Introduction to Finite Element Method
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Lecture - 32

Yeah, let me summarize what all we did in the last class, I think that will give a good picture of this whole procedure for nonlinear finite element analysis. Let us start with, I hope all of you understand what we mean by load step, time step. Load step and time step are synonymous, I said that. That is very important to understand, because you will see that in many of the softwares, you will get messages in the sense that time step is not sufficient, minimum time step attained; this kind of messages is what you will get, do you want to specify the minimum time step and so on? So, you should understand that what he really means is that when he says minimum time step attained, convergence not attained; this is something like a message that may come up. Many people do not understand this message.

What it really means is that he has gone to give a minimum time step in sense that minimum load that is possible.

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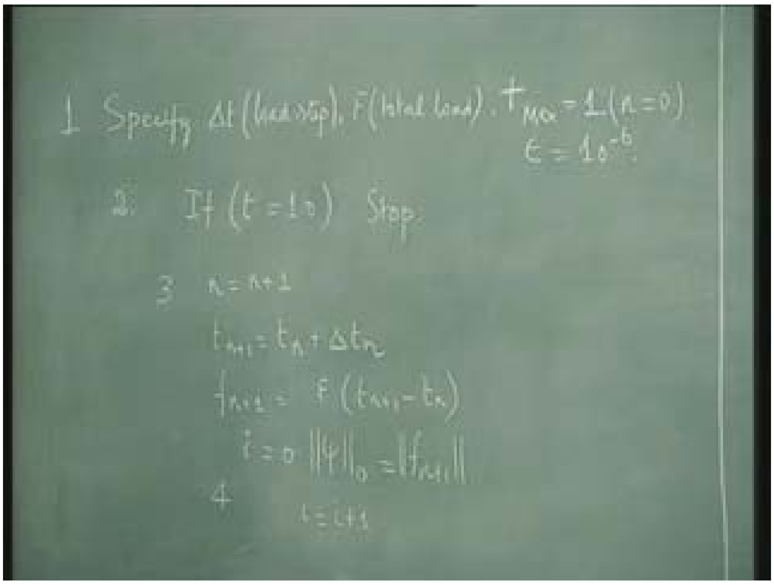

I told you yesterday itself that we are going to split the load into a number of smaller and smaller loads. What the software means to say is that he has reached the minimum possible say 10 to the power of minus 8 Newton. He says that he has reached that and still there is no convergence; that is what he means. So, both of them are synonymous. I will elaborate the point once we go through the steps. So, the first thing that you do as soon as you start the finite element analysis or before you start it, as an input you give what is the delta t initial. I would say initial delta t or initial load step. I already told you that we are going to normalize the time to 1, which means that when I say initial load step of say delta t, say, let me say that this is a naught naught naught naught 1, seconds and all have no meaning.

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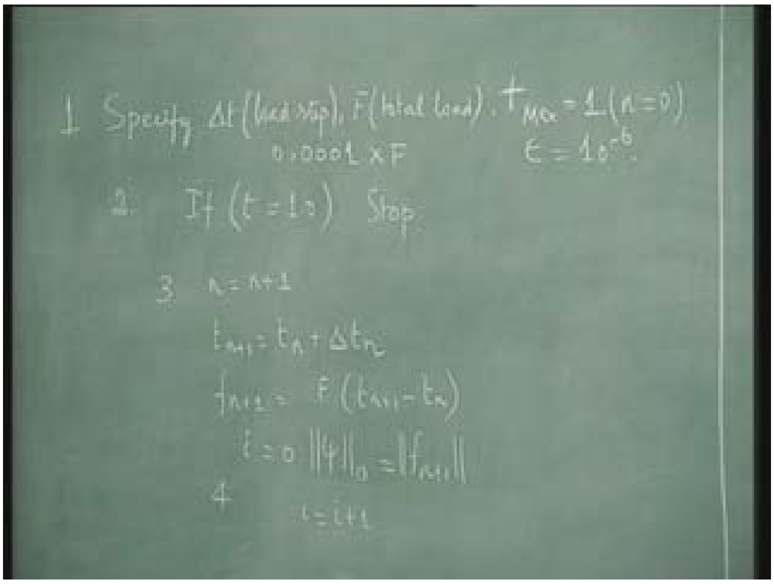

I can say seconds, but it literally has no meaning. What I mean to say is in my first step, apply this multiplied by the total load, you apply this; that is what it means. You apply naught naught naught 1 into F.

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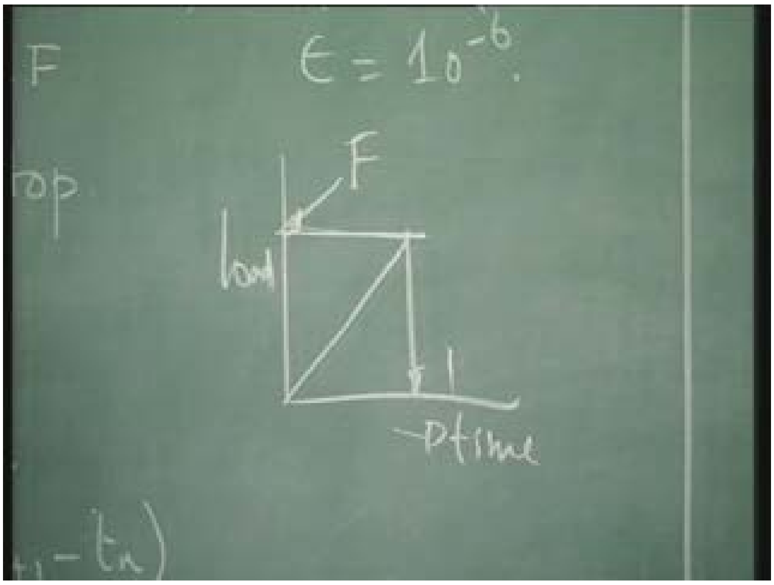

In other words, there is a distribution; is something like this between the time and the load. The total load, say F, is reached when time equal to 1. You can also view it like this; that the total load F is reached, when the time is equal to 1. When I say the initial time step is naught naught naught 1, what I mean to say is the first load that you are going to apply is that multiplied by the total load that I have given. Is it clear? Let me get into the loop. See, there are three loops in fact. There are three loops to, if you want to look at it in non-linear finite element. One loop is the incremental loop.

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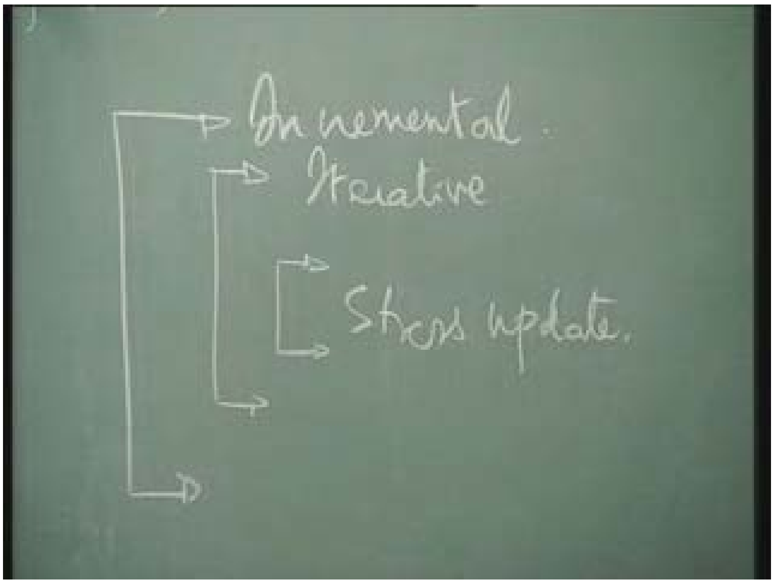

That is the first loop. The second loop which is inside is an iterative loop. Sometimes we use an algorithm to do stress update; I will come to that and there will be a stress update loop. Many softwares today use what we call as implicit algorithm for stress updates, so there is one more loop inside. We may not have time to go into stress update for plastic, plasticity and so on. At least for 1D I will do it; I may not have time to do it for other cases. But nevertheless, I want you to understand that there are three loops. What does it mean? Look at the time that is going to be involved in doing a problem. We can straight away compare this with an elastic problem; elastic problem just one loop, that is it. Just put F is equal to 10,000 and run the problem; that is all, every thing you calculate in one go.

Now, look at that. You have to increment and inside every increment you have to iterate and inside every iteration, you have to do a stress update that itself which is very involved; especially when you go to plasticity, stress update itself is involved. So you have this kind of three layers. What is the result of this? The result is that, we require very powerful systems or the time required to run this kind of problems are quite high. That is what gets out of this kind of non-linear analysis; that you need much more powerful systems and the time taken to run a problem is also very high. Is it clear?

Let us come back and look at these loops here. What I have done is I have essentially summarized what we did yesterday so that you can look at it more closely. The first thing what we do is see whether time t is equal to 1 or in other words what does it mean?

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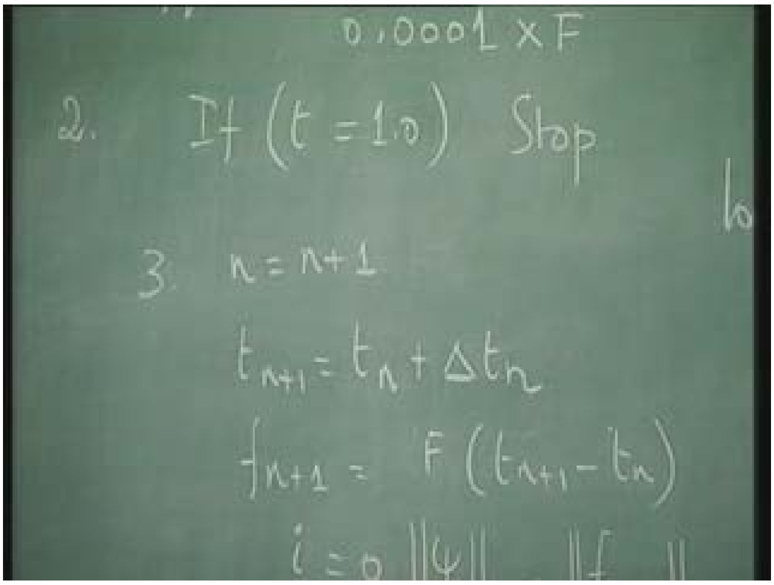

It means whether I have exhausted, I have exhausted all my load. Initially of course, I am not going to be exhausted. I would not have exhausted the load. So, if it is exhausted, stop the program; we are through it. If not, get into the big loop, outside loop, incremental loop; from here starts the incremental loop. So, I say n is equal to n plus 1 and t_{n+1} is equal to t_n plus delta t_n. Note this delta t_n.

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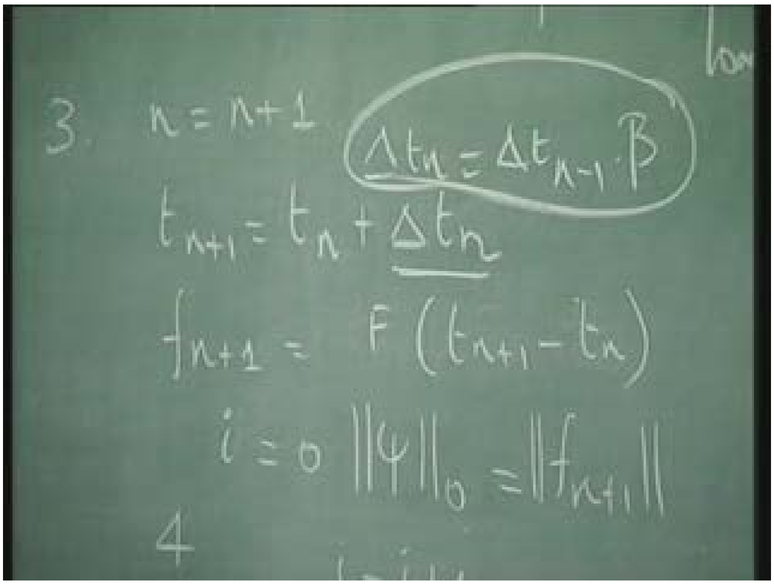

That means the delta t is not a constant. Many softwares does a small optimization of delta t_n. In fact here you may see that delta t_n is equal to delta t_{n-1} into beta. They would do something like this; beta is factor say, 1, 1.5 and so on. Beta is also another factor, I will come to that later, but let me say that the first thing, first lesson is that the time step may not or will not be a constant in most of the softwares and that the time steps are adjusted so that we reach the total load in the shortest time. Because, when I start I may not know whether it is going to converge or in other words, even when you start, it is going to be difficult for convergence to take place. So, what we usually do is we start with a very conservative estimate. We start with say, naught naught naught 5 or something like that, but later when we pick up we will realize that time steps can be increased.

Please note this carefully; this may also be due to the nature of the stress strain curve. This also is due to the nature of stress strain curve. In other words, it is not that when I start with low value, I can immediately jump; I may not be able to jump also. If it is going to be an elastoplastic problem, for example, what will happen? Initially it is going to be elastic. Assume that everything is elastic; then, my time steps can be larger. K_T is not going to change; it is something like resolving an elastic problem. So, there is no problem. I will get convergence in two steps, keep on going. But, when I come to plastic region, then this may not be the case. I have to cut down, because the tangent stiffness may vary. This is what we have seen from the 1D curve itself and in which case, I have to give a variable time step and the time steps will get adjusted depending upon the rate of convergence. Is it clear? That is why people give a variable time step. So t_{n+1} is equal to t_n plus delta t_n updating the time step. Then I calculate f_{n+1}. f_{n+1} is the load at the n plus 1th step; that is the total load multiplied by delta t_n.

That is the, that is the start of the big loop. Then, I am getting into the iterative loop. Next loop is the iterative loop. Is there any question, is it clear? Now, I will start in the iterative loop. When I start into the iterative or go into the iterative loop, obviously, my error is nothing but the norm. I told you that this is what is called as the norm in the last class itself. My error norm is the same as that of the load norm. Clear? f_n plus 1; okay, okay, okay, sorry; yeah what is the question? Yeah, no, no, no, no; please note that f_{n+1} is the increase, is the load that is applied during this step. Do you understand? I will just put that yesterday's stress strain curve here; simple stress strain curve.

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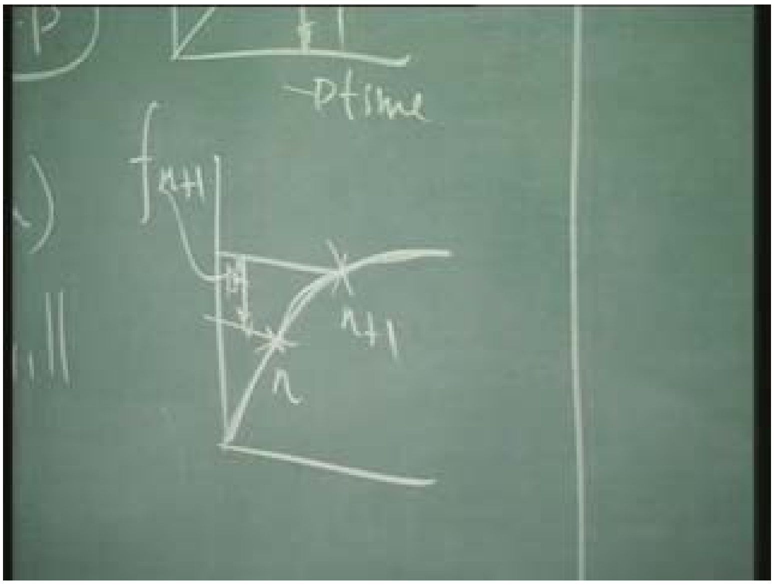

Say, suppose stress strain curve is something like this or load deflection curve is something like this, we are moving from here to here. That is at n that is at n plus 1; that is for the one dimensional case f_n plus 1 or delta f if you want to call it or f_{n+1}. So, initially that is the f_{n+1}. Is that clear? So, initially this stress which is within the element will not be able to equilibrate obviously this f_{n+1}, because that is good enough only to equilibrate the total load at n. Please note that it is good enough to equilibrate only the total load at n. So, the error which is the difference between externally applied to the internally generated loads would obviously be equal to f_{n+1}. So, I start with that as the error and get into the iterative loop.

Please note this carefully. Now, I get into the iterative loop. So, I calculate first the tangent stiffness. Note that, get that. So, first I calculate the tangent stiffness.

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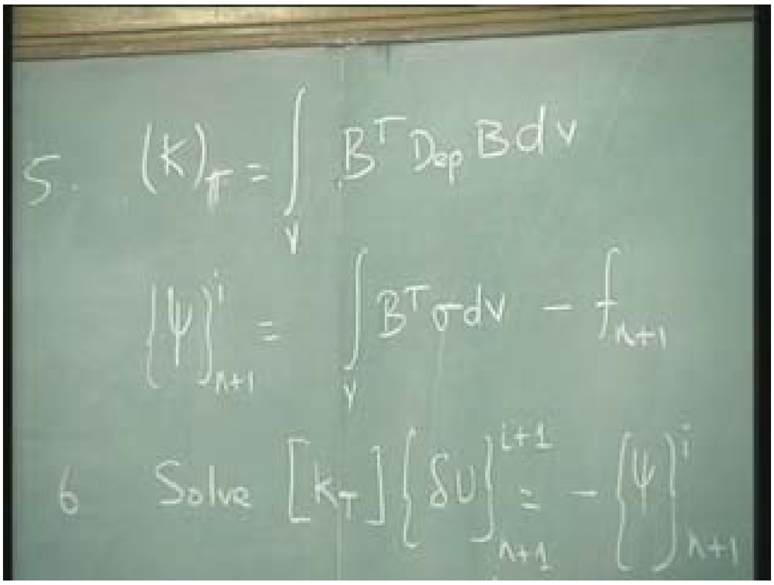

Now, how do I calculate the tangent stiffness? What I did yesterday? I have just written down what all I did yesterday as the tangent stiffness. Then what is the next step? Please look at that. What all we did yesterday, have a look at that. No, you do not jump; displacement comes after two steps. I calculate the error, because that is going to be my right hand side. Then, I calculate the displacements; that is delta u, del u. This is for one increment. Is this clear? So, I calculate del u. After this, what am I going to do? I am going to add, note this step carefully.

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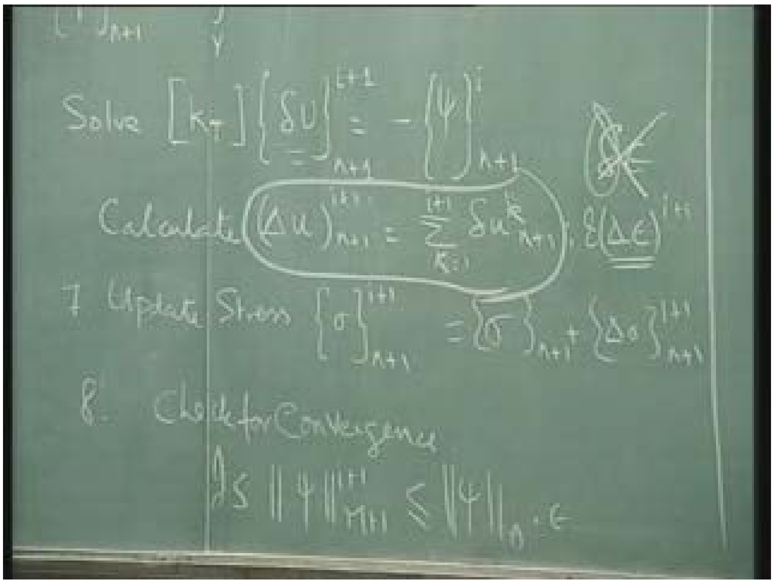

I am going to add all this delta u's in the iterations. All the delta u's of this iterations I am going to add to get delta u inside this iterative loop. Is that clear? So this is the total delta u till now inside the iterative loop. Then I calculate delta epsilon. See, note this carefully, this is a very important step. I have to calculate delta sigma, but I do not calculate delta sigma from delta epsilon. No. What is the difference between, del, let me call this as del epsilon and delta epsilon. What is the difference between the two, what is difference between the two? You just note that. What is the difference between delta epsilon and del epsilon? Sorry, del epsilon and delta epsilon. Del epsilon comes from del u. Sigma of del u's are delta u.

If you want to look at it figuratively or I mean through a picture, what we mean is, suppose it is something like this. I am here; again same thing, solve it.

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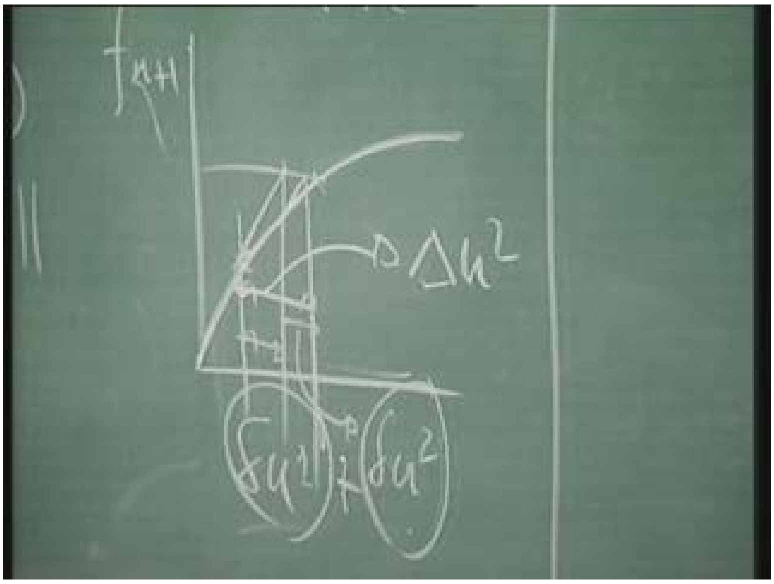

This is del u, del u for the first iteration, so, 1. Then again from here I go, this is del u 2. Is it clear and delta u 2 is this; sum of this plus this, that is equal to delta u 2. So, del u 2 and delta u 2 that is a difference. Are things clear, any question? Fine. Now, from delta u 2 or delta u i plus 1 calculate delta epsilon or it does not matter whether you calculate from here, it does not matter. But, you have to be very careful. B into delta u you keep adding; anyway you calculate delta epsilon. So, please note I am interested in this only; not this. Because, look at the next step. Update the stress; look at the way the stress is updated.

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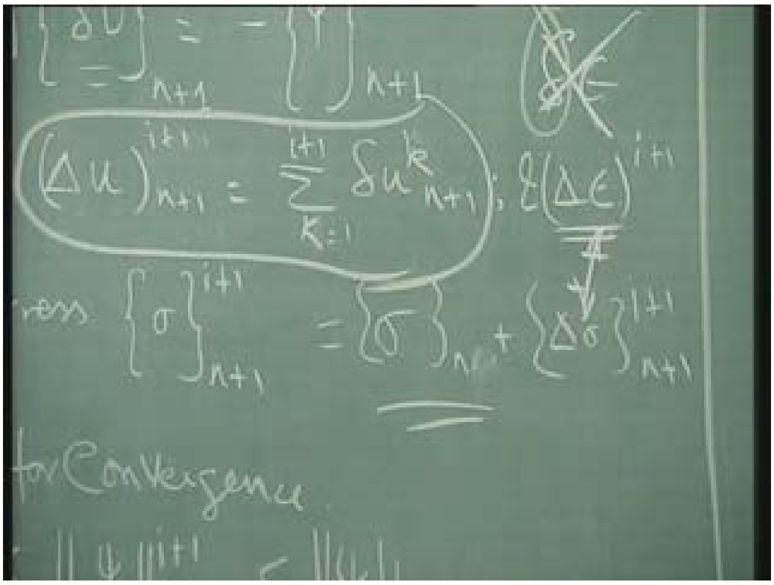

The stress is updated from actually at converge stress at n, sorry not n+1, converge stress at n plus delta sigma, delta sigma; delta sigma being arrived at from delta epsilon. Is that clear? Now, what is the difference? What I am trying to say? See, for example, I will just write it here. For example, I could get sigma from, there are two ways in which I could get sigma. From sigma converged, what is sigma converged? That is sigma at n.

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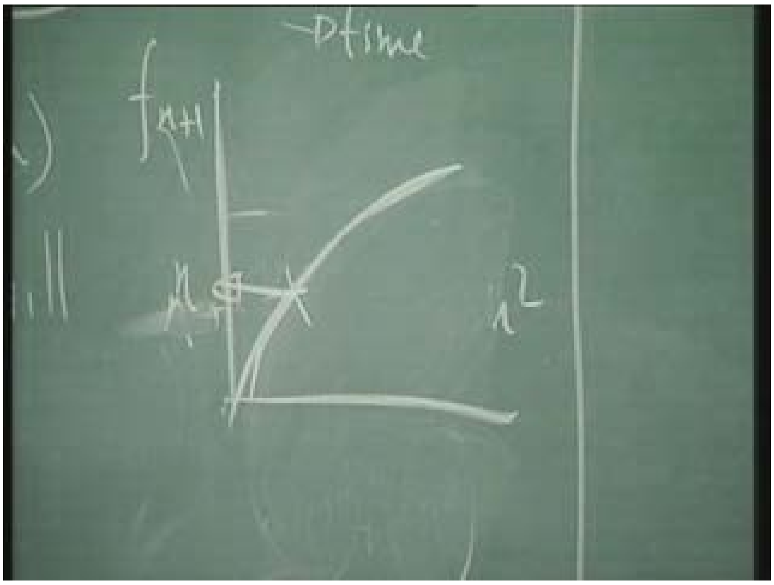

That is this, here; that is sigma at n. This is the place, last step it had converged there, n. I mean, I will remove all these and note it again. That is the sigma, sigma at that place.

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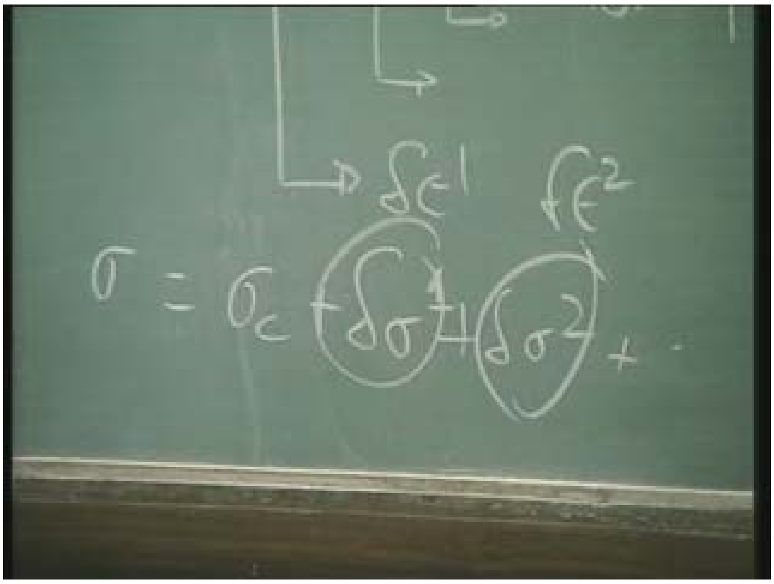

See, I can say that sigma converged plus delta sigma at the end of the first iteration plus delta sigma at the end of second and so on. I can write it like this also. Sorry del sigma I mean, I am sorry; the delta and del, there is a small confusion. Del sigma 1 plus del sigma 2 and so on. I can write it like that, del sigma 1 being calculated from del epsilon 1, del sigma 2 calculated from del epsilon 2 and so on. Is it clear? But, we do not do like this. We do not do like that, but what we do is instead of this, I remove this and put this as delta sigma, delta sigma being calculated only from delta epsilon.

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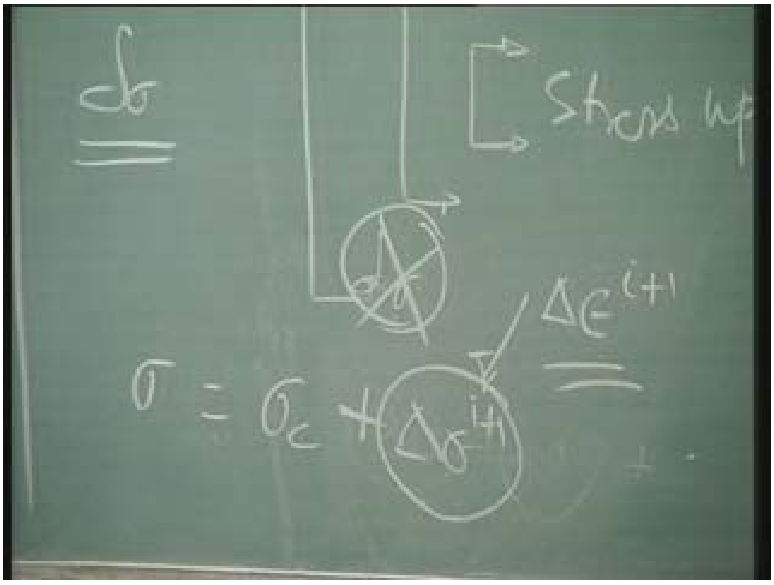

So, delta sigma say at i plus 1th iteration is calculated from delta epsilon only. Basically, this is a very important point that you have to note. Let me ask you a question. You understand now the two ways in which I can update the stress? One through del sigma, another through delta sigma through delta epsilon. Why do you think I favor this? No; much more important, physically much more important than that. Please note that my del sigma’s, del sigma’s are not, are not actually physically realizable quantities. They come out from my approximate calculations.

What has a meaning is only the converged stress and converged stress, converged stress has to be updated only ultimately. When I calculate like this del sigma 1, del sigma 1 has no meaning. It has no meaning right now. So, I do not have that path, but I calculate completely the del sigma. If you do in the other fashion, if you do through del sigma instead of delta sigma, if you do like this, then you get into what we call as spurious oscillations and you get into all sorts of troubles. So, the next step, 7th step is stress update. Of course, when you are using a package this is done internally by a package. But, if you keep getting I mean, non-linear analysis is one where I will not say that every package is very good one and so on. But, if you keep getting in a package some oscillation of stress, then rest assured that they have made a mistake somewhere here, there are good chances. I do not want name packages, but there are good chances that there may be an error in the stress update algorithm in which case, there can be an oscillation and then you get it into all sorts of troubles. So, that is an important step. So 7th step is stress update.

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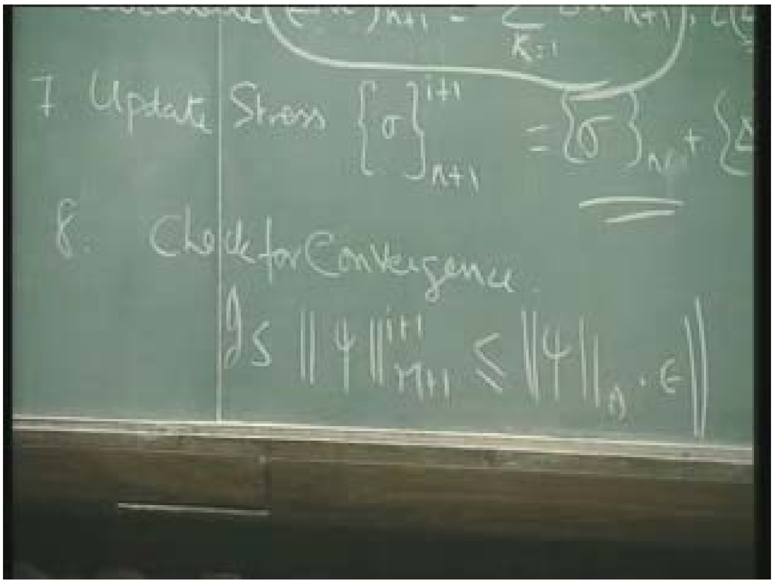

Once I know, once I update the stress, the 8th step is to look for convergence. How do you look for convergence? Very simple; now calculate, now that I have got my new chap sigma, I have to now see whether he is able to battle the outside external force that has been applied. If the internal force The battle is over, when the internal force is or is good enough to equilibrate the external forces. So, I calculate the error and compare it with the original error multiplied by some tolerance which is say 10 to the power of minus 6. So, I calculate that. Yeah, any question, clear?

So, now where are we?

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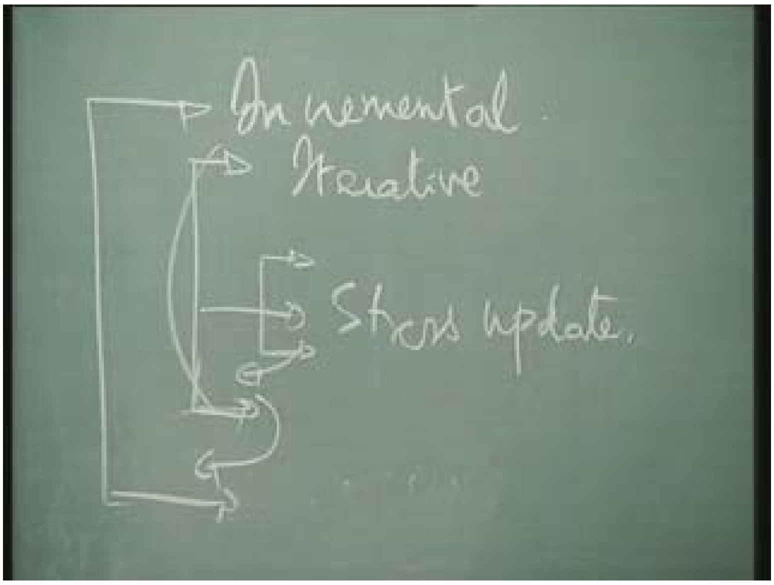

We are in this loop, we are in this loop. We went inside the loop here, came out and we are in the iterative, increment loop; sorry, iterative loop inside the incremental loop and with that, if the convergence is attained we go to the, go out of the loop. If the convergence is there, then, one minute, when convergence is there, I come out into the other loop. So, I go to my initial step and see whether I have completed the problem. Is it clear, any question on it? Now you know how to run it. If given a problem, you know how to run it. The only thing I want is that I will stop with this, the general procedure. But, I want two things. I want to do two things. One is for simple say, a truss element, I want you to calculate the stiffness matrix. Just have a feel and we will discuss in short the issues in stress update. With that we will close the non-linear analysis.

See, what we have actually said is for small strain. For a large strain, the procedure is exactly the same, it is not going to be any different. But the strain displacement matrices and all that will change. So, the B matrix would change. There will be a non-linear term, there will be a linear term and so on. So the procedure, whether it is geometrically non-linear or geometrically plus materially non-linear or materially non-linear or contact, whatever it is, the procedure is exactly the same. This is how it is done. But, stiffness matrix may change; load may change, because when there is contact or stress update may change when you move from one material to the other and so on. But the procedure, the overall procedure remains the same. Is it clear?

Now, I will give you this exercise. Please see, next five minutes let us see whether you are able to get me the stress update, sorry, the stiffness matrix for a truss element.

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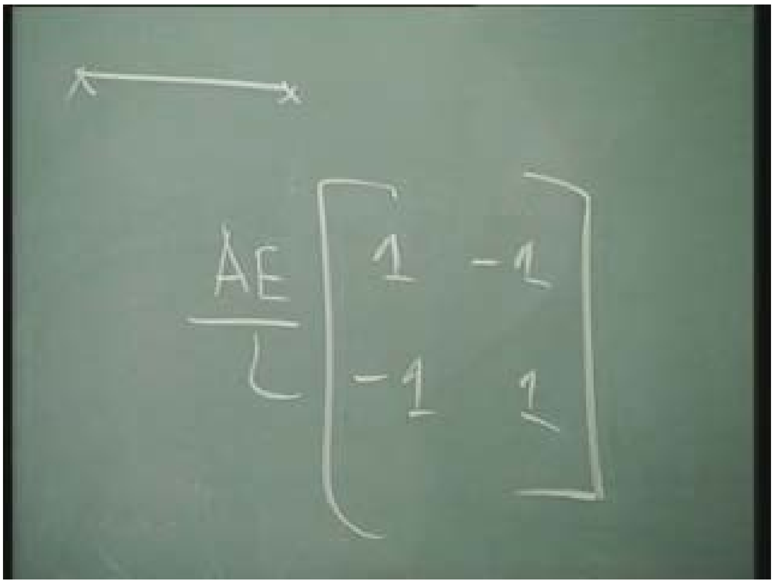

If you want, you just look at these steps and see whether you are able to do it. What is that you require? If you want our good old friend, the stiffness matrix for a truss element, I have also written. Looking at it, I want you to say what it is. You can very easily say this, it not a big problem. I am not interested for you to go through the tool but nevertheless, an important point. The isoparametric concept, please note this carefully; the isoparametric concept which I had introduced for plane strain, plane stress, axi-symmetric, solid are, of course, we have not talked about shell and so on; this concept of isoparametric element is obviously applicable for this. The procedure for calculating stiffness matrix, the natural coordinate system, the integration and all those things are applicable here and we have to, when it comes to this, the elastoplastic analysis, I will say that. In a minute, let me finish this. There are some issues, that is also very important.

So, I want to, I want you to calculate the truss for a non-linear problem. Now, for non-linear problem what do you think you have to do for a truss element? It is very simple. You need not think so much. What is that you should do? Fantastic; so, E is not a constant. So, the attack is right on E; fine, then attack is on E. Then, what do you do with E? Pardon; take it inside. Why do you want to take it inside the matrix? E is not a constant. That is very good.

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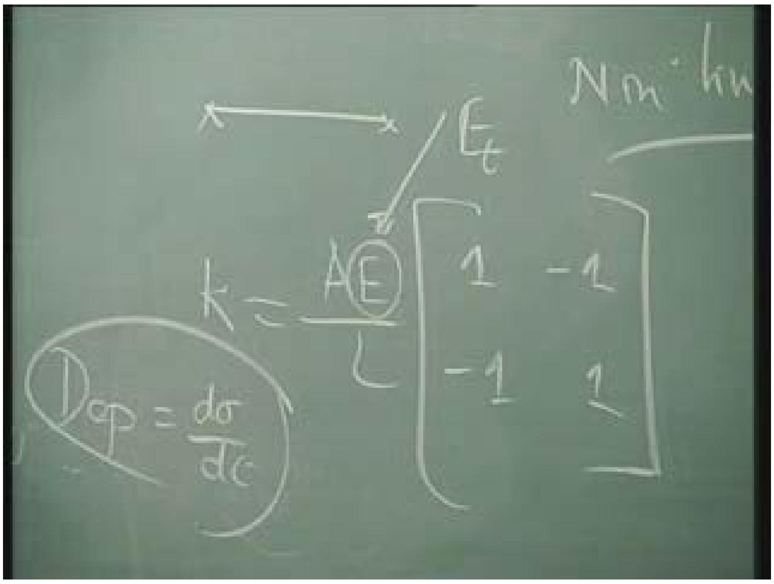

This is the stiffness matrix k, but that is not a constant. So, you have to attack that fellow. Dep, I have put there; you look at that, my steps there. B transpose Dep b I have put. So what do you do with this? D; what is Dep in uniaxial case? We are looking at uniaxial case. ep is plastic, D non-linear whatever you want to call it. So, what is E now? No. What is D? What did I call Dep as, yesterday? d sigma by d epsilon; remember that. I have said that, that is the Jacobian. Sometimes we will call this as Jacobian and d sigma by d epsilon. I said that in the last class. So, what is that you will replace E by? That is all. There is nothing there. I will replace it by what we call as tangent modulus, say E_t; that is all.

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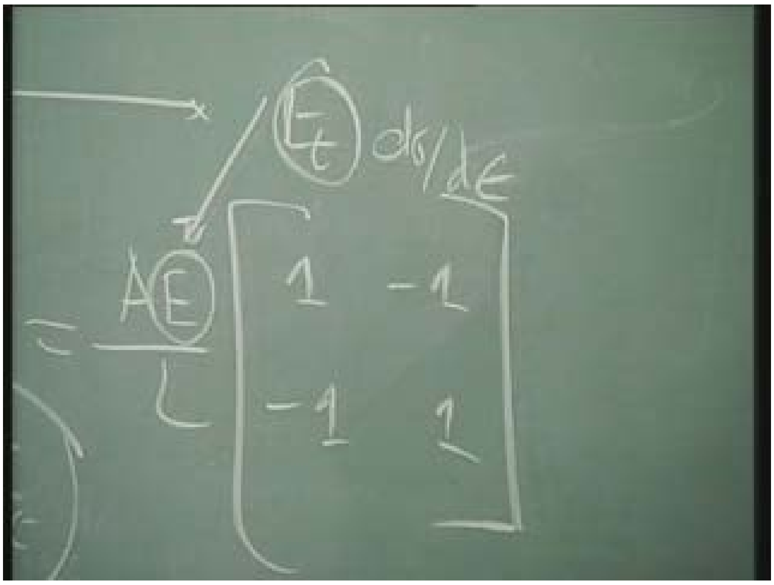

In other words, the one-dimensional equivalent of Dep is E_t, which is d sigma by d epsilon; one-dimensional equivalent of it is d sigma by d epsilon. Now, how do you calculate this for, say for example, an elastoplastic analysis? What are the assumptions that you will make? There are number of assumptions. I do not think we need to go into it.

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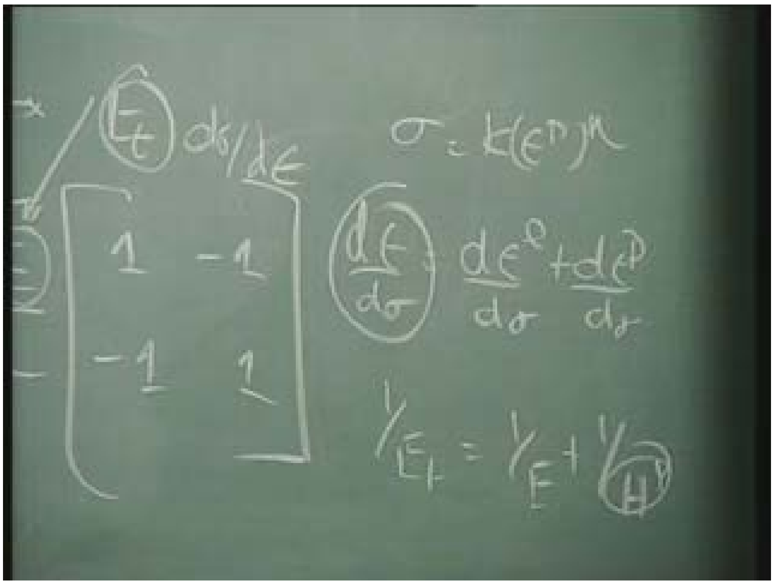

But only thing, one of the major assumptions is that d epsilon is equal to d epsilon e plus d epsilon p or in other words total strain can be split into an elastic part and a plastic part, so that dividing by d sigma throughout that is 1 by E_t is equal to 1 by E plus 1 by H prime; H prime is D sigma by d epsilon p. Is it clear? So, you can calculate E_t from simply this equation. This H prime is obtained from experiments and that is nothing but d sigma by d epsilon p. Suppose, I give sigma is equal to some k epsilon p power n; say for example, I give a sigma is equal to k epsilon p power n, then you can easily calculate the stiffness matrix from such a equation. I am not going to go into the details of the Dep for a multi axial case, but I just want you to understand that the calculations are very similar; only thing is equations involved are very much higher. Is it clear? Can you expand on E_t a bit? Let us see, what you get out of it? From that equation I want you to, can you tell me more about E_t from that equation?

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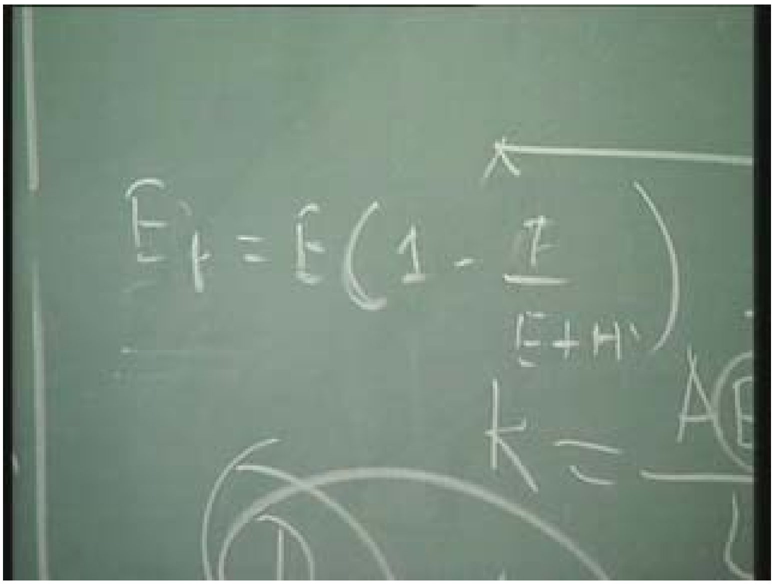

In fact, you can write E_t as E minus E divided by E plus H prime. E into 1 minus 1 by E plus H prime, something like that you can write, which means that there is a loss of the stiffness due to plastic strains. That anyway, that is a simple say some jugglery that you can do and get to the final step. Is this clear, any question on it, any question on it, on whatever we have done? The only thing that remains for me is stress update algorithm. I am not going to the details of it, because as I told you it is very involved and so, just I want to tell you certain issues that are involved.

If you want to really look at stress update algorithm, we have to study what we call as plasticity theory, for example for plastic analysis. If it is a non-linear elasticity, then it is straight forward. d sigma can be updated straight away from say d non-linear. I just call it is Dep; you can call it is d non-linear, you can just say, straight away you can update. But in plasticity you cannot do this, because you have to satisfy what we call as consistency condition and other things. Even here you may have problems, but for the first approximation this may be better. In other words, just update algorithm is very crucial. Lot of research has been done because, that algorithm should be such that it should satisfy, the updating should satisfy conditions called consistency.

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That is in other words, what it really means is that when I have a stress strain curve for a particular strain, I have only one stress. I cannot have multiple stresses or I cannot have a stress which is above or somewhere like that. I have to lie always on the stress strain curve, which means that I have to be consistent in the stress strain relationship and so, stress update algorithm should satisfy consistency. I think we will stop with that, as far as the stress update algorithm is concerned. You cannot do much in this course, but when you really read a literature or a package manual, you can understand stress update means that it is updating the stress at this point.

See, apart from all these, there are lots of other issues in non-linearity. See, many, many materials, whether it is elstoplastic or rubber, which is of course a non-linear solid has other very important properties. For example, they are, many of them are incompressible. For example rubber is incompressible. In the plastic region, for plastic strains, the material, the metals are incompressible or in other words they are nearly incompressible, if you take elastic and plastic strains into considerations. The other things, apart from satisfying convergence and other things you also have to satisfy what we call as volume conservancy, volume has to be conserved. This is done by so many means. For example, people use penalty function approach. Again, it is a big topic. But nevertheless, what I want to tell is that many softwares take care of this, but there are many algorithms that attack this problem of volume conservancy. Is it clear? So, that is one thing. That is, that again makes the problem very intricate. That is one thing which makes the problem difficult as well. There is one more issue which people have not pointed out, but I would like you to see, do it looking at these equations, because there are a lot of nice things that you can get physically looking at equations.

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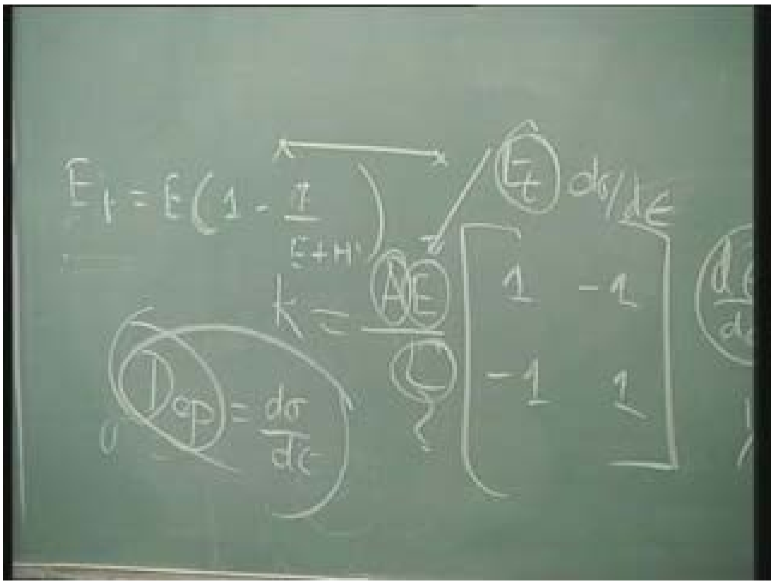

Look at that, AE by L; E, I have now replaced by E_t. Everyone agreed, but you all kept quiet after that. But, look at that equation and comment on my observation. In other words, we are in the non-linear path. How do you say that A is the same? What is the A? A, you will start with one A initially and then you will keep going with A, but you can always say that look I am not very happy with that equation, you have to do something with A. Where do we get this concept from? From your engine;

fantastic. So, it is from true stress and engineering stress or nominal stress. So, immediately you can react that what happens to this and this?

For a small, I mean, for an elastic problem it is fine. We are not worried about, we do not apply true stress for an elastic problem. We will keep A itself. A is equal to A_0, we call it in other courses. But in this particular case, we are looking at plasticity and are we justified in saying that A and L we will take the initial areas and initial length. No; now, when you start touching that part A and L, then we are into, what? Into geometric non-linearity; so, when I touch both these fellows, as well as wonder whether A and L are the same guys as I started with, then I am in the realm of geometric non linearities. Then, I have lot more issues. Then what do I do with A and L? I am now in the incremental stage. So, do I keep updating my A and L as I move in the increments?

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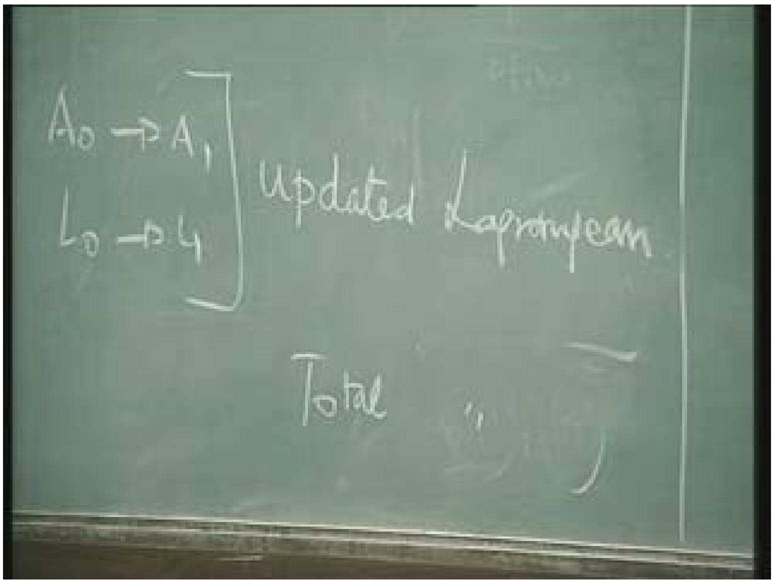

I go to say first step, I then update my A_0 to A_1, update my L_0 to L_1 and so on. Do I keep updating like this, is my next question. Yes; you can update like that or in other words you get a new reference for your problem or for your geometry and in which case for the geometrically non-linear problem, you are following a technique called updated Lagrangean. In updated Lagrangean technique, which is followed for a geometrically non-linear problem, it can also have material non-linearity. When I have material non-linearity and geometric non-linearity, I will worry about all these terms here; geometry as well as the E_t. I will worry about all the fellows there. So, in that kind of problem, either I start with the reference, I apply a load say delta f_1 after the first step, then I will update my A, L and so on and then go ahead, keep on updating so that my reference configuration, my reference configuration keeps getting updated as I move or march in the time steps. If that is the case, then I follow what is called as updated Lagrangean. But, if I do not, do not do that, I am still okay, because the way you now define stress, strain and all are different. You go to a large stress; I mean large strain, different type of stress and so on.

If my reference configuration is the same as A_0 and L_0, but calculate my stress, strain and all the other factors with respect to my old reference configuration, then I follow what is called as total Lagrangean. Please do not get confused between updated Lagrangean, total Lagrangean and the Lagrangean elements or Lagrangean that is used for calculating the stiffness matrix and so on. Please do not get confused between the two, they are different. This is an updated Lagrangean, total Lagrangean for geometrically non-linear problems which states how you are going to update the configurations. Total Lagrangean; see, suppose this is the, this is the body.

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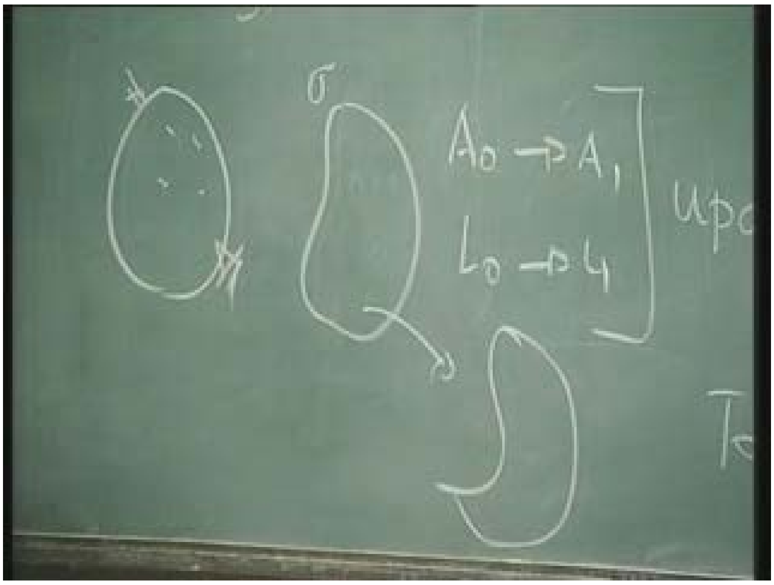

Now, after the first step I apply the loads; there is some boundary conditions and so on. After the first step or first increment my body is something like this. I have some co-ordinates here. Now, if I now update the coordinates according to the displacements and keep this as my reference and go to the next step so that deformation is say something like this, I update my configuration here and go to the next step and so on, so, what do, what do I do actually? I keep updating my reference configurations. See, in a much cruder fashion what it means? It is not, what I say is not correct, but it gives you a good picture. What I mean to say is I calculate first stress. Now, I calculate say sigma.

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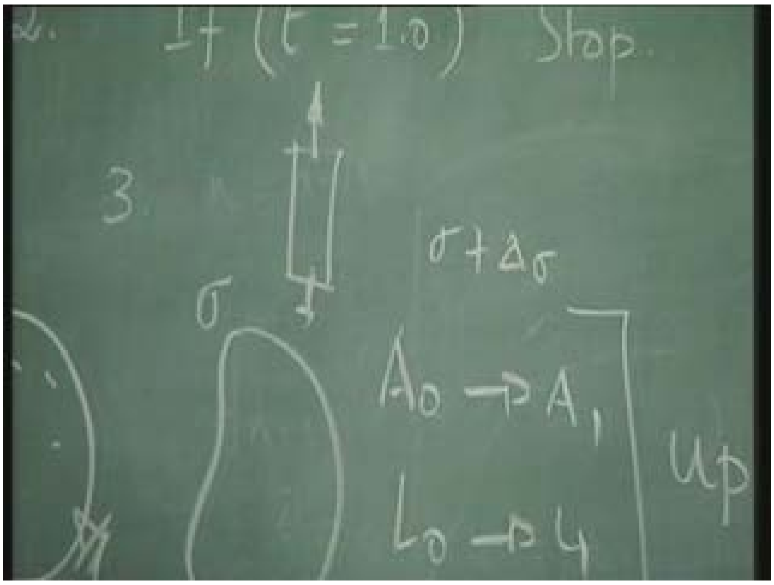

Let me take a bar. I apply the load. After the first increment, I get some sigma. So, after the next increment, in order to calculate sigma and in order to do all those things, I use A_1 and L_1 which is what I obtain after the first increment, so that I will then, I will calculate some delta sigma based on A_1 L_1. Then A_1 L_1 is my, then I go to A_2 L_2. Then, in order to go to the next increment A_3 L_3, I will use to A_2 L_2 as my reference. If I keep updating my reference, I would have forgotten A_0 L_0. I will go to A_1 L_1, from there I will move to next; I will move to the next one and so on. If I keep updating my reference, roughly speaking that is what we call as updated Lagrangean.

But, if I say that no, no, I do not want to update like this; my intermediate configurations I have no interest, even when I go to the next step my reference is only this and I will work on techniques in such a fashion that my reference does not change, then I am following what is called as total Lagrangean. Now, what do I do? Actually it is something like the way I handle stress. For example, the stress that you may handle for updated Lagrangean would be sigma that is close to true stress. But, the stress that you may handle in total Lagrangean may be close to the nominal stress, but I will make adjustments in my strains and so on in such a fashion that I do not consider an error.

In other words, in other words the difference between total and updated Lagrangean is mathematically, it is not, there is nothing to do with the error in the finite element program or the error in procedure. It is only mathematical. It is only a reference configuration, but I will adjust my stress definition, strain definition in such a fashion that I take into account, this thing is at the back of my mind; what is my reference? So, updated Lagrangean, total Lagrangean does not mean that I have some error. Though there are claims that updated Lagrangean is much more natural than total Lagrangean in elastoplastic analysis and so on, but mathematically both of them are very rigorous. These are two procedures that are used for, when are they used? Please note this, when we have both material as well as geometric non-linearity.

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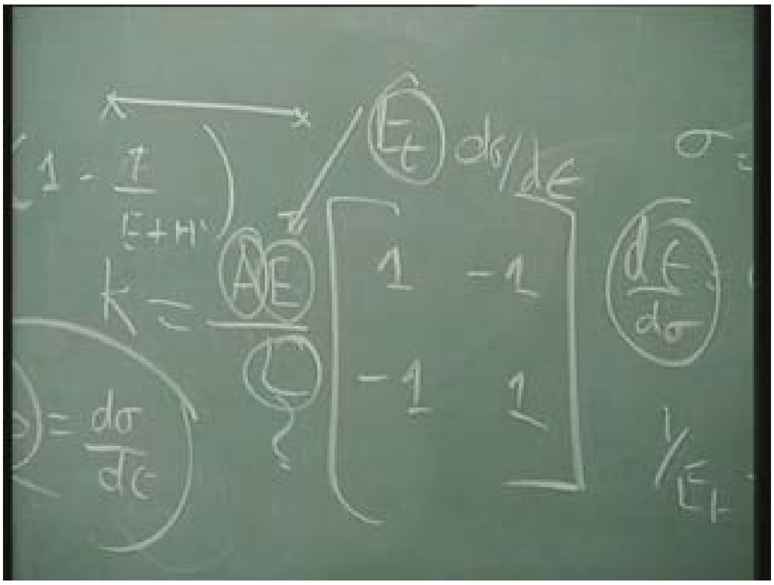

That is a short excursion into non-linear analysis. Is it clear? I know that I have, though I have put a lot of mathematics, it is not very rigorous, because the updated, total Lagrangean and all are mathematically very intense. We are not going into the details of it. At least I want you to know these names, because that will be very helpful when you run using a package a non-linear problem. Because, as I said, as far as non-linearity is concerned, the aim of the course is not to make you develop codes. You know, that is not, that is not my aim. To appreciate at least first linear part, yes, I would like you to go to an extent of even developing a code, but as far as non-linear part is concerned at least you should appreciate what are the terminologies, how things are done, what is convergence, what is time step and all that; how things are done, so that you will be able to handle the problem, the non-linear problem, using a software or a package. We will meet in the next class.