**Introduction to Finite Element Method**

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**Lecture - 3**

In the last class we saw a number of applications of finite element analysis. We saw really the breadth of applications. We saw a number of case studies from different fields and I am sure that you are all convinced that this is a very good technique which can be applied to lot of practical problems. From now on, we will study some of the basis or theoretical basis of finite element analysis. You may have to have background in various fields in order to understand exactly how this technique works. But as far as possible, in this course, we will see to it that all these things are covered and the course is completely self contained. We stopped with the definition of stiffness.

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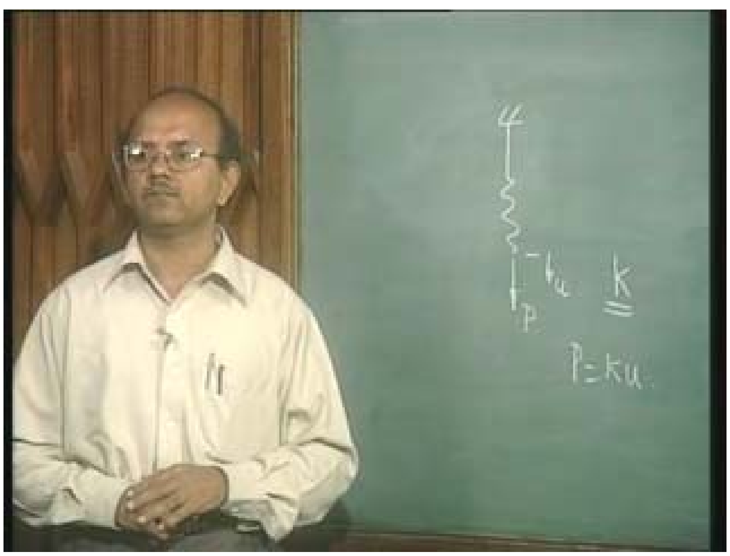


We said that or we said that it is easy to understand what stiffness is as far as the spring is concerned, but how do you define stiffness for other components? We considered that there is force P that is acting on the spring and if the displacement at this point is u, then we said that it is possible to define a term called stiffness K such that P is equal to Ku. Now the question is how am I going to extend this concept to other components or other bodies which are not necessarily nice springs like this?

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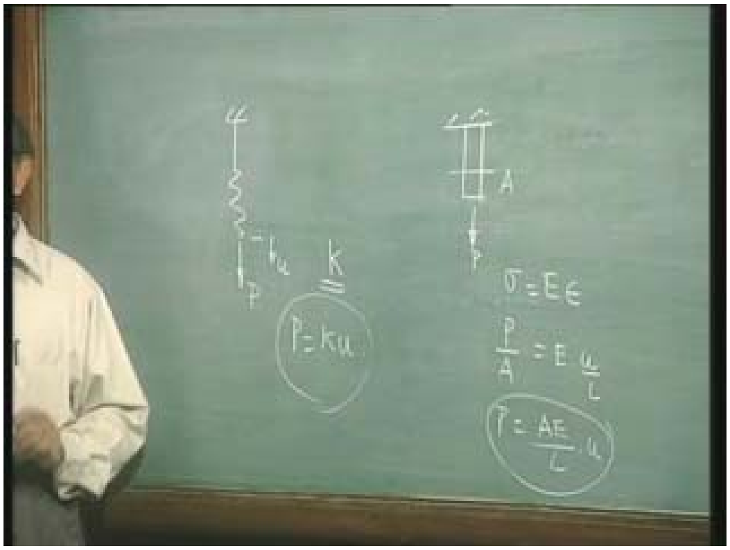


As a first case let us consider a simple bar and let us say that I apply a force P. Can I talk something as stiffness for this particular bar? Is it possible that we can define it? Yes; how do we define it? May be all of you are familiar with it. We know what is the relationship between stress and strain? What is In a very elementary level you would have defined stress as P by A where A is the area of this particular bar. So, P by A is equal to E into change in length. What is change in length in this case, say for example?

No; as far as the strain is concerned what is change in length by original length; that is how you would have defined. Let me call this change in length as u, in a similar fashion as we say that the displacement of the spring is u divided by L. It is very simple to now say that P is equal to AE divided by L into u; AE divided by L into u. Now compare this and this. It is quite clear that I can call AE by L as the stiffness of the bar. Stiffness straight away is not a concept which is restricted only to a spring but can be extended to say for example a bar. We are going to extend this for all other types of components as well.

Now let us look at another problem.

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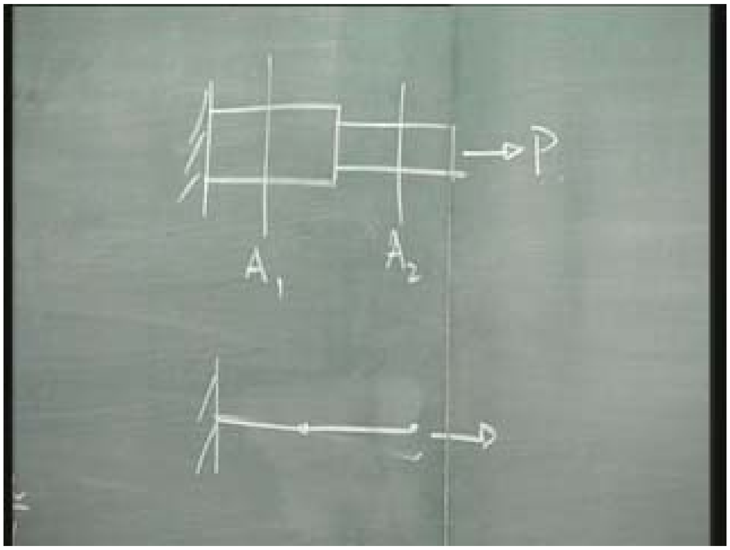


Let us see whether we can extend this kind of concept to a problem, where a simple case a very simple problem, where I have a P, force that is acting on a bar whose, for example, cross sections are say $A_1$ and $A_2$. This poses slightly more difficulty when compared to this problem or it is slightly more difficult. You can, looking at it immediately say that "Sir it is made up of two bars"? It is made up of two bars. So, I can say that why not you consider it as two bars? Once I start considering it as two bars you come into the realm of discretizing this particular piece; you have discretized it into two bars.

Now on we take over or we come into the realm of finite elements. Yes, that is nice answer, but there is a small problem. What is the problem? Here again, like this case, we had stiffness to be only one number AE by L. Because one end was fixed it was nice to me to say that the displacement is considered only at the bottom of the bar and we arrived at this point. On other hand look at this; it is not like that. I have now two bars. If you look at these two bars which are joined at this point, this point starts moving, this point also starts moving. One end is not fixed; easy to handle, but slightly more complex than this.

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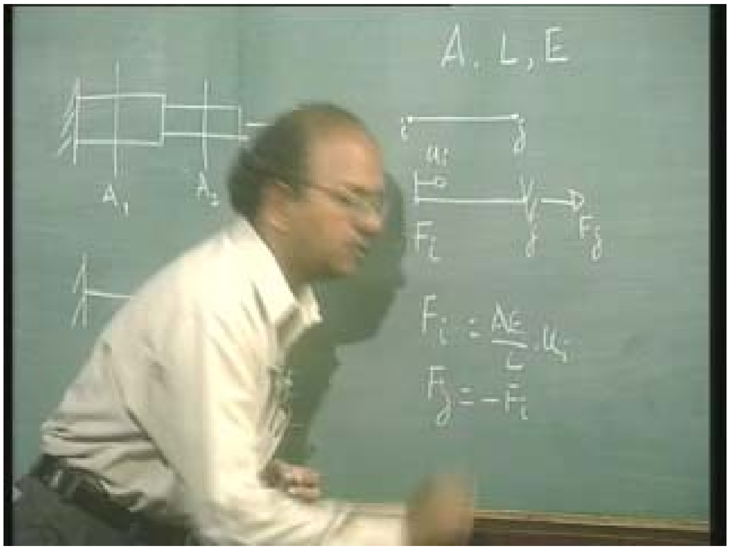


You can look at another problem where there are more number of bars. There is a technique to handle this. In order to use that technique first let us define what is called as a bar element? What is a bar element? Let us take an element. I now started calling this as an element. Let me take that element and put it out. Immediately you see that there are two guys sitting on either side which are called as nodes. So, let me call those nodes as say i and j; a very generic term say i and j. Now let me define or determine the stiffness matrix. Note this, we are now going to define stiffness matrix of this particular bar which are defined by these nodes i and j. What is also important is the area. Let me call this as A, area of this bar as well as the length, this length which I would call as L and the Young's modulus which I will call as E.

Let us start with what we know. In this bar, let me apply a displacement which I call as say $u_i$ and fix j. Now let me call the force that is required to cause this displacement as $F_i$. From my previous result I know what $F_i$ is. What is $F_i$? AE by L into $u_i$. When I fix this j, this j is going to take a reaction or in another words there is also a force that is acting at j. Let me call this as $F_j$. From elementary strength of materials I know that $F_j$ is a reaction and that it is, what is that? Opposite to $F_i$ and of equal magnitude. That's right; so, $F_j$ is equal to minus $F_i$; that's the first step.

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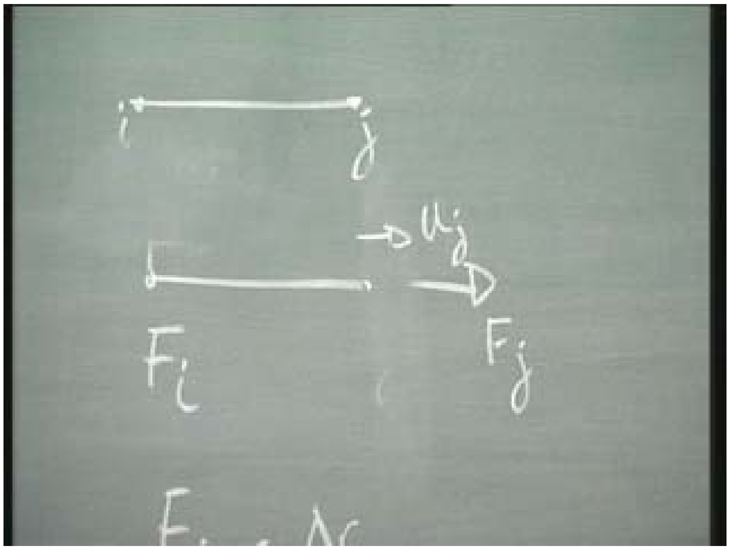


Now let me again repeat this with j free or applying a displacement of $u_j$ and fixing i. What is that I am going to do? I am now going to release this ….. and say that let j move from the original position by an amount $u_j$ and let me fix this position.

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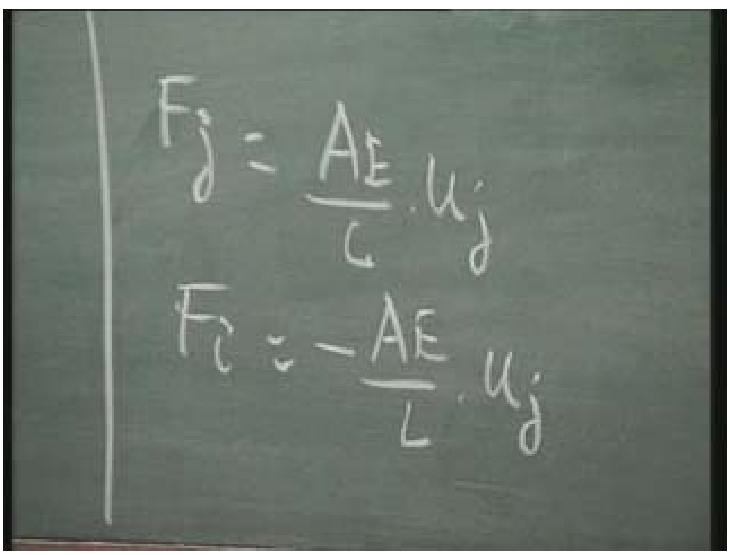


In this situation what happens to $F_j$? That is straight forward and that happens to be AE by L into $u_j$ and now what happens to my $F_i$? That is minus of $F_j$ or in other words that is minus AE by L into $u_j$. Is that clear? Now coming back to my problem here, this particular bar, this particular bar now has two nodes and is in the position where both the nodes would start displacing or $u_i$ and $u_j$ exist for this element. Since I have derived in a very simple fashion what is $F_i$ and $F_j$ when one or the other are fixed, it is possible for me to straight away get to a situation where both $u_i$ and $u_j$ exists. It is nothing but a superposition of these two results.

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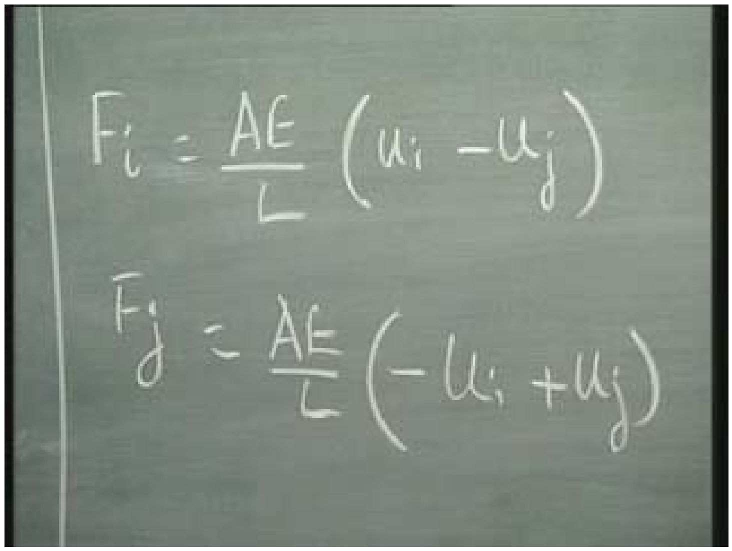


So, when both $u_i$ and $u_j$ exist, I can say that $F_i$ is equal to AE by L into $u_i$ minus $u_j$. Is that clear? What happens to $F_j$? $F_j$ is equal to AE by L into minus $u_i$ plus $u_j$. Is that clear? This can be expressed in a matrix form. What do I mean by that?

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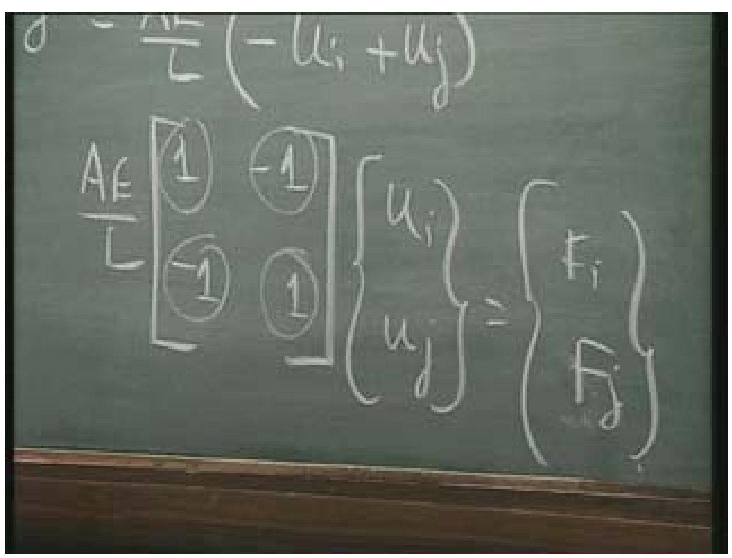

I can express this as AE by L into 1, minus 1, minus 1 and 1 multiplied by $u_i$ $u_j$ is equal to $F_i$ $F_j$. Look at this particular equation quite closely. You can immediately recognize what is that you can recognize? You can recognize that there is a force on one side and there is a displacement on the other side. In other words this is a relationship between force and displacement and that relationship has been brought about by this particular matrix. Is that clear? Comparing that equation with my first equation here where it is P is equal to Ku for a simple spring where I called K to be stiffness and similarly there is a relationship between force and displacement, I can call that matrix which I derived now to be what is called as the stiffness matrix. That is the stiffness matrix.

Let us understand what this stiffness matrix means. Can I extend the concept of stiffness, the physical behavior or the physical understanding of stiffness what I have for a spring, to the stiffness matrix which we derived? If I had asked you to define stiffness and not write down an equation, you would have said that stiffness is equal to force per unit displacement. There was no problem because there is only one place where force is applied and one place where the displacement happens to be, there. What about in this situation? How do you now define here or how do you physically get a picture of what the stiffness matrix is? That is the question. How do you define it? Can someone tell that or I would call this as a $K_{11}$ $K_{12}$ $K_{21}$ and $K_{22}$, along with AE by L.

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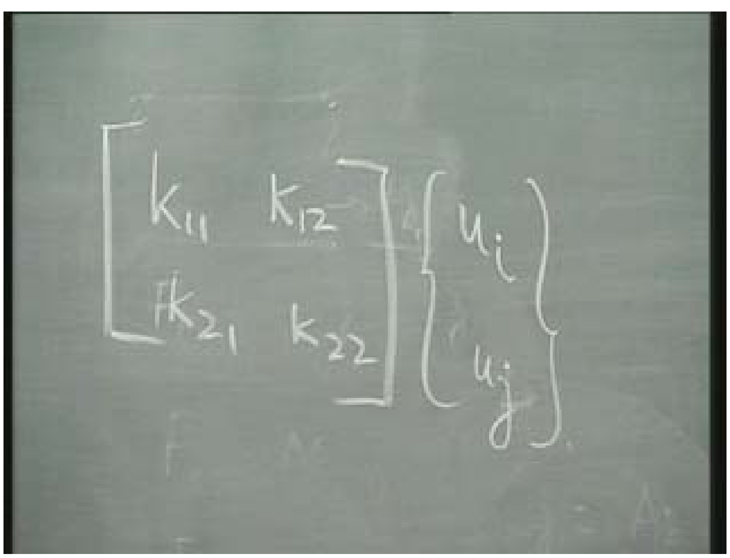

The same matrix can be written for us to interpret as $K_{11}$ $K_{12}$ $K_{21}$ and $K_{22}$ that multiplied by say $u_1$ and $u_2$ where i and j's I have replaced by $u_1$ and $u_2$. If you want you can keep this as i and j itself. If you are getting confused you can keep it as i and j itself. My question is what does $K_{11}$ indicate? What is the physical meaning of $K_{11}$? What does $K_{12}$ indicate and so on because here I said K indicates force per unit displacement.

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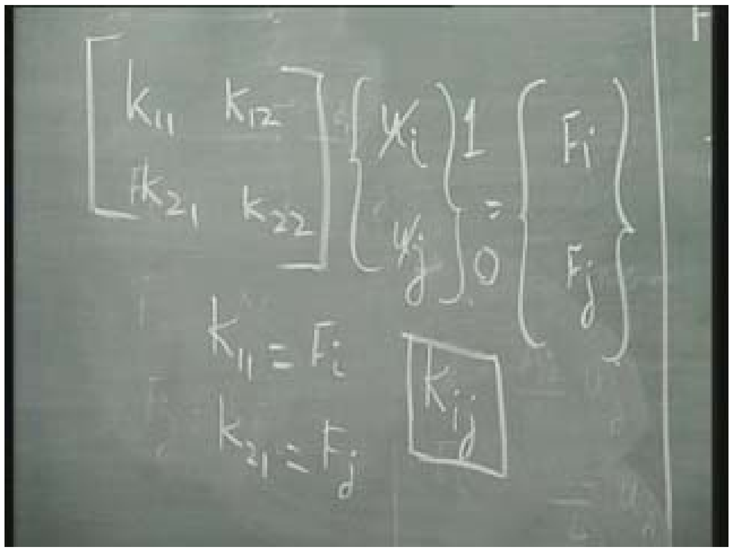


In order to understand that, in order to give you a clue let me say that I apply a unit displacement. Here it is force per unit displacement. Keeping that spirit let me say that $u_i$ is equal to 1 and let me fix the other side $u_j$ is equal to zero and then look at the complete equation now. What is that in the other side? $F_i$ and $F_j$. What does the first equation indicate? First equation; $K_{11}$ is equal to $F_i$ and what does $K_{21}$ indicate? $K_{21}$ is what? $F_j$. Is that clear? Can someone give me an interpretation of what is $K_{11}$ and what is $K_{21}$? One of you; yes, so it is nothing but, yes, in other words just to repeat that what it means is $K_{11}$ is the force that happens to be present at i or the force to be applied at i in order to have, note that, unit displacement at i and zero displacement at j and $K_{21}$ is the force that is required to be applied at j in order to have a unit displacement at 1 and zero displacement at j.

The same thing, same logic can be repeated for j as well; you can put this as zero and this as 1. You can again interpret this $K_{ij}$. I can have a physical interpretation of $K_{ij}$'s which are entrées here. i is equal to 1, j is equal to 1; it becomes $K_{11}$. i is equal to 2, j is equal to 1, it becomes $K_{21}$ and so on. So, I can have physical interpretation for $K_{ij}$. What is the physical interpretation? What does $K_{ij}$ indicate? It is the force that has to be applied at j in order to keep ….. Now look at this and tell me. See, $K_{21}$ means, what is the displacement I have applied for i? It is unit displacement. So, if I have to apply a unit displacement at 1, and a zero displacement at j then $K_{21}$ is obtained from $F_j$. If there is a unit displacement at j, then $K_{ij}$ indicates the force that has to sit at i. It is the force that is required to make or to give a unit displacement at j and the force being measured at i, the force that is required at i for a unit displacement of j. So, that is a simple physical interpretation. Yes, zero displacement at i; force that is required to be present at i for a unit displacement at j, zero displacement at i.

Now let us look at this matrix closely. What are the other things that you see in this matrix?

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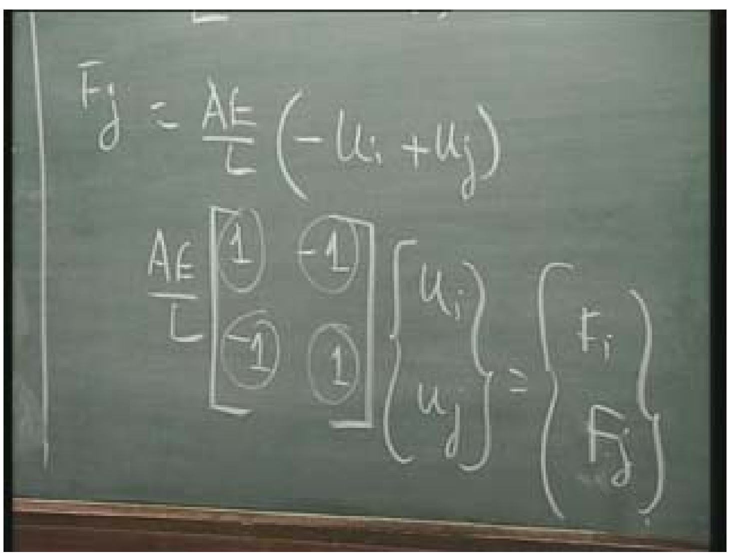


What are the other things that you see in this matrix? Symmetric, very good; so, this matrix is a symmetric matrix. It is not necessary that every matrix that we are going to come across is symmetric matrix; but this particular matrix because we are just dealing with a linear elastic problem, this matrix happens to be a symmetric matrix. Very rarely do we come across matrices which are not symmetric and in this course most of the problems except when we go to contact we will only be dealing with symmetric matrices. Some of the large deformation problems have non-symmetric matrix, let us not worry about it. But as you correctly said, the next thing that you observe is that this matrix is a symmetric matrix.

It is very simple now to understand that I can have say two bars which are here or two sections in a bar to have different A's, different E's and different L's.

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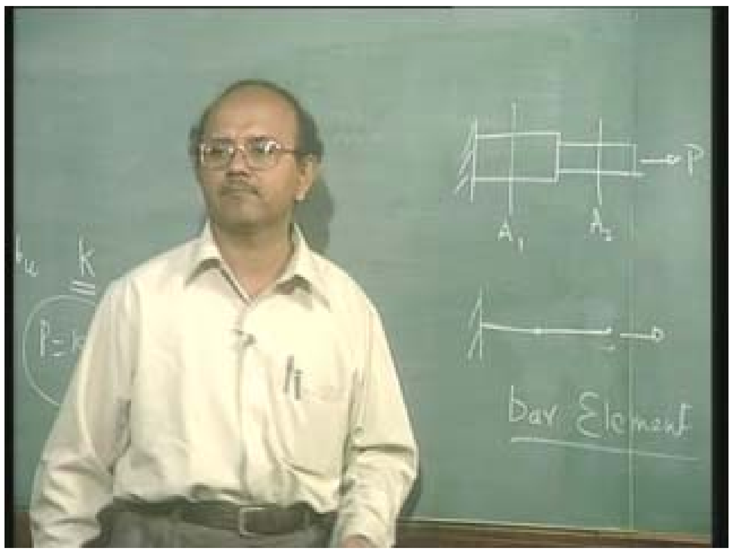


The only thing is that A, E and L would change. It will become $A_1$, $E_1$, $L_1$; $A_2$, $E_2$, $L_2$ and so on. Now let us see how we can determine a composite or an assembled matrix of a bar which looks like this.

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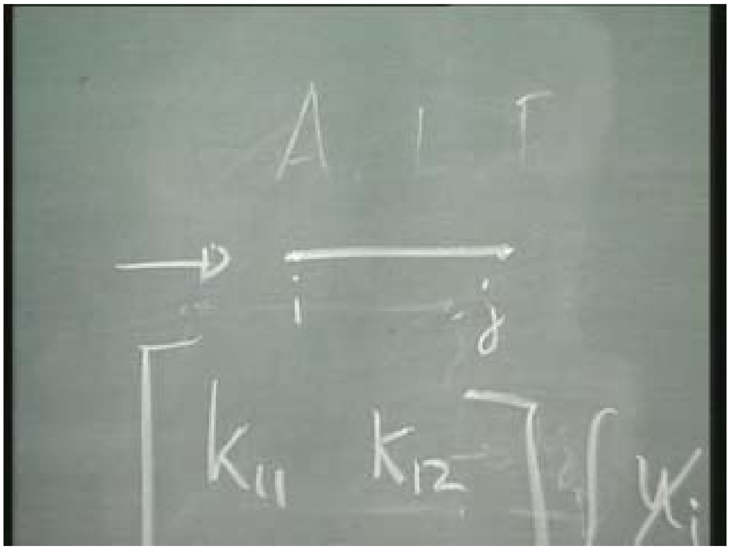


This particular bar which we considered as i and j have what are called as how many degrees of freedom? Two degrees of freedom or i and j whose displacements are $u_i$ and $u_j$ give the freedom for this bar to be displaced in the X direction. So, there are two degrees of freedom or if you consider each node there is one degree of freedom. What do I mean by degree of freedom? It means that this particular node i can be displaced only in the X direction and j can be displaced in the X direction. As an element this has two degrees of freedom, one degree of freedom per node.

Now let us see how this concept can be applied to a composite bar and how assembly can be done so that I get a composite stiffness of this bar. For that the first thing I have to do is to determine the stiffness of individual bar. But before that I have already done one job. What is it that I have done? I have already discretized this particular component into two elements. I did it by looking at it and seeing that there is a change in A, change in E and so on, if it happens to be there, and so I have used that knowledge in order to discretize it. But there are lots of other considerations when I discretize it, but that we will see as we go on. But right now it is important to understand that discretization is very important and can depend upon a number of factors, one factor being a nice physical picture or an engineering sense and that is what has helped me to discretize it.

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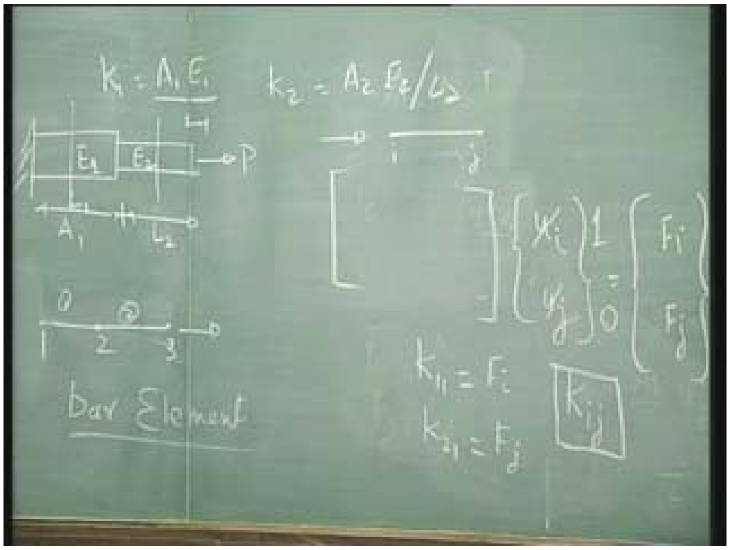


Let me now call this as element 1 and let me call this as element 2 and let me right now worry about fixing this particular node. Let me call that node as 1, sorry 1, 2 and 3. Is that clear? Let me call $A_1$, This is say $L_1$; let me give some lengths to it. Let me say that it is $L_1$ and this length to be say, $L_2$. Just to generalize this problem, let me call the Young's modulus to be $E_1$ of this bar and this to be $E_2$. Since AE by L comes up as a unit, let me define $K_1$ to be $A_1E_1$ by $L_1$ and $K_2$ to be $A_2E_2$ by $L_2$. 2640$K_1$ is equal to $A_1E_1$ by $L_1$ and $K_2$ be $A_2E_2$ by $L_2$.

First what I can do is to write down the stiffnesses of individual elements and look at this kind of assembly for individual elements or how does this equation look like? How does it look like for the first element? I can change that here itself.

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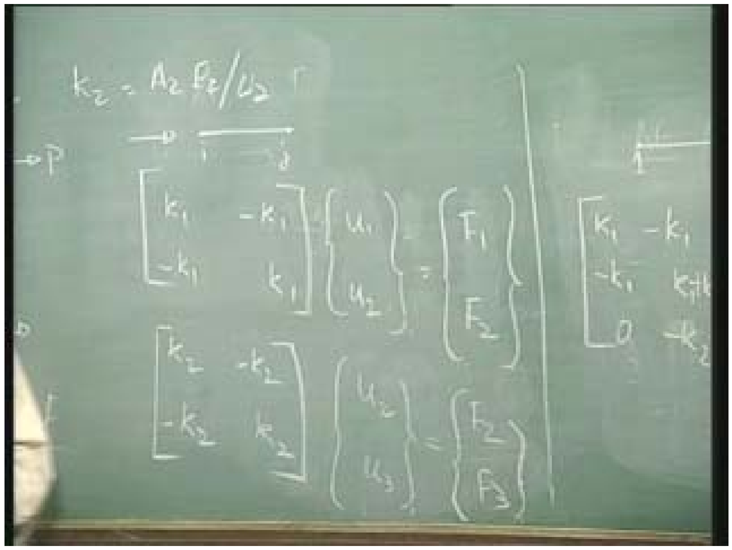

How does it look like? It will be, for the first element, $K_1$, minus $K_1$; $K_2$ does not come here, we are only looking at the first element minus $K_1$ and again $K_1$. Now what happens to i and j? Now what happens to i and j? $u_1$ and $u_2$. What happens to $F_i$ and $F_j$ and that happens to be $F_1$ and $F_2$. I am not worried about whatever the values of $F_1$, how it looks like and so on; just I am repeating it for this particular element. Any questions? i and j was generic indices, now I have replaced it with the actual index.

Let us now look at the second element. The second element can also be written in the same fashion. I will do it in the same matrix or maybe we will write it separately so that things become clear. What do I get? Instead of $K_1$, $K_2$; so, $K_2$, minus $K_2$, minus $K_2$ and $K_2$. Instead of now 1 and 2 what are our i's and j's? $u_2$ and $u_3$; what happens to this guy here? $F_2$ and $F_3$; $F_2$ and $F_3$. Now I have to assemble them. Please bear in mind what are the steps that we are doing? We discretized it first step. I will write it down at the end of this exercise but nevertheless as we go along please bear in mind what we are trying to do. We discretized it in terms of what we called as elements. In this case, the behavior is bar. We will come to what is behavior in a minute, but in this case we call this as bar element, so, we discretized it. The next step is we wrote down the stiffnesses. That is the next step; we wrote down the stiffnesses of individual element. The third step is I have to assemble them. How am I going to assemble or what does this term assemble mean?

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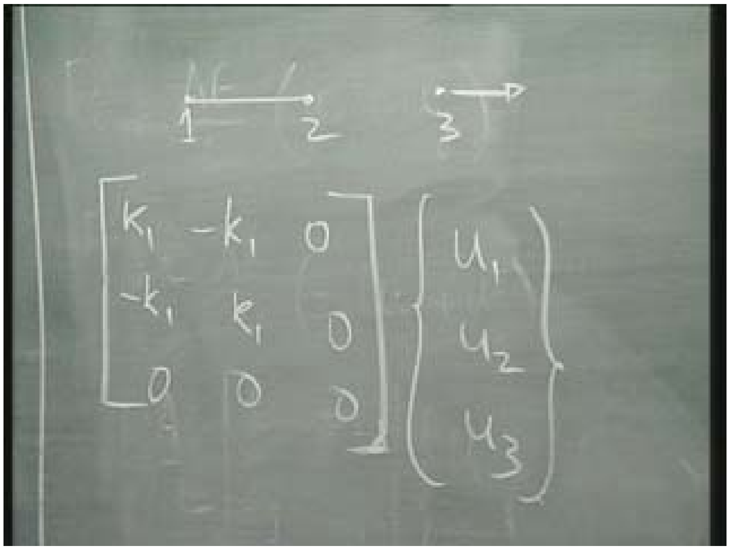


In order to assemble them let us see or let us imagine that we are building this bar. Let us not worry about the actual manufacturing process; it may be turning …… (00:31:11) let us not worry about that. We will assemble them as if first I will put this and later put that. In fact, it may well turn out to be a welding process because $E_1$ and $E_2$ are different; like we may imagine that they are welded together. So, first I take this piece, the first element, put it there and identify these guys, these nodes there. Let us imagine that that is what happens. Since I have already identified a node at 3, it is imperative that it also has a displacement $u_3$.

Let us say that it has a displacement $u_3$, though more importantly this element does not support any force there. If I apply a force here, the force happens to be in the air and this particular element does not support a force. In other words there is no stiffness. See, when can you apply a force? Only when there is a resistance to it or in other words there is stiffness to it; stiffness to the body on which you are applying. Since it is in vacuum, let us imagine that; let us not worry about anything else. So, I cannot put a force at that place or even if I put a force, there is no resistance due to this bar. If I have to now expand $u_1$ as $u_1$ $u_2$ and $u_3$, how will my stiffness matrix look like from our physical understanding of K?

Let us see how it would like? So, I will have $K_1$, no doubt; I will have minus $K_1$. These two guys are for $u_1$ and $u_2$ and what will happen to my third term there? Very good, because you may imagine or remember our $K_{ij}$ definition; so, here I will have minus $K_1$ and then followed by $K_1$ 0 0 0 0; that is very good.

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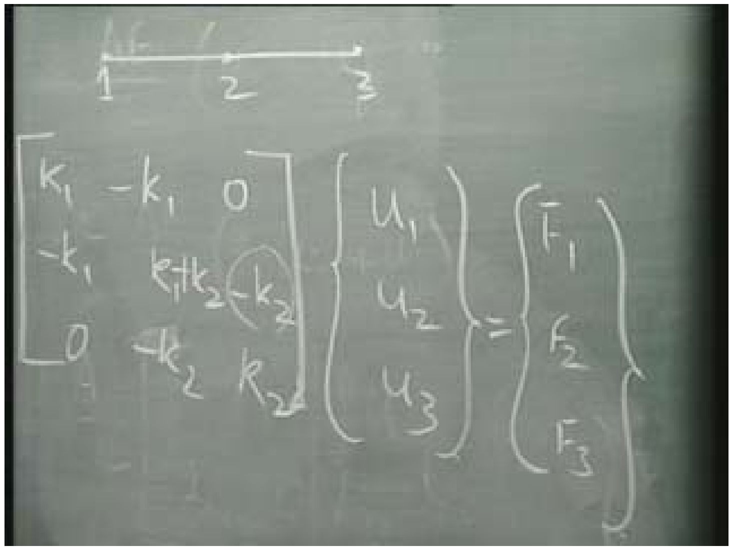


Let me go and put the other element here, weld them together or in other words weld them together. As soon as I do that, this force what we have mentioned here will be supported. Let us not worry about that right now, but the process is quite clear and hence some of the zeroes are now going to be replaced by the stiffness we are giving, by introducing the next element which means that I will have $K_2$ coming to the aid of the first element; so, $K_1$ plus $K_2$, minus $K_2$, minus $K_2$. Please note that this is the last term in the matrix that is in the second row and 2, 3 and $K_2$.

One of the things you had not noticed in the last stiffness; when I said that you just said symmetric. I will look at it but you have not noticed one very important thing in this stiffness matrix which I will come to in a minute but what is important here is that we have assembled the matrix. First thing is that look at the matrix size. Now there are three nodes and there is a 3 by 3 stiffness matrix and there are $u_1$ $u_2$ $u_3$ that are not known and what is the right hand side? What is the right hand side? Correct. It is $F_1$ $F_2$ and $F_3$; remember that they are nothing but the forces that can be applied. It is not necessary that they should exist but that can be applied to 1, 2 and 3. So, this is the complete equation which I am interested in after assembling. What is my next step? I have to apply what is called as boundary condition. I have to apply what is called as boundary condition. But before we go to boundary condition let us look at this matrix closely and observe one more point.

Again physically there is lot of mathematical implications to it. At this stage let us not worry about the mathematical implications but physically let us look at it and see whether we know why it happens to be in this fashion. Look at that matrix. You will see that there are some positive terms and negative terms. $K_1$ is positive, $K_2$ is positive; so, there are some positive terms and there are some negative terms. See that there is no negative term in the diagonals. All the diagonal terms happens to be positives. Apart from that you also see one more thing. What is that you see? Zeroes; so, there are positive terms, negative terms as well as some zeroes and it is very clear that the diagonal terms are positive, off diagonal terms, there are negative terms as well as zero terms. In this particular problem there are negative terms as well as there are zero terms.

The question is, is it that these terms that are sitting in the diagonals are positive only for this problem it happens to be positive or is it by design or it is by chance? Is it that it is positive only for this problem or for any other problem? Is the question clear? What do you think will be the answer? One minute; let me give you a clue. In order to answer that question, let us follow the same logic by applying say unit displacement at say for example 1 and then zero displacements and so on. Now you can see or you can say whether this is positive by design and not by chance.

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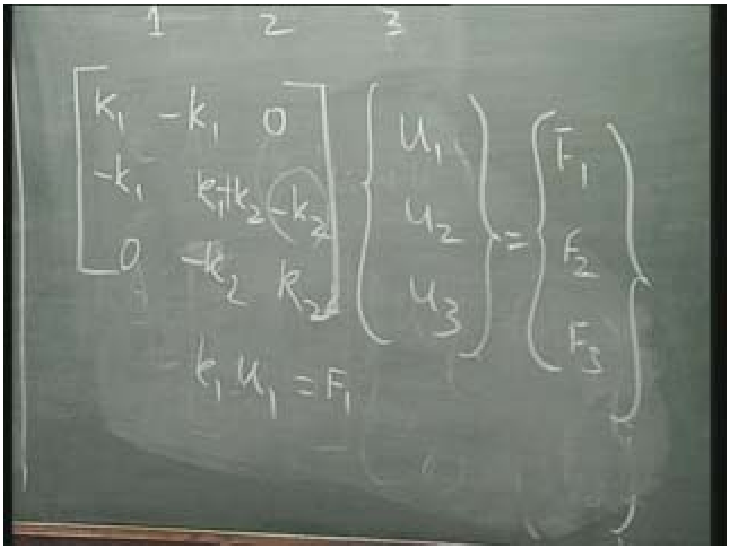

Let us look at my first equation, say $K_1$ say $u_1$ ……. (00:38:40) not a unit displacement at least $u_1$ is equal to $F_1$. So, $u_1$ is equal to $F_1$ by $K_1$. $u_1$ is equal to $F_1$ by $K_1$.

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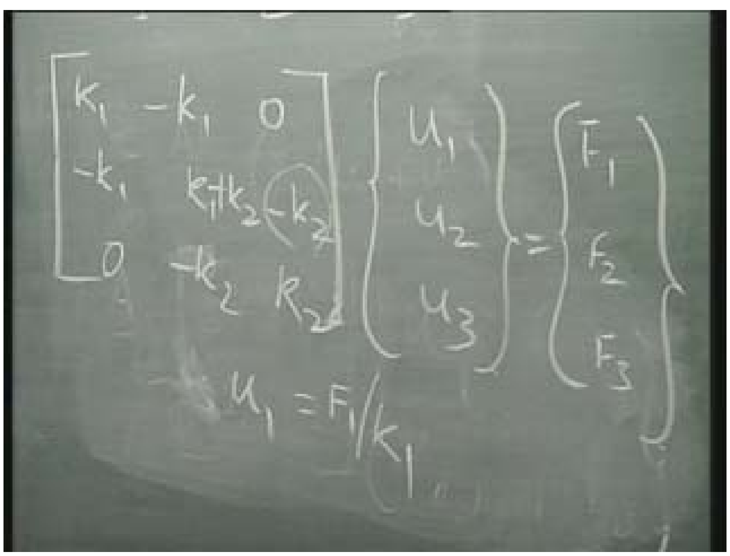

What does this tell? If I apply a force $F_1$, then $u_1$ would follow the force in the same direction as long as $K_1$ is positive. So, if I apply a force I know that the displacement has to be in the same direction; it cannot be in the opposite direction. So that particular condition is satisfied as long as $K_1$ is positive. On the other hand if $K_1$

happens to be negative, what happens? Say for example I take this bar and then I start applying a force here. It will move in this direction if $K_1$ is negative. That is not possible. The bar has to move in the same direction. What does it indicate? It indicates that $K_1$ has to be positive and the next thing is that, is it clear?

That is very simple thing. On the other hand can $K_1$ be zero; then $u_1$ becomes infinity. So, necessarily these diagonal terms have to be positive. We will come to a much larger condition of positive definiteness of stiffness matrices later. Let us not worry about it because we have lot to learn from the physics of the problem before we go into the mathematics of the problem. I will not define all those things right now. Ultimately we will come to that. What is my next step? What is my next step, what do you think is the next step?

Boundary condition, very good; so, what is the boundary condition of this problem?

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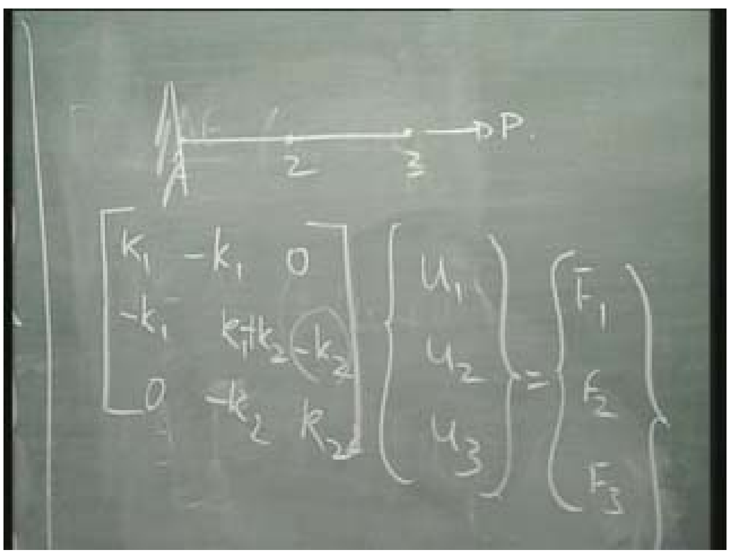


This problem states that one is fixed and that a force is applied at this point and that is the secondary thing; we will come to that later but first thing is that $u_1$ is fixed. Please note one thing carefully that when I say boundary condition and say $u_1$ is fixed, I just state that $u_1$ is known. It does not mean many students make a mistake, that boundary condition means $u_1$ is equal to zero; a particular degree of freedom, $u_1$ or whatever be the degree of freedom that happens to be zero; not necessary. It just means that the displacement for that degree of freedom is stated. In this problem, it is stated to be zero.

Suppose in another problem I say that this displacement is 0.5 mm; that is also a boundary condition. Boundary condition simply means in this particular problem displacement happens to be zero. What are the types of boundary conditions and what are the ways in which to express them, all these things we will see as we go along. But, for this particular problem it is important to understand that the displacement happens to be a boundary condition and that, that displacement happens to be zero. Why I am stating this point is because a same configuration, as far as geometry is concerned, the same configuration can be looked at as a beam as well.

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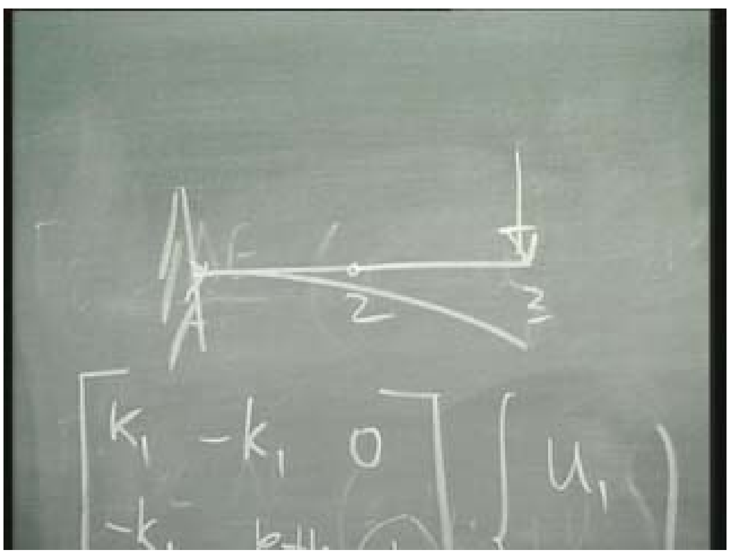

The only thing is if I now apply a force in this direction, it is no more a bar, a beam; it becomes a beam which means that it can bend. A bar can take only axial loads; you know from strength of materials. On the other hand a beam can take a bending load. Is that clear? Again, I have different types of displacements, a displacement, which looks something like this and I have different types of boundary condition. What are the boundary conditions? Look at the question, what are the boundary conditions? The previous case for bar we defined what is the boundary condition? There was only one; now we just modify the problem and say that what are the boundary conditions? Displacement and rotation that is right. So, displacement as well as rotation, both is specified at that place. In other words the same node or the same position, same point is specified in a slightly different fashion; the geometry happens to be the same. But, why is this difference? That is because of the governing differential equation for these two problems; they happen to be different. Before we go further, before we go to that point let us now worry only about this problem and get a complete physical picture of finite element analysis. So, we will continue with this.

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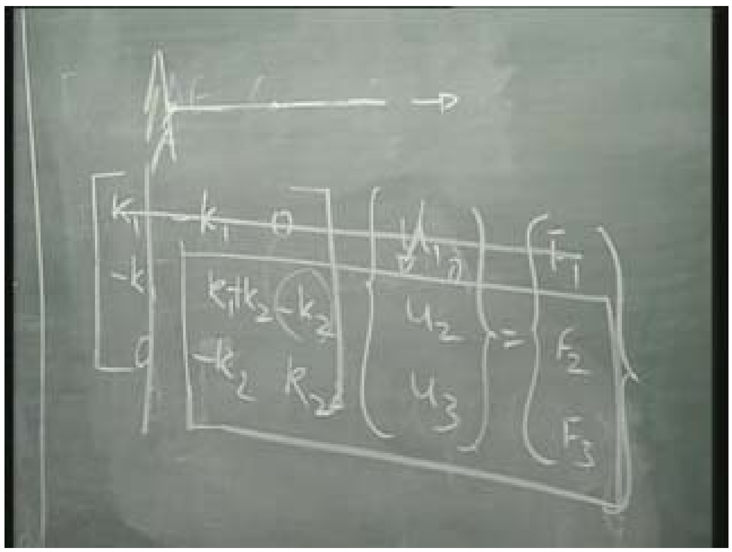

As you correctly said $u_1$ is specified and $u_1$ happens to be zero. So, how many unknowns are there? Now two unknowns; $u_2$ and $u_3$ and $u_1$ is already known and happens to be zero. Since $u_1$ is equal to zero, I eliminate these two lines or in other words I eliminate that row and the column and restrict my attention to this particular piece. Now I know something else also in this equation. What is it that I know? What is it that I know in this equation?

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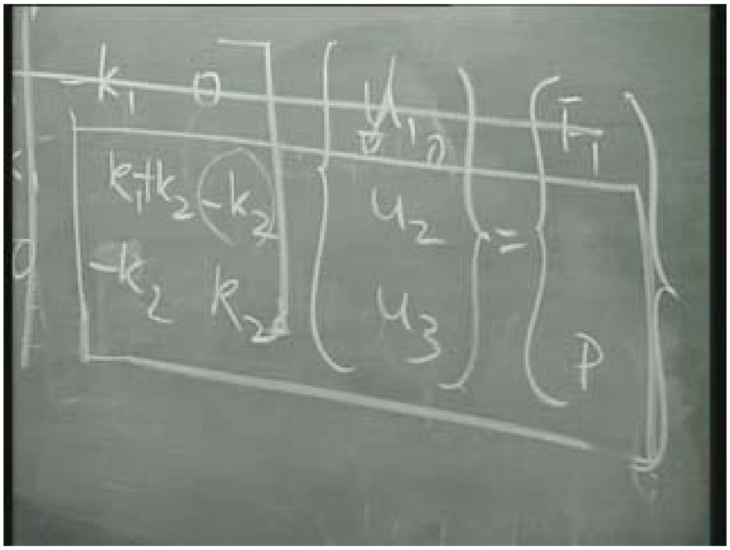


$F_3$; correct. So, I know that $F_3$ happens to be the applied force P and what happens to $F_2$? Do I know it or I do not know it? Is it that we do not know? Note that carefully; this is the mistake many students make. No, no; please note very carefully. Look at the problem again, look at this problem again. Look at this problem, look at this. Now, where is that node? Node is sitting here.

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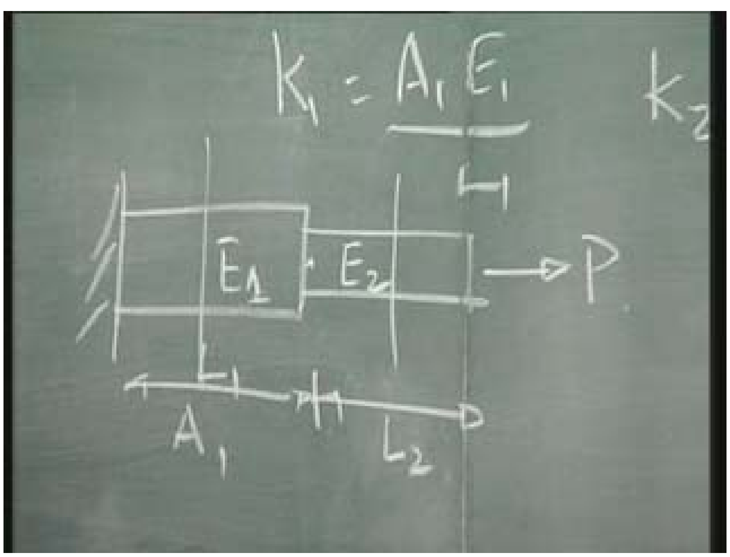


It is nice that you have made that answer because this is usually the confusion. Please note that what we are doing here is to put down the external forces that are applied.

We are not worried about the corresponding internal forces that are developed; we are looking at the external forces. These are the external forces. Please do not get confused between that and the internal forces which are developed due to stresses. This is the usual confusion for students and hence this is an external force, P is an external force.

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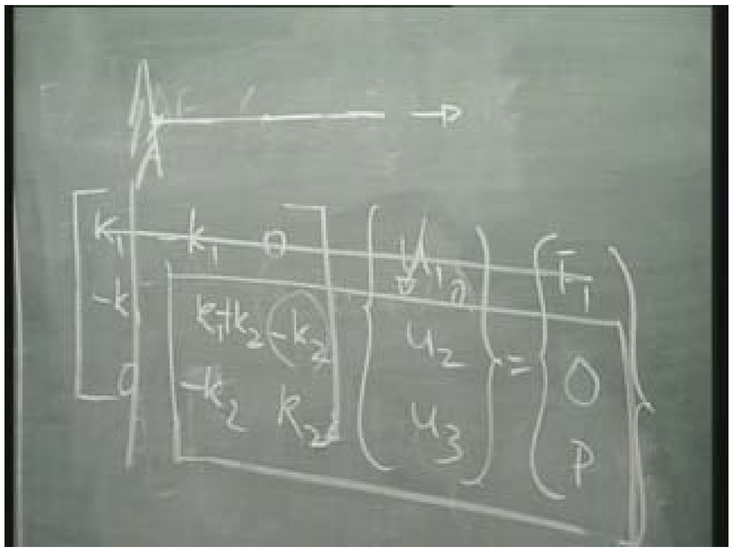

Correspondingly is there an external force at node 2? No; so, this happens to be zero. Now I have two equations and two unknowns.

Just to summarize, what did we do? We discretized, we divide or in other words we divided this into a number of elements. Then we wrote the stiffness matrix for each of these elements. We assembled them, that is what we did here and then we wrote down boundary conditions carefully and also put down the forces that happen. One other thing before we go ahead and solve this problem; it is very important to note from this equation. What is it? What is that you see; another thing that you see?

You please note that at the place where we give a boundary condition we are not specifying the force. It is not possible to specify both a boundary condition as well as a force. What does it mean? What is it mean? It means that when I fix a particular node, the force that acts on the node is decided by the system. So, if I say P, it is very clear from this problem before even solving it that the reaction for P has to be taken at the place where I have fixed. So, the force that is going to happen or going to be present at this place is going to be decided by the force that you give here. I cannot specify both a force as well as a boundary condition at the same point. When I specify a force at this point, the displacement at this point is decided by this system, by the stiffness matrices and when I specify a displacement at this point the force that is required to sustain that effect is specified again by the system, by equilibrium and so on.

We will have more interpretation of this equation and how to solve them or how to solve this equation in the next class.