Introduction to Finite Element Method
Dr. R. Krishnakumar
Department of Mechanical Engineering
Indian Institute of Technology, Madras

Lecture - 5

In the last class we were enquiring about a quantity called stress.

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We said that stress is the result of forces. Remember that we said the forces can be classified into what are called as surface forces and body forces. That is where we stopped. We started defining what is stress? What was the difficulty we had? We said that stress, in our earlier classes rather that stress is nothing but force per unit area. ….. divided this or normalize the force by area and it was very simple. But when the body takes the shape of a very complicated connecting rod or something else like what we saw in many of our examples or for example the sheet of the side wall of a coach and so on, then it may not be very easy for us to define or to use this simple formula. It is okay as far as a bar is concerned; as far as the bar is concerned.

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If I apply a load like this, then, yes, it is quite simple to state that the stress can be defined as force per unit area or sigma is equal to F by A. But the problems or the real life problems rather are not as simple as this. So, we have to have a much more rigorous understanding of stress. Let us see the definition of a stress from a more rigorous three dimensional setting. Now let us consider a body.

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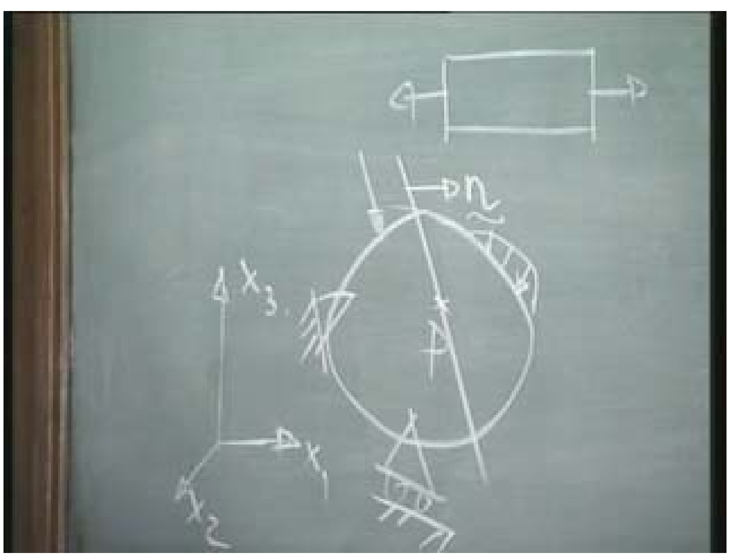

Let us put some coordinate system for it. Let us call this as X_1, X_2 and X_3 and let us say that there are forces that are acting on this body and that there are some, what are called as boundary condition. Remember, we had already talked about the importance of the boundary conditions in one of our earlier classes. Let us take this body out and let us cut this body by an imaginary plane. Our interest is to define a stress at say a point P. Why is that we are interested in a point P? Remember, when you saw our results, you saw lot of colors there. So in other words stresses were defined throughout the volume of the body. Hence we have to define stress at each and every point of the body. So, let me take a point P and let me cut the body with a plane which passes through this point P. Let us say that the normal of this plane is equal to n; all of know that n is a vector. That is the normal. Let me know keep these two body separately. Forget for a moment all these forces and the boundary conditions and let us consider the two bodies separately.

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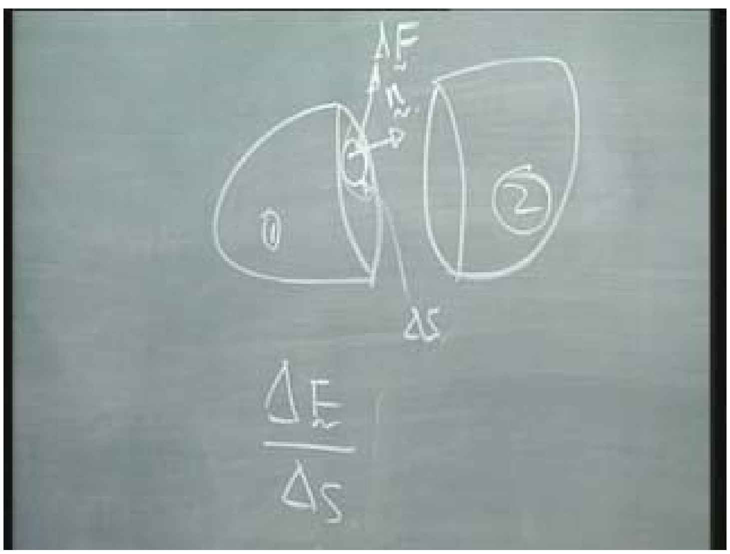

My point P is here. Let us say that we take an area around this point P and call this as say dS. Remember that the normal at this point is n. The normal at this point is equal to n. It is obvious that if I call this as say one part of the body and the other as the second part of the body, then there is an influence of the second part of the body onto the first part of the body. The influence is through a force. Let us say that the influence at this point P is equal to some F or if you want you can say delta F. That is a force that is acting. Now, let us stick to our old definition of stress, but the only difference now is that we are going to talk about area, which are around a point P or let us now say that the normal, the average of these forces that act at this point P can be given as say delta F by say delta S; for a moment to stick to our delta F here let me call this as delta S as well.

The average force or the average force is due to the influence of the other body at an area surrounding the point P is given by delta F by delta S. When I started this problem, I said that let me take a small area around P and now what I have done is to take an average of delta F by delta S. There are two questions that may come to our mind. One is that how small is small, because as I take larger and larger areas, the force distribution may not be very uniform. So, I will get different values of this average. Remember, this is force, area. The definition is still there; our old definition is still there. So, one argument is that if I keep taking different areas around this, because you have not defined how small is small, so, I can take one area, then another area and so on I may get different values at this place. Because I am averaging out in an area and especially when the force distribution around this point P as I cut it and look at the influence of this body onto this body the force distribution may not be very uniform. If it is uniform there is no problem. If it is uniform then delta F by delta S would remain the same. But, if it is not uniform, the average may rise up, go down or whatever it is. So, I have a small problem there.

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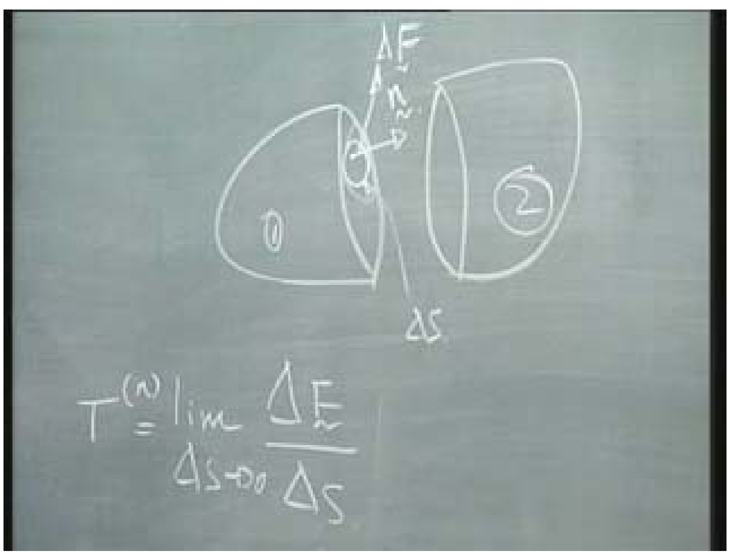

In order to alleviate that problem, I define a stress as limit as delta S tends zero delta F by delta S. This is what I define as a stress say T. Since I have got this stress by cutting with a plane whose normal is n, I call this as T_n, with a subscript n. We will come to this in a minute as to why this n becomes important physically and then more mathematically. We will come to that in a minute, but before that let us look at this quantity. Please note that this quantity is a vector. It is a force. The influence of this body over this at that point which is of interest to me. In other words this is a force, influence comes out as a force. This quantity is a vector because this is nothing but an area. So, this is a vector.

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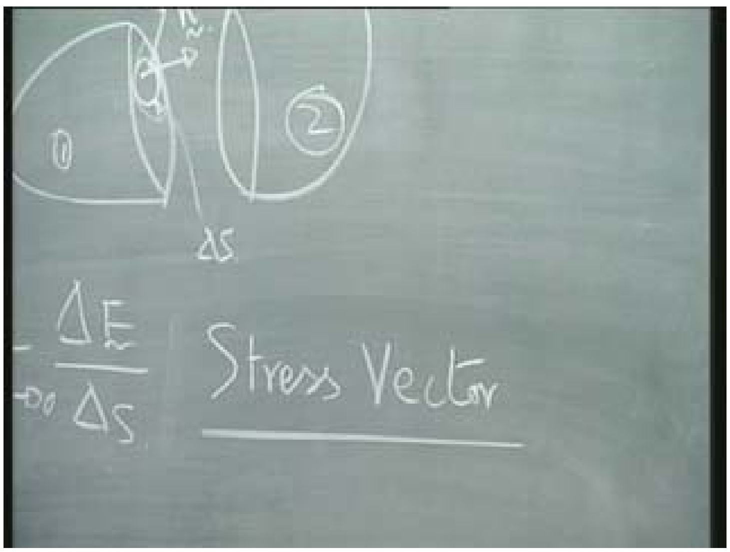

We call this T with such a definition as stress vector.

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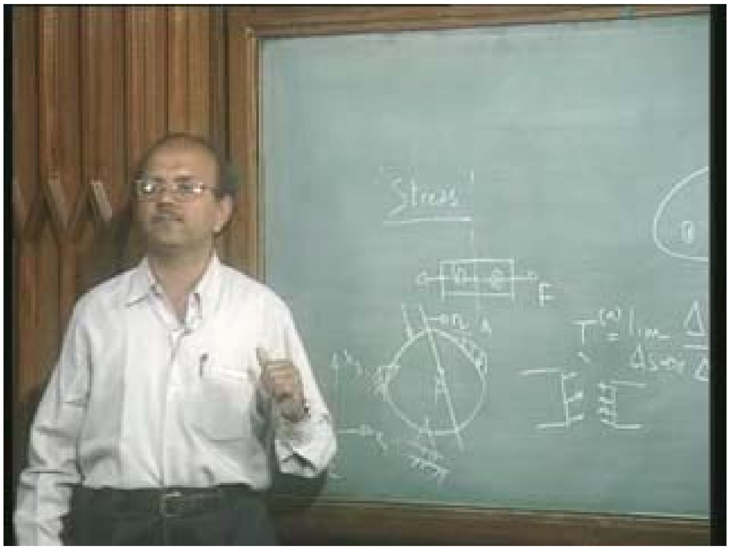

If you remember, if I had this particular bar and in fact if I cut this piece, if I cut it into two pieces like 1 and 2 if I had asked you to draw a free body diagram for this, you would have drawn in this side something like that and this side a corresponding force that is acting in the opposite direction and you would have told me that this is nothing but the internal force which is the result of these stresses. You would have said that this part is sigma A, that part is sigma A and you would have said that this is under tensile load. Just extend that here. The only difference now is that as soon as I say what the stress is here, you had immediately a plane in mind which is 90 degrees to the axis of the bar and we would have taken this plane here. But here what is the stress of this point? Then I have to specify the plane. Now, physically what does it mean? Physically what does it mean?

We have defined a stress at this point as say F by A where F is the force that is acting and A is the area and we had actually looked at the equilibrium with respect to the outside and the internal forces. Suppose I had taken a plane like this, what would be the stress with respect to that plane? With respect to this plane if I cut a body like that what would be? It would be zero. Or in other words at this plane the stress would be zero whereas on this plane the stress would because F by A or in other words physically we know that stress would depend upon the plane. How exactly does it depend upon we will see that in the course of this class but it is very clear that the plane becomes important and that poses a problem. What is the problem? This definition of stress vector is for a particular plane whose normal is n. So, suppose I ask you a question what is the stress at a point P, at this point P which can be related to my point in the actual component? Then the question that you will ask me is what is the plane that you are talking about? Going back to this picture here, I had chosen a plane n.

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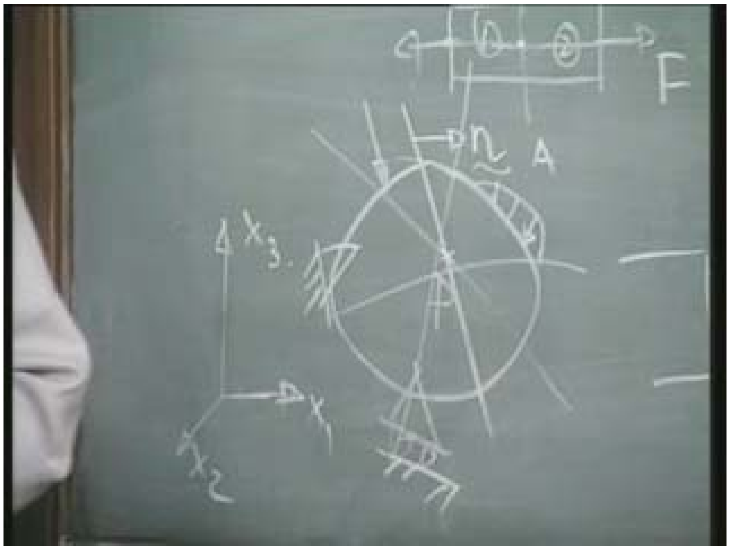

The question you will ask is whether you want the stress at this point P along this plane or along this plane or along this plane or along this plane and so on or in other words this stress at a point P becomes meaningless because that is also wedded to the normal that you can draw or the plane which you use to cut it into two pieces and look at the interaction at a small area around P. So, infinite number of planes I can cut. There are infinite numbers of values of stress that you would talk about or infinite values of stress vectors that you would start talking about. That is quite confusing, isn't it; that I cannot talk about infinite number of stresses.

What is the remedy? One thing is clear that the stress depends upon the plane; another thing that is clear is that I cannot just talk about stress at a point P.

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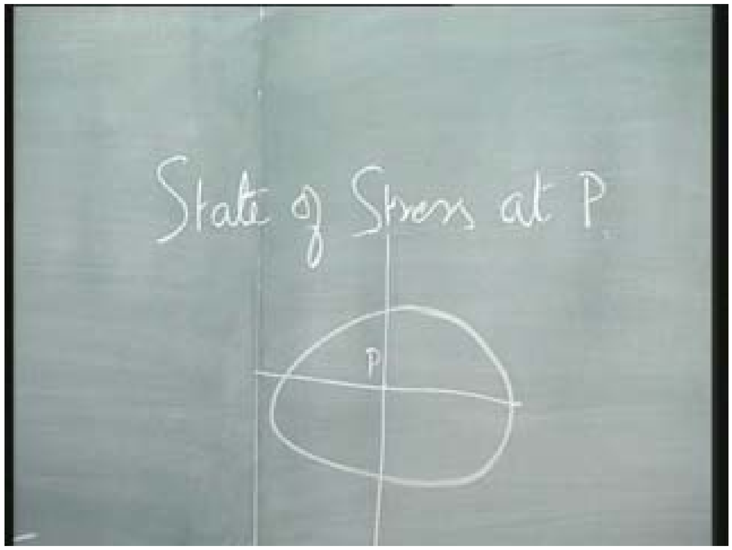

So, the solution to this is to talk about what is called as state of stress at a point at P. What is state of stress at a point P? From this definition it is clear that if I am able to define certain planes very nicely, then I can say that state of stress is related to certain well defined planes or in other words what is a plane which is easy to define for you and me? Yeah, so, that plane is defined by my coordinate systems. For example in the same figure if you want the state of stress at a point P on the same component, we want the state of stress at a point P, then I can define three planes, 1, 2 and another plane perpendicular to the axis of the board such that the normals of these or in other words, the normals of these planes pass through x_1 x_2 and x_3.

These three planes are well defined planes and the first thing I will do is I know that it is difficult to define infinite number of planes. So, the first step is let me define them in these three planes. It turns out as we will see that these three planes are useful, very useful to us because if I properly define stresses in these three planes as I am going to do now, then it is possible to relate these stresses to any other plane that you would define by means of a normal n.

If it is possible for me to say what is the stress in this plane, I can derive a formula to say if you give me what is n then I would tell you what would be the stress in this plane? Now, let us go ahead and define the stresses in these planes. Remember that right now we are talking about stress as a vector; it is stress vector. It has both magnitude and direction.

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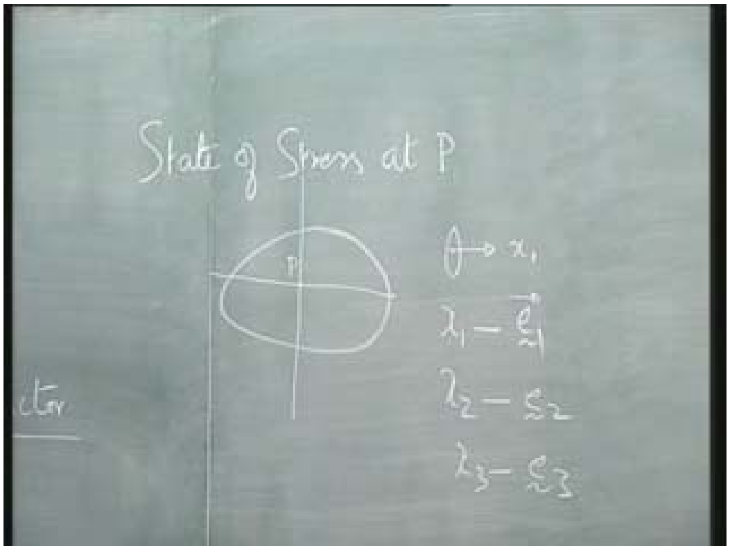

Let me take this point P and then expand what happens in these three planes. Let me take that P here and let me draw a plane whose normal is say along x_1. Let me call the base vectors along x_1 x_2 and x_3 as e_1 e_2 and e_3. That is x_1 x_2 and x_3 each have base vectors as e_1 e_2 and e_3. If it is not visible let me write it quite clearly. For x_1 we have e_1, x_2 I have e_2 and along x_3, I have e_3. Note that I put a squiggle here in order to denote that they are vectors. We will put an arrow at the top on its head like this or in text books you would see that this letter will be written as a bold letter. It not easy for me to write a bold letter here, so, I am putting a squiggle at the bottom to denote that they are vectors.

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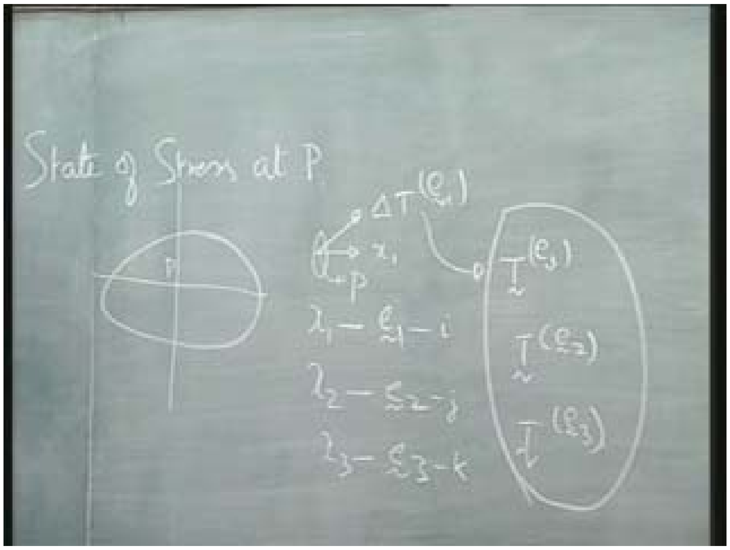

e_1 e_2 e_3, in your earlier classes, you would have been exposed to this. In your earlier classes you would have defined e_1 e_2 e_3 as i j k; that is right i j and k. Because of certain reasons which will become clear maybe towards the end of this class, I have defined them as e_1 e_2 and e_3. Now let me say that this force that is acting on this particular plane is equal to say delta T. How do I put here? e_1 and the corresponding stress due to this as T e_1. What is this? It is exactly the same as this. Here I have taken some plane with a normal n and defined this stress vector. In this case, I have taken a plane whose normal coincides with that of e_1, along with the same direction e_1, and then I have defined a stress.

I do the same thing with respect to this point say P; we are considering that point is P. I am doing the same thing or I am going to do the same thing with respect to the point P with the plane taken such that its normal is along x_2 in which case I would get one more, please note this is different, one more stress what? vector, which I would write as e_2 and lastly I will take another plane. How do I take another plane? Very good; so, the normal with normal in the direction of x_3, which I would call as whose unit vector is called as e_3. So, I will get one more stress vector which I will call as e_3. So, I have defined three stress vectors. As soon as you see a vector what is that which comes to your mind? You say that this vector can be resolved in three directions. I can resolve this vector in three directions so that I can write this as, how would do I write?

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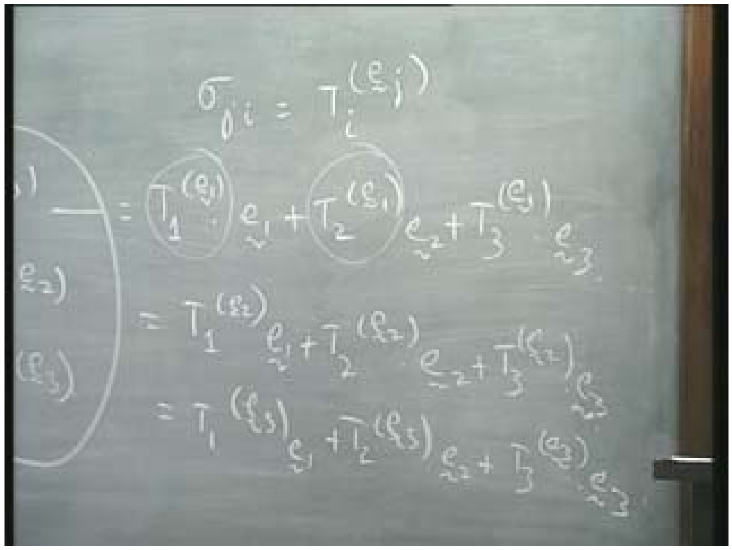

T_1 in the 1 direction e_1 into how do I get? Correct; e_1 that is it, plus T_2 of e_1 into e_2 plus T_3 of e_1 into e_3. This vector is resolved in the 1 direction, 2 direction, 3 direction. Note this very carefully that this vector has e_1 as the plane. So e_1 e_1 and e_1 remains the same and this 1 2 and 3 indicates the directions and e_1 e_2 e_3 are the unit vectors along the three x_1 x_2 and x_3. Similarly I can write this as T_1 e_2 into e_1 plus T_2 e_2 into e_2 plus T_3 e_2 into e_3. You can write the last …. again as T_1 e_3 into e_1 plus T_2 e_3 into e_2 plus T_3 e_3 into e_3. Already I know that you are complaining too much to write; I wantedly wrote this because I know it is every time you write like this it is going to be difficult. So, you would immediately request me to write it in a short hand. I mean, you have to write so many things, it becomes difficult. We will device a short hand technique to write such a complicated equation or so many numbers, which has so many indices and so on. We will device a very simple method of writing it in a minute.

It is usual or it is customary to write T's in terms of what is called as sigma. Sigma_{ji} with T such that j indicates the plane and i indicate the direction; j indicates the plane and i indicate the direction. To understand this if I put Te_{11} that is in other words what is this? Sigma_{11}; this is sigma_{12} and this is sigma_{13} and so on. See, some of the books would define this as sigma_{ij}, a few of them. For example Kolsky, very famous book on Stress waves in solids, he would define this as sigma_{ij}, i is the direction and j is the plane. But it does not matter because you would shortly see that this sigma is symmetric or ij is equal to ji. But, usually it is the plane and the direction, which are the ones that are used to define sigma.

Yeah, any questions? Yes, now the point is that I have now how many quantities? 9 quantities, 9 quantities to define state of stress at a point. When I show you sigma_{ij} is equal to sigma_{ji}, then it will reduce to 6 quantities but, as such you can see that we have 9 quantities. So this sigma_{ij} is defined as the state of stress at a point.

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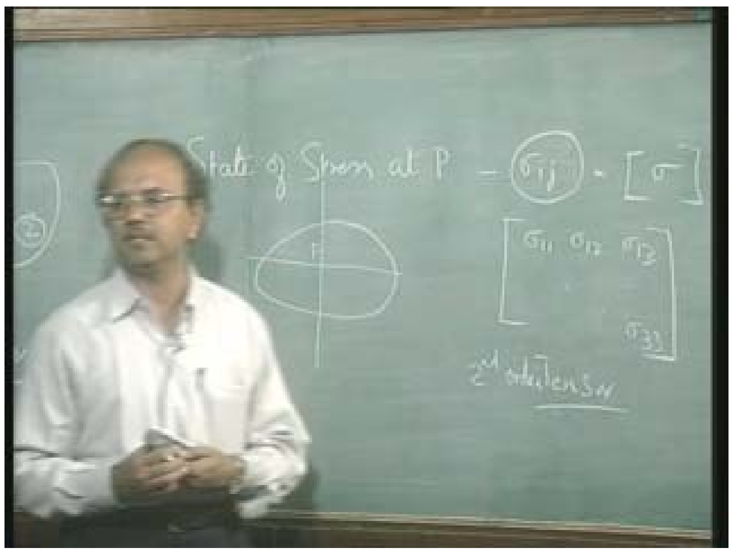

So, state of stress at a point is defined by means of sigma_{ij} and which you can say is matrix of sigma. They are nothing but sigma_{11}, sigma_{12}, sigma_{13}. So these 9 quantities are the crux or the clue to stresses at this point. Please note that we started with stress vector and resolved that stress vector in the three directions, do you remember? That is what we did. The question is then, what is this sigma? Is it a vector? Is this stress, state of stress defined as a vector or is it something else? Can you call this as a vector? No; obviously no, because from the vector, stress vector, quantity only we have come to this and also how many quantities are there? 9 quantities.

So, this sigma state of stress is not a vector but what is called as tensor. To be precise it is a second order tensor; it is a second order tensor. Let us not right now worry about this word tensor; you need not be too much taken aback by sudden jump in mathematics, I will define tensor if necessary later in course more precisely; what is tensor and so on but nevertheless you should understand that stress can be expressed as a vector or a tensor. So, you cannot say, see for example, certain things like for example speed; what is speed? When I say speed, it is scalar. When you say velocity, you would define this as vector. But, when we say stress, then you have to be very clear. If I mean state of stress at a point then it is a second order tensor. Stress can also be expressed as a vector. There is a relationship between the two; we will come to that in a minute. Is it clear so far?

Now what is this relationship? Why have I defined this state of stress at a point? In order to understand that let us do a simple derivation and see whether we can utilize this definition in a much better fashion. We will take back or let me take you to the same point P.

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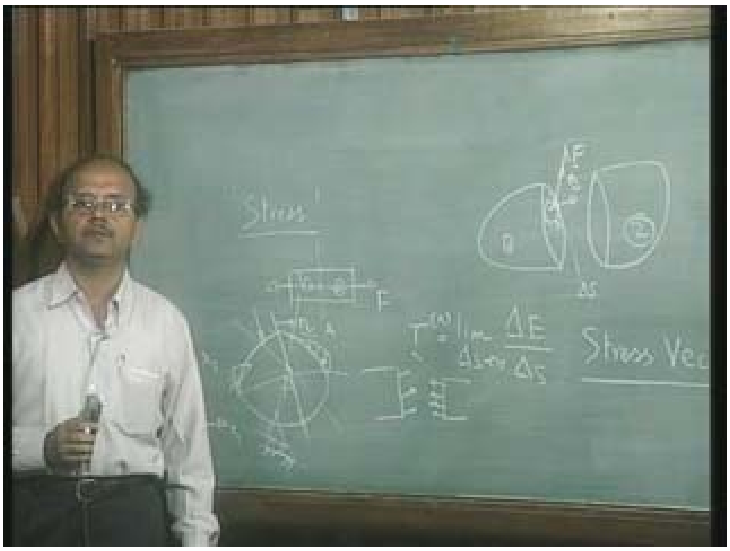

Now, let me see how to determine the stress vector along a normal n at this point P. In order to do that, let me construct a small tetrahedron with the height h, with height h about this point P. How am I going to construct a tetrahedron, small tetrahedron at this point P.

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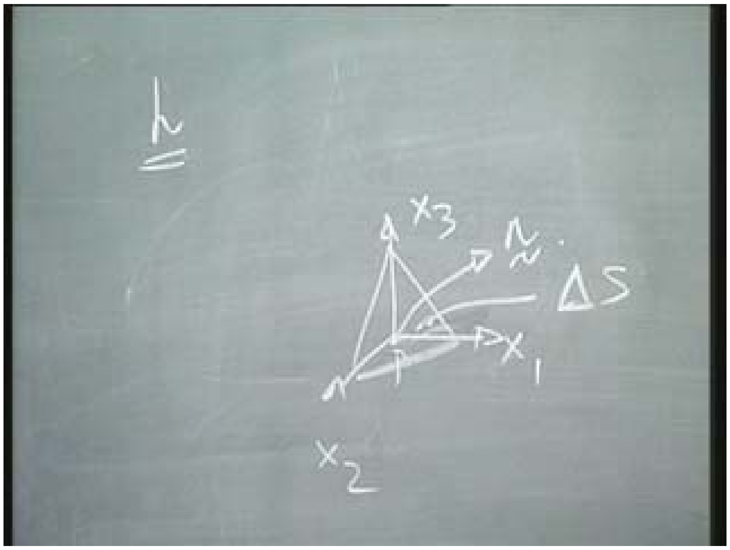

Now, let me shift the origin or rather let me draw the coordinate system say at that point P. That is the point P and let me the draw the plane of interest whose normal is n at a distance h. So, this perpendicular distance, the perpendicular distance between this point P and this plane is called as h, the height of the tetrahedron. The tetrahedron is now made up of these are the axises. What are they? They are say x_1 x_2 and x_3. This tetrahedron is made up of planes which are normal to x_1 x_2 and x_3 as well as the plane of interest whose normal say is n, whose normal is equal to n. Now, let me say that the area of this plane which I have taken to construct the tetrahedron be say delta S; the same concept as before, only thing is I have just shifted it a bit.

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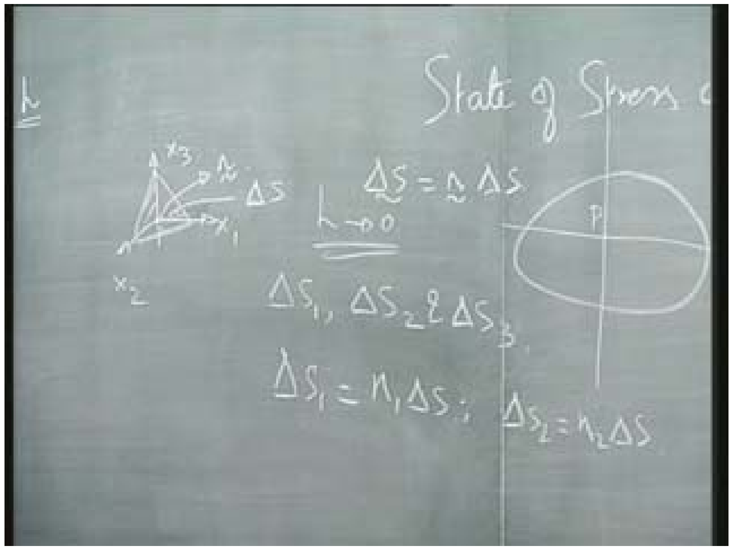

Yes, I know you have an objection, so, let me say, let me look at the result as h tends towards zero. What is the result as h tends to zero? When h tends to zero, this plane whose normal is n will pass through the point P. Let me say that the three planes whose normals are in the x_1 x_2 and x_3 direction has an area equal to delta S_1, delta S_2 and delta S_3 respectively or in other words delta S_1 is the area of the plane that is formed, that plane whose normal is along x_1 and delta S_2 is the plane whose normal is in the two direction and delta S_3 is the plane whose normal is in the, which direction? Third direction.

The relationship between delta S_1 and delta S, delta S_2 and delta S and so on is very clear. Please note that you can define the area also as a vector and call this as delta S is equal to n delta S, so that delta S_1 can be defined as delta S_1 can be defined as n_1 delta S. Resolving this in the one direction, I will get delta S_1 is equal to n_1 delta S. Similarly delta S_2 is equal to n_2 delta S. What is n? Please note that it is normal which results in the direction cosines n_1 n_2 and n_3. I am sure, yes, all of your familiar with this; so, I am not going to repeat what it is. So, delta S_1 is equal n_1 delta S, delta S_2 is equal to n_2 delta S and delta S_3 is equal to n_3 delta S. Now, please note these three planes, the n_1, n_2 and n_3 planes are also planes which I have used to cut the piece. So, I have taken this piece out. Actually, what is that I have done? I have taken the piece out from the original body. Now, in these planes also forces will be acting, the body is to be in equilibrium, so, forces will be acting in these three planes as well.

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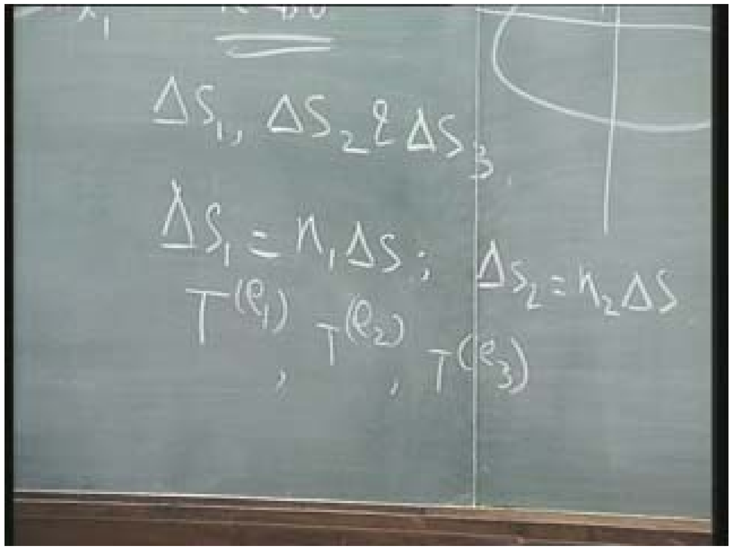

Let the force that acts on these three planes be denoted by T_1 T say Te_1 Te_2 and Te_3. Let us now look at the equilibrium of this body; let us look at the equilibrium of this body. How do I now look at the equilibrium of this body? What do I mean by equilibrium of this body? What is that I say for equilibrium of the body or what is the condition that I should apply? Yes, correct. So F sigma of the force vector should be equal to zero. What are the forces that are acting on this body? What is the body that is of interest to me? It is this body, it is this body that is of interest to me.

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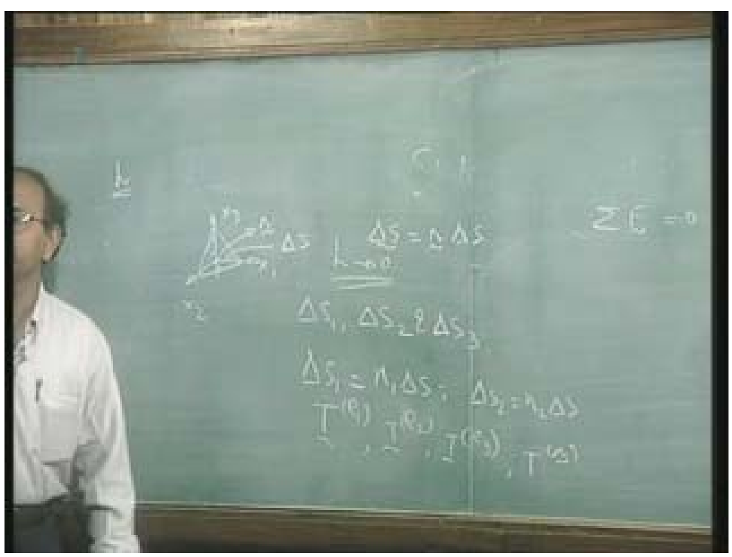

This body consists of three planes and an inclined plane. The forces that are acting in the three planes are Te_1 Te_2 Te_3. They are vectors. I need not do it every time plus a force that is acting on this plane which is say T_n. Apart from this there can be one other force. What you thing is the other force that can act apart from these three? Yes, they are what we called as body forces. The gravity force is one type of body force, correct. So, apart from these forces the other force that can act is body force, which typically is a gravitational force. Now, what is gravitational force? How is it defined?

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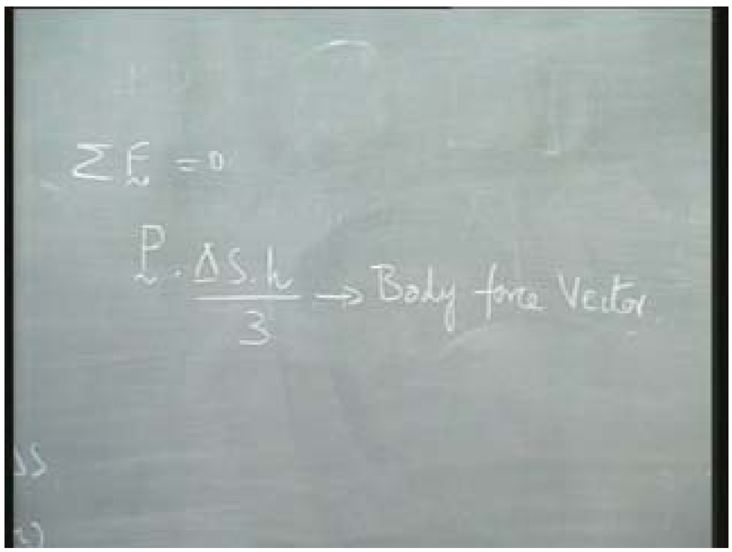

Let me say that gravitational force is defined by say vector; let me call this vector as P, vector P and defined as body force vector per unit volume. In other words the force vectors, these are all vectors. So, the body force vector is defined as P into volume of the body which can be said or which can be determined as delta S into h divided by 3; one third delta h into S or is it delta S into h? gives me the volume of the body. This is the body force vector. That is the body force vector. So, sigma of F is equal to zero.

What does it mean? It means that this T plus this T (Refer Slide Time: 39:40) that is Te_1 plus Te_2 plus Te_3 plus Tn plus that is equal to zero. All the sigma of F or sigma of all the forces are equal to zero. That is what is meant by the equilibrium of forces. Just wait a minute, we will write that and before that let us make a small approximation so that we need not write it again and again. What is that approximation? It is not an approximation, actually we remove that h tends to zero.

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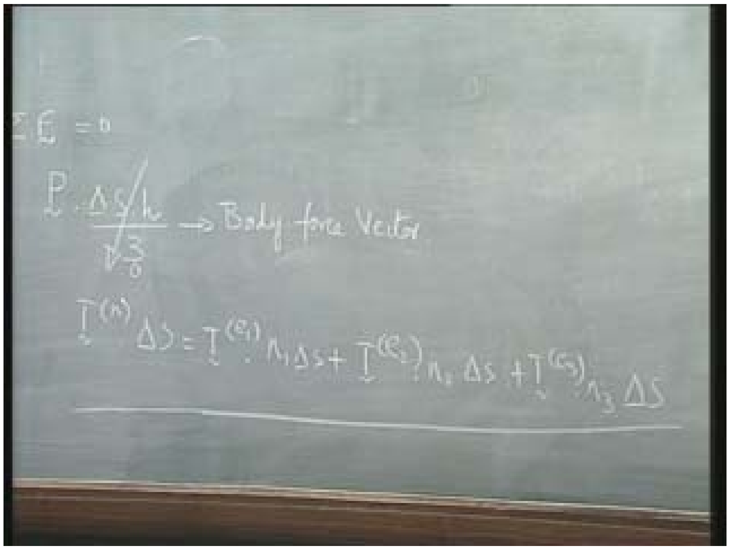

What is that force which will be affected by h tending to zero? Body forces; so, this will go to zero when I put h is equal to zero. Let me express then Tn in terms of other three forces. The directions are taken care of; let us not worry about that. We know the directions are taken care of are T of n into what? This is stress, this is the stress vector; that multiplied by area, delta S is equal to Te_1 into what is delta S_1? What is delta S_1? n_1 delta S; correct. So, into n_1 delta S plus, what is that next term? Te_2 into n_2 delta S plus Te_3 into n_3 into delta S. This equation is the equation which depicts the equilibrium of this small tetrahedron which I have taken to bring in my stress tensor. Immediately the trick is out of the bag. What is that you are going to do now? What will you do now here?

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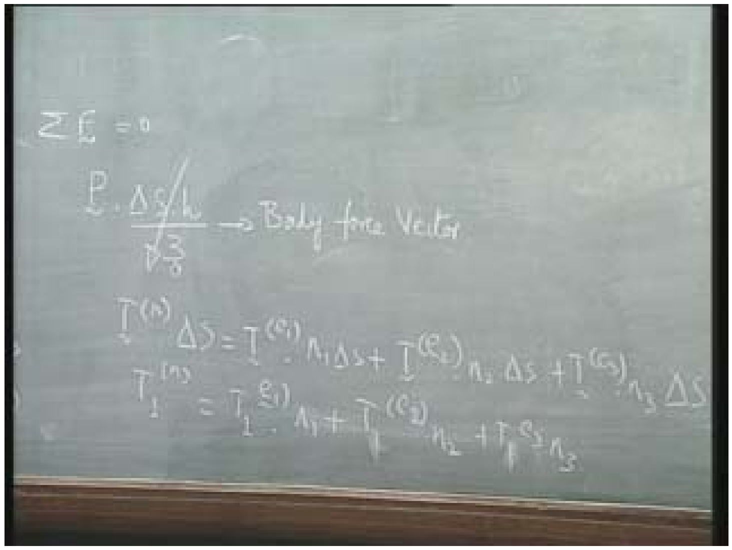

Yes; that is it. So, you resolve this in x_1 x_2 and x_3 direction. So, resolving this in the three directions say for example in the first direction what is that I will get say T_1, T_1 n and canceling delta S throughout on left hand side and the right hand side what is that I will get? T_1 n is equal to T_1 e_1 into n_1 plus T_2 e_2 into n_2 sorry e_1 into n_2 plus T_3 e_1 into, no, no, e_2 e_2; both of them are T_1; T_1 e_2 and T_1 e_3. Yes, that is good; so, just as we did in the previous one, same, there is no difference between the two. The same way you can write T_2 and T_3.

Before we go to the next one let us rub this off. We will have the result written in a slightly different notation. What is the notation we introduced for this? Sigma. What is this? What is this? $Sigma_{11}$. What is this? $Sigma_{21}$. What is this? 31, correct.

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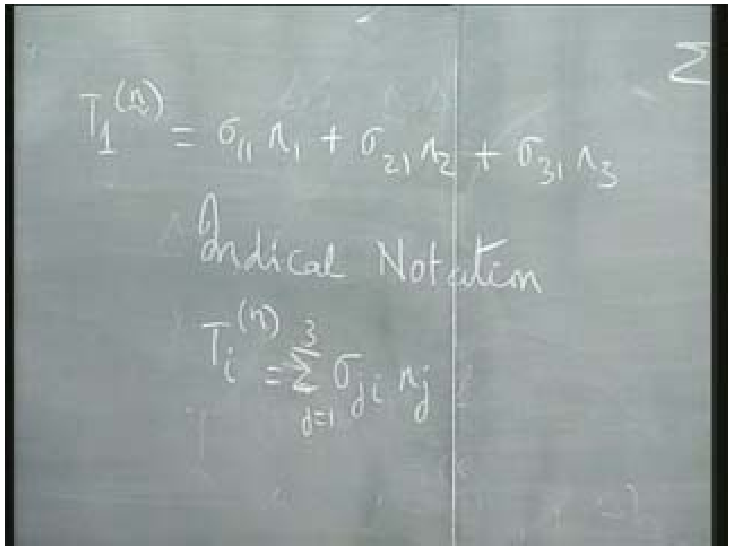

Hence I can write T_1 of n as $sigma_{11}$ n_1 plus what is the second term? $Sigma_{21}$ n_2 plus $sigma_{31}$ n_3. Same way I can write T_2 and T_3. How will you write that? What will happen to this? n_2 and so on; $sigma_{21}$ n_2 and so on. I am not quite lazy to write all this. So, the first question you will ask is how do I simplify this? I simplify this using what is called as an indicial notation, by using indicial notation. In indicial notation I will write, I will explain that may be in next class; let us write this first. In indicial notation, I will write T_i is equal to $sigma_{ji}$ n_j. In your earlier classes you would have studied about summation convention. So, you can say that summation from j is equal to 1 to 3. For example when i is equal to 1, I get using the summation convention, what is that I get? I get the equation which I have written here.

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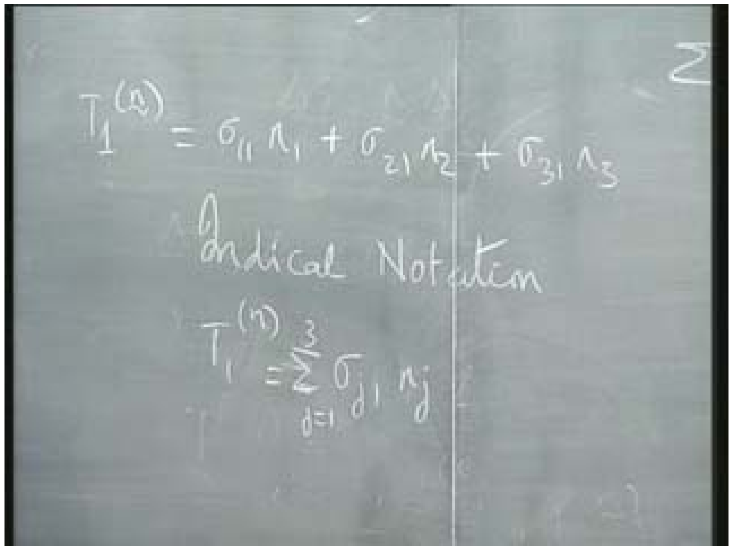

For example from what you know already you would have written this equation as T_1 and 1 here. The next equation when I want to write, you would just replace this 1 by 2 and so on. In other words by your summation convention, you would have written three equations. Einstein introduced a simple notation called indicial notation and he introduced a convention using the indicial notation. This convention is called as Einstein convention. I am not sure whether really Einstein introduced it, but the convention goes by Einstein's name.

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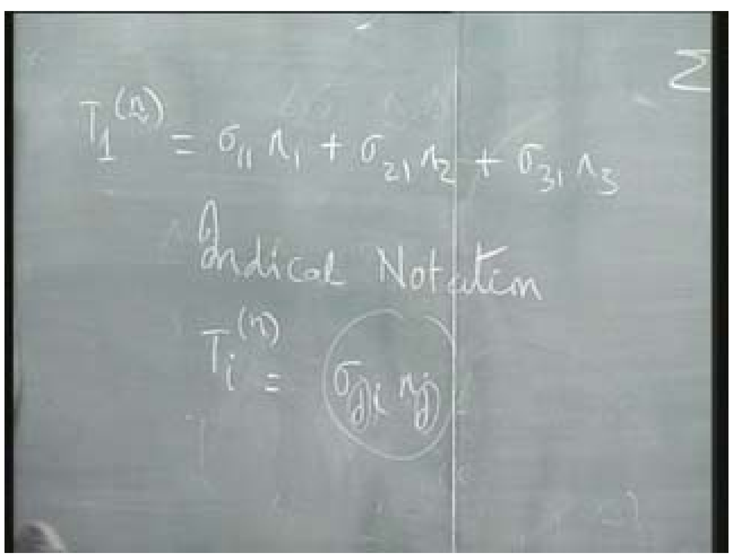

Under that convention we do not write down the summation, but we put certain rules for summation. As you see it, you immediately see one thing. There is a term here with repeated index. Index is repeated; j is repeated in that particular group or assembly and i appears on the left on side and the right hand side. Some index are repeated and some index are free. We will learn more about these two indices in the next class. What these indices mean and how we use the summation convention, we will see that in the next class.