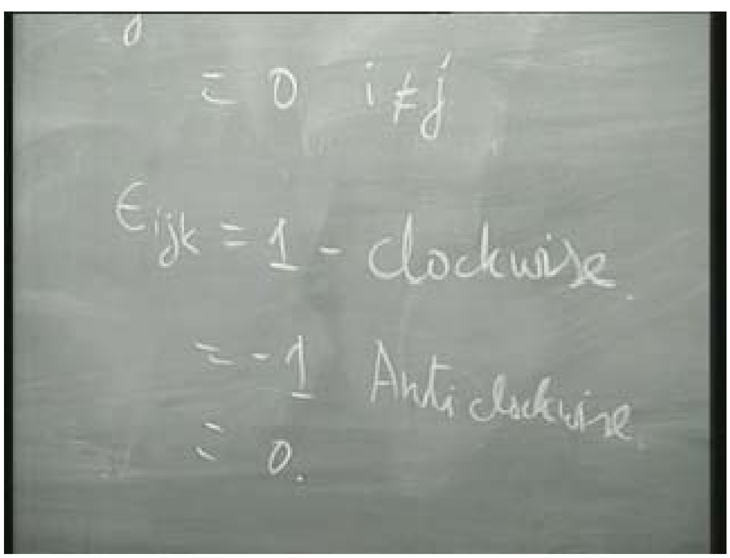

This is for clockwise 1 2 and 3. Whenever I have anticlockwise, instead of 123, 213, 321, and so on, this permutation symbol is deemed to have a value. It is just an assumed value. We say that this will have minus 1 for anticlockwise. Otherwise, whenever I get this index to be repeated, for example epsilon_{ijk} becomes 112, it is a part of it; 112 or 122 or 121 and so on, whenever this index gets repeated it is supposed to have a value of zero.

We will do two simple problems to understand, where how we use this. Before we go further we will just try to understand. Is there any question on indical notations? Any question? Now, I will give you a problem in just two minutes so that just think about it and we will discuss it in a minute. I hope these questions are clarified. Are all these things okay? Whatever we have done so far okay? Yeah! I know there are one or two more doubts, I will clarify it towards the end of the class. We will just follow what we are doing now. Some doubts of what I have done in the previous classes I will clarify it towards the end of this class.

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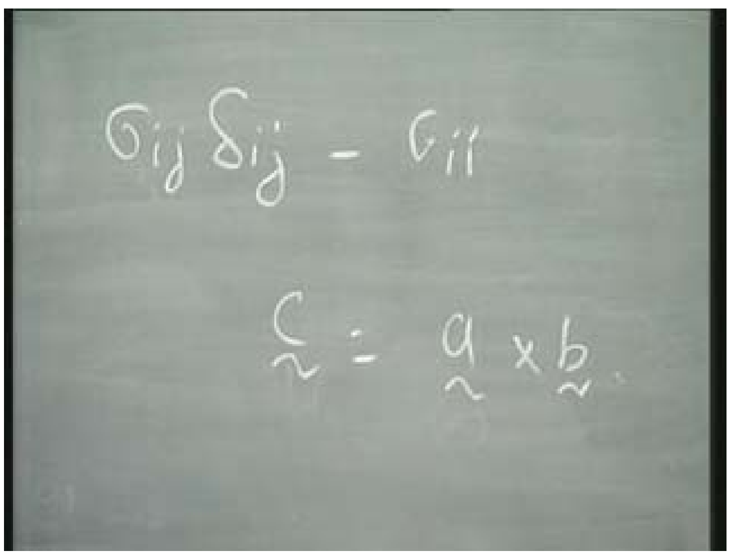

Now the first question is what is sigma_{ij} delta_{ij}? Sigma_{ij} delta_{ij}, what is this? Sigma_{ii}, is it diagonal matrix? No, it is a scalar, very good. So, it is sigma_{11} plus sigma_{22} plus sigma_{33}. What is that you do? Simple; when I look at i and j for a delta, immediately I know that when i is not equal to j, I know it is zero. So, the only terms which will now be available for me to sum up are terms where i and j are the same. I just go ahead and put for this indices i and j to be the same, in which case $sigma_{ij}$ $delta_{ij}$ will become $sigma_{ii}$; so, $sigma_{ii}$.

I will give you a tougher problem, slightly; wait for a minute. Let us say C is a vector and that is formed by our famous cross product. I want to now write How many quantities are there? C_1, C_2 and C_3 or C_i; similarly a, similarly b. I want to write this in terms of C, rather C_i's, in terms of a's and b's. Let me not use a_i's and b_i's. Let us say a's and b's; a_1 a_2 a_3, b_1 b_2 b_3. Can you do this by using epsilon? That is the question. Can you do that using epsilon? What you do is, write it down as you know in terms of C_1 in terms of a_1 a_2 a_3, b_1 b_2 b_3 and see whether you can introduce now epsilon. If you have questions, you want time, do you want a minute or want me to tell the answer? Just try it out for a minute.

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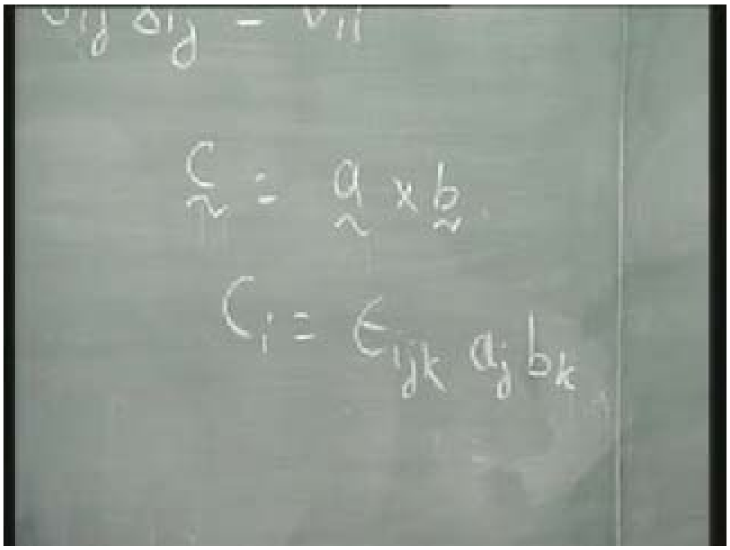

Let me give you a clue; C_i. Just verify; C_i can be written as $epsilon_{ijk}$ a_j b_k. C_i is equal to $epsilon_{ijk}$ a_j b_k. Note the repeated indices j and k. What happens to C_1? You write it down in terms of j and k. Just check up whether we will get the same result. Yes, so how did you get this value first? One is that C_i I will keep this as 1; so, C_1 sorry $epsilon_1$, when I say when I put 11 what will happen to this? Zero. Then 112 again 0; 113 again 0; so, the one, the first fellow who will have the value here is 123; 123 which will have a value of 1. So please expand this and check up whether you get the value, the cross product that you know of. I am not going to do it because it is very simple. As an exercise, you please do that and now you will understand that the use of epsilon$_{ijk}$ is also there to easily write down expressions like this. Such expressions can be very easily written down and you will have more and more use of these quantities later in the course. Any questions?

Having learnt now the indical notation to certain extent, we will go to study what is called as equilibrium equations; we will study equilibrium equations. All of you know equilibrium equations, but what I am going to do is to express it more rigorously; I am going to express this more rigorously. What is an equilibrium equation? For a body which does not deform, immediately you will say sigma F is equal to, sigma F what is that? Zero. Now sigma M is also equal to zero. One of the questions that can come to your mind is why is that I am always putting delta F and then looking at it as a sigma?

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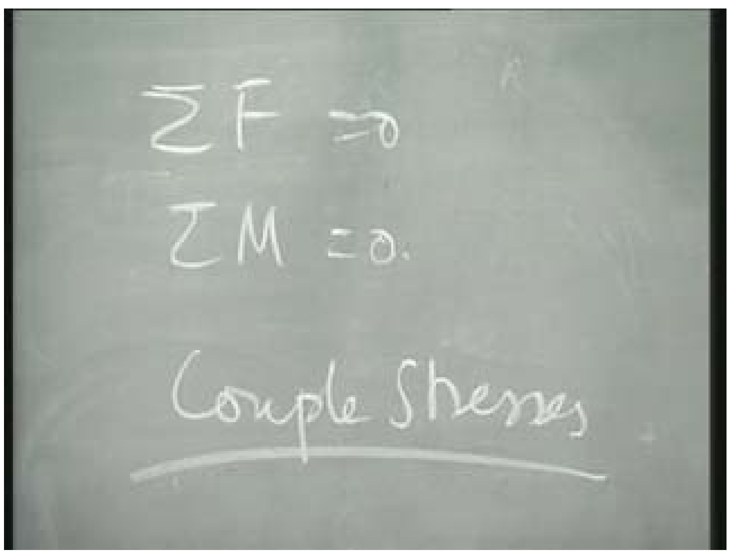

Because for an equilibrium sigma F is equal to zero, sigma M is equal to zero; moment also should be equal to zero. You remember that when I had cut this body by means of a plane, remember we did that, and I had a point P and I had put only F there. I had not put M there. In normal continuum mechanics and in solid mechanics, we do not consider M, moment. When you want to consider moment also, it gives rise to what is called as couple stresses, which we are not considering in this course; we are not considering that in this course. So, let us not worry about … When I cut it, it does not mean that the body is under equilibrium even when sigma M is not equal to zero, it does not mean that. When I cut it and look at the influence, we assume that M is equal to zero at that point and we do not consider M at all. So, let us now first consider what happens to a body when sigma F equal to zero.

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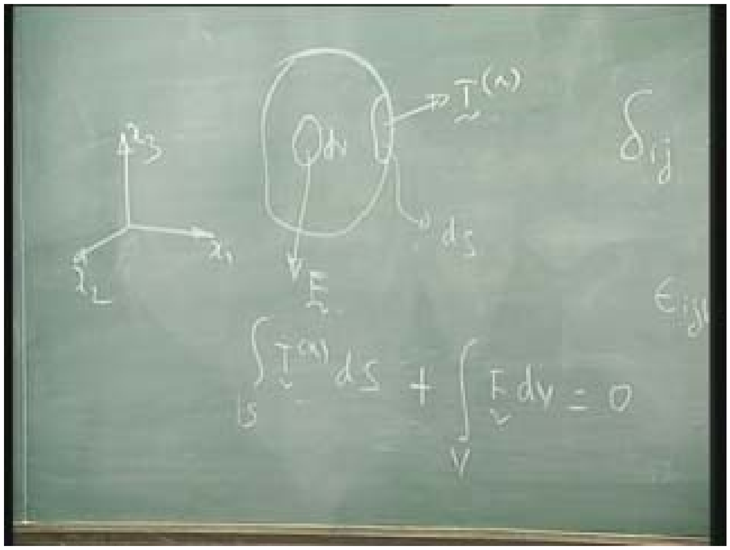

Let us put the body. That is the body of interest to us and the body has two types of forces that are acting. What are the two types of forces, body forces? Let us say that is the volume inside, sitting inside dv and that there is a body force that is acting and let me call this body force as say F. Please note, it is a vector and there is also a surface force that may act on the surface, say dS and that surface force we can call it as, say T. What is that I have left out? n? 4240Now, the first thing that I am going to do is to write down an equation which all of you know, sum of these forces is equal to zero. How do I write that down? I will first integrate all T's, surface forces. So, it is a surface integral designated by S T n dS.

Please note that it comes directly from my definition. So, T n dS or rather I should say that it is a vector, I should I mean sorry stress. I should not call this; always I have a tendency to mix force and stress. So, I hope when you are following it up, we follow this as a or a traction; I should call this as traction and a stress vector that is acting. So, this would be the total force. I am sorry for that; this is the total force. Whenever I put T and say force what I mean is actually the stress that is associated. The context will tell you what I really mean. This plus what is the other thing? Body forces, where

I have to take a volume integral. Always the students have a tendency to get scared about, you know volume integral and surface integral and so on, but what I am going to deal in this course is only the concept. Do not worry that you are going to do lot of integration and you have to study this lot more and so on. I am going to only look at it conceptually.

With respect to volume, how do I write this? Fdv is equal to zero. That is the equilibrium equation. It is possible to write this down in an indical notation. Though I would not like to do it quite often, but it is possible to write this down in an indical notation as well just for clarity because we are just starting on indical notation. Please note that this equation is valid for or I can resolve T and F in three direction, valid for all the three directions.

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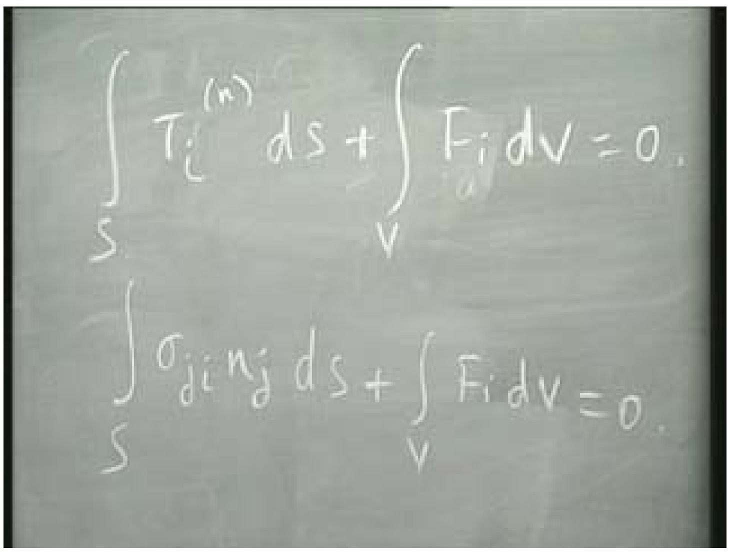

As a common factor for three directions, I can write it as, in terms of i as integral S T_i, i will take values from 1 to 3, n dS plus volume integral F_i dV is equal to zero or in other words T_1 and so on. What is T_i n? $Sigma_{ji}$ n_j; so, I can write this as $sigma_{ji}$ n_j dS plus integral V F_i dV is equal to zero. At this stage, I see that one is the surface integral and the other is a volume integral. Now I want to convert surface integral into a volume integral. How do I do that? Divergence theorem; let us not worry about the intricacies of divergence theorem or how it is derived and so on. We will just write down the result of the divergence theorem; divergence theorem is used to convert.

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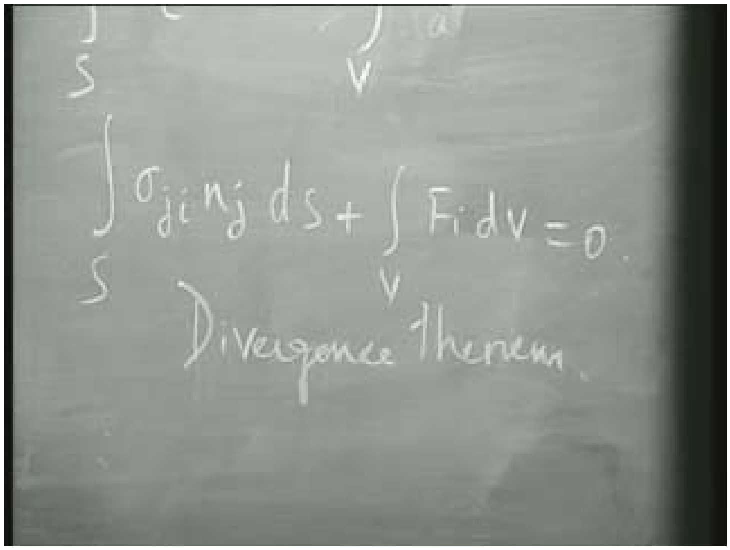

Divergence theorem is used to convert the surface integral into a volume integral. Yes, I know some of you might have forgotten divergence theorem, so, I suggest that you just have a look at it and come back. We will continue this derivation from the place where we left in the next class. But, before we close I want to clarify one or two things and introduce one small notation in order to write down the divergence theorem.

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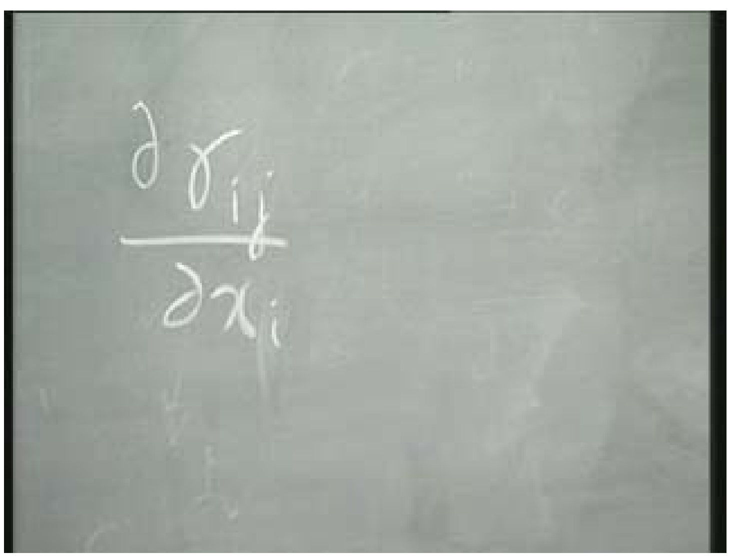

It is possible that we differentiate some of these quantities. Sigma$_{ji}$ or ij can be differentiated with respect to say x_1, partial differentiation with respect to the x_1. So, I can write this as a dow by say dow x_1 or x_2 and x_3. In other words I may differentiate this with respect to i and so on and then sum it up.

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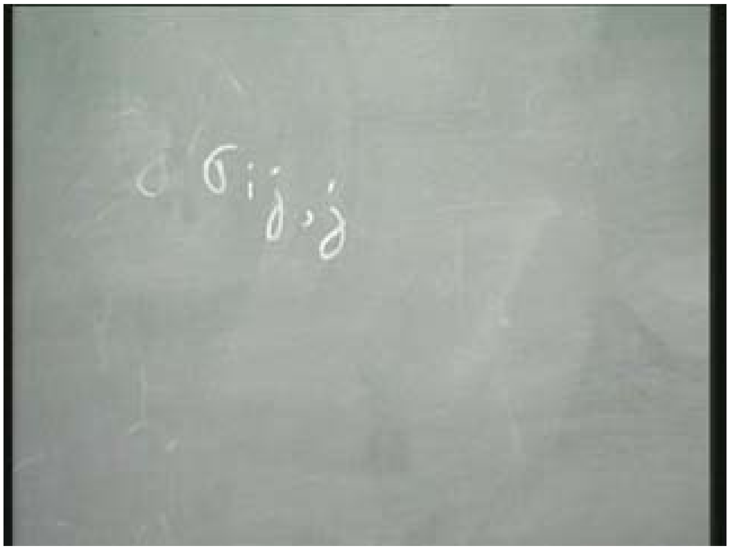

Whenever there is a differentiation like this, I write this down as dow sigma$_{ij}$ comma, sorry sigma$_{ij}$, comma j or comma i or comma k which means the order will increase now. Please note that depending upon what I write here, the order will increase or decrease. This comma indicates that I have differentiated it with respect to x_j and j gets repeated. We will use that for divergence theorem in the next class and maybe we will start the class with one or two clarifications which our friends have sought on what I have done before. I did not want to do it because I did not want to stop the flow of the stresses, that concept of stress that we are dealing with and hence I just carried on with it.

We will come back and recapitulate what all we did in this class and the previous classes. We do not lose track of it and go over to further definitions. If there are any questions, we will start with those questions in the next class.