Introduction to Finite Element Method

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Lecture - 26

We had a very detailed introduction for thermal problems yesterday. Now, I am sure all of you are clear as to what the thermal problems are that we are interested in and what is its application is and so on. Let us get back to the important derivation that you want to do.

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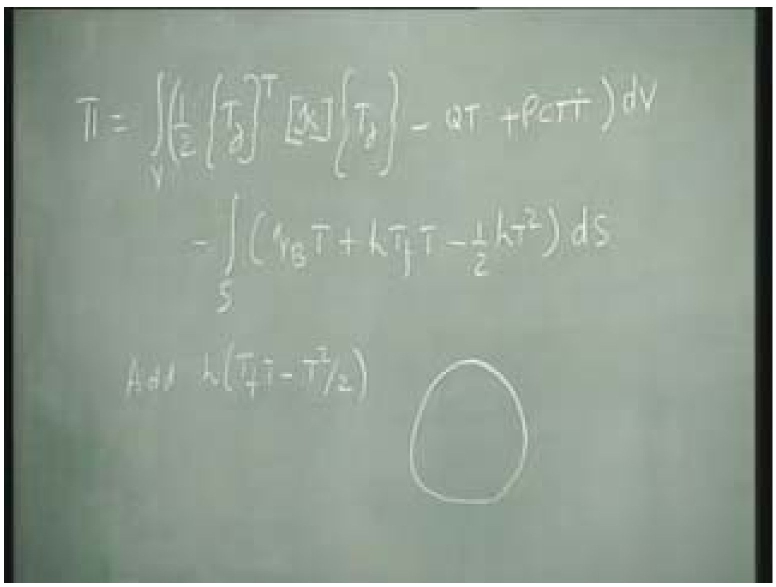

In other words, we have to look at the finite element implementation of the theory, probably what you have learnt in your earlier classes on heat transfer. We start first with of course the differential equation and then we go over to the functional. Yes, this route, I am not deriving this route from the differential equation to this functional, because that is again quite involved. Let us just skip that, because that you may have to spend about one to one and a half hours on that. But if people are interested, there are standard text books that are available. That has more to do with calculus of variations and so on. Let us assume that for that differential equation, we can put down a functional of this form. That part let us assume and go ahead. People who are interested, they can look at J N Reddy Finite Element Analysis or Applied Functional Analysis by J N Reddy. That book gives you as to how to convert; given a differential equation, how do you convert it or how do you get a functional? These things are explained in that book. If you are interested, you can have a look at it.

Let us start here. Of course this has to go with the boundary conditions which we have discussed. Let us look at small, small but important point, when we convert this into a two dimensional problem.

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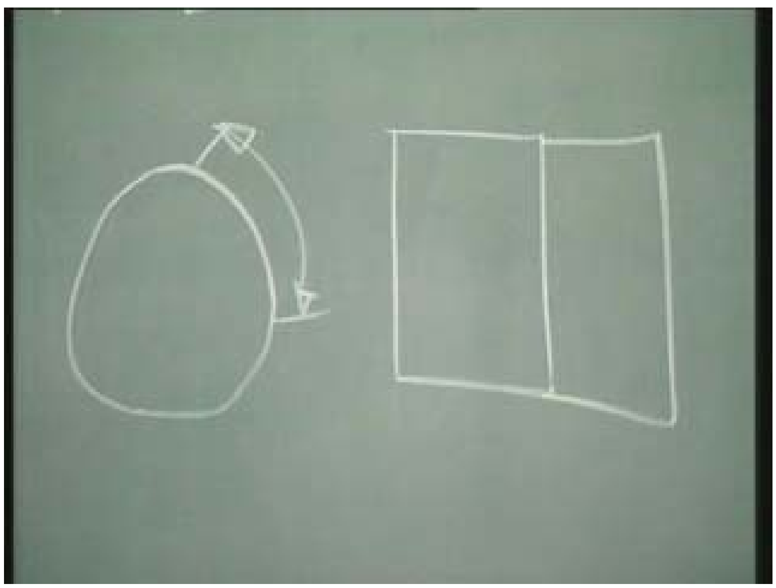

I already told you that we were looking at the sheet, yesterday we said that we were looking at the sheet and I just skipped the issue as to how to specify heat transfer and do it still in 2D. For example, I am going to weld these two sheets. Strictly when I consider a problem of two dimensions, then the surface becomes here, what? Line; so, boundary conditions I would say are specified only along this. In fact I had mentioned that yesterday and the functional is also valid for that strict two dimensional boundary condition or two dimensional process where for example, heat transfer takes place outside. This is for example very close to a plane strain problem, where for example because of the geometry, because of the geometry there is no stress, sorry or rather the strain perpendicular to the plane of this board and in a similar fashion when you are looking at a very long piece and you are looking at temperature distribution, then there will not be any heat transfer perpendicular to this board; there will not be any heat transfer perpendicular to this board and hence this kind of boundary condition is fine.

On the other hand, I am going to look at a problem like this, a thin sheet, say for example, I want to find out the temperature distribution due to welding. Then how does the heat transfer take place in this problem? Yeah, that is correct. Basically it is perpendicular to the plane of this board, the lateral direction perpendicular to the plane of the board. Then, I have problems of equating this or treating the problem in the same fashion as that of this problem.

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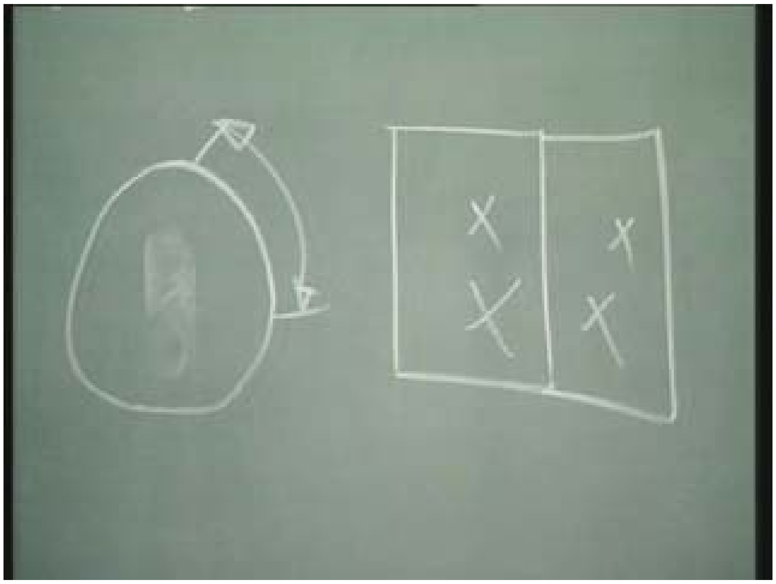

Here there is no heat transfer from here, but here all the heat transfer takes place only in this, from this particular face. It may be one face or it can be both the faces, this face as well as the face that is behind, both the faces, in which case you have to modify this functional with an additional term for one side. if there are two sides, two terms with an additional term which is given by this equation.

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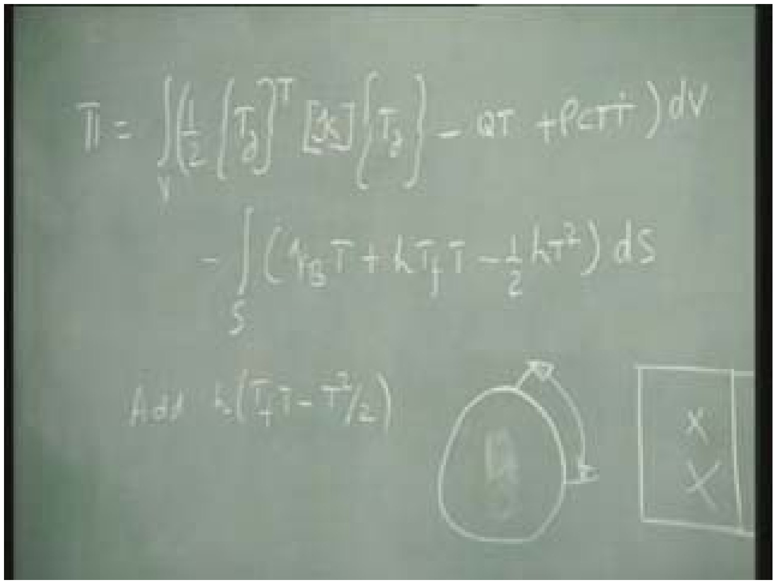

Look at this equation, so that equation or this is added to this. Is that clear? We will keep that aside for the time being, because it is very simple to add it. It is no big deal, because how we are going to do is, very logically we are going to proceed. So, there is not a problem. But look at this equation. You would recognize that there is q_B, h both of them are there. In other words, there can be a heat flux due to, for example, welding, which I had said q_B, heat flux due to welding. There can be a convection term. That convection term is given by this multiplication of h. Both of them can co exist in this particular form. Is that clear? Now, we will go ahead and put our, time being we lift this, we will add it ultimately. In fact you can do that as an exercise, if that term is included, how would the, how the complete finite element formulation looks like? You can do that by adding it later.

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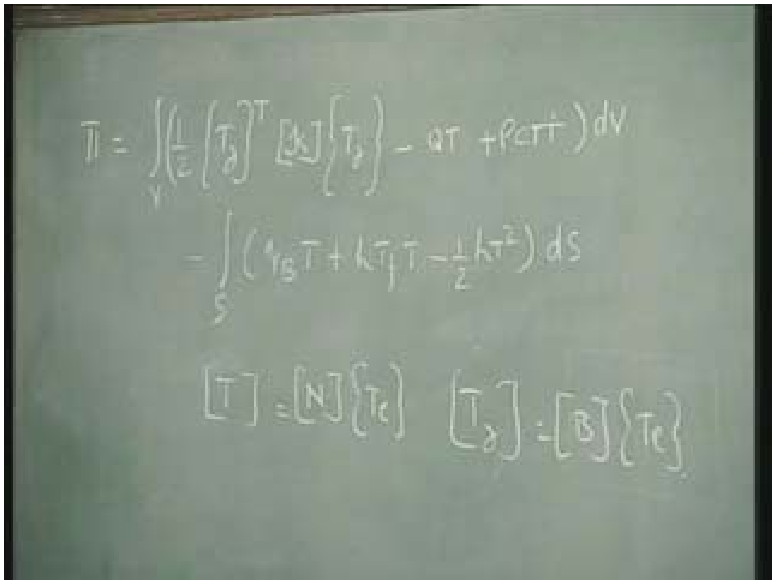

We already saw that I have to write down T is equal to N T_e and that this T, that is equal to B T_e. Is that clear? Now, I have to substitute this into this expression. Is it clear? Let us see, what is that I am going to get? What is this matrix, by the way? Kappa matrix, call this as Kappa. Do not confuse this with k, Kappa matrix for isotropic material where k is the same in all directions. What does it become? k 1 zero zero 1. So, k k in the diagonal, It is a 2 by 2 matrix, so, k becomes, k takes diagonal terms and off diagonal terms go to zero. This is for an isotropic material. Is that clear? Now, how do you write this equation with the help of this? You have to do a small jugglery.

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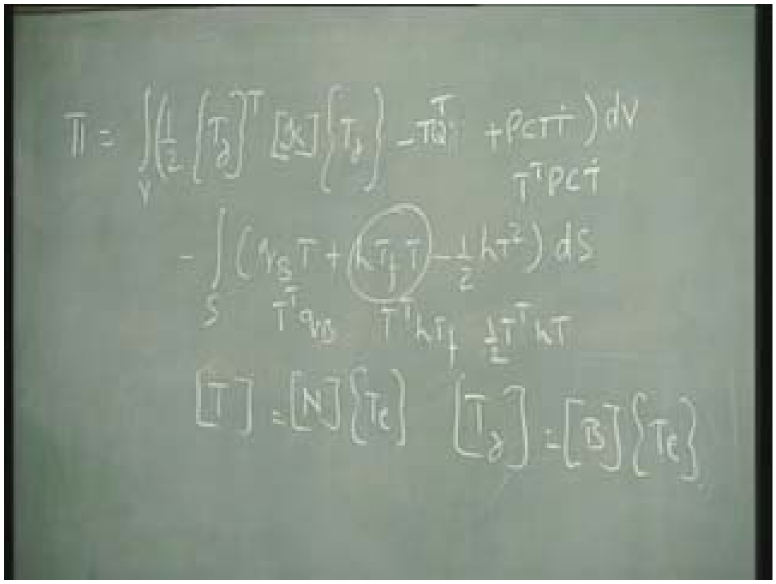

Q T, write it actually as T transpose Q, rho C T T dot write it as T transpose rho C T dot, write it like that, so that it is easy to now write. Similarly this you write it as T transpose q_B and this write it as T transpose h T_f and this write it as T transpose hT. Is that clear? Because, that makes it easier to write; I mean these terms are just replaced, all these terms replace this. Substitute these two expressions into my original expression, my pi. Yes; can you do that? May be you can tell me how it looks like? Is this clear? Up to this, I am sure all of you are clear. We follow the same isoparametric and so on.

For the time being, just to make things clear in this particular derivation let us only look at 2D. That means this v right now, is replaced by S, surface and the surface is by the boundary line. That is all. Just you can replace these two for the time being. I mean it is very simple, if you want to extend it. That is not a big problem, you can just extend that. How do you write it, what are the things that are there? What happens to this?

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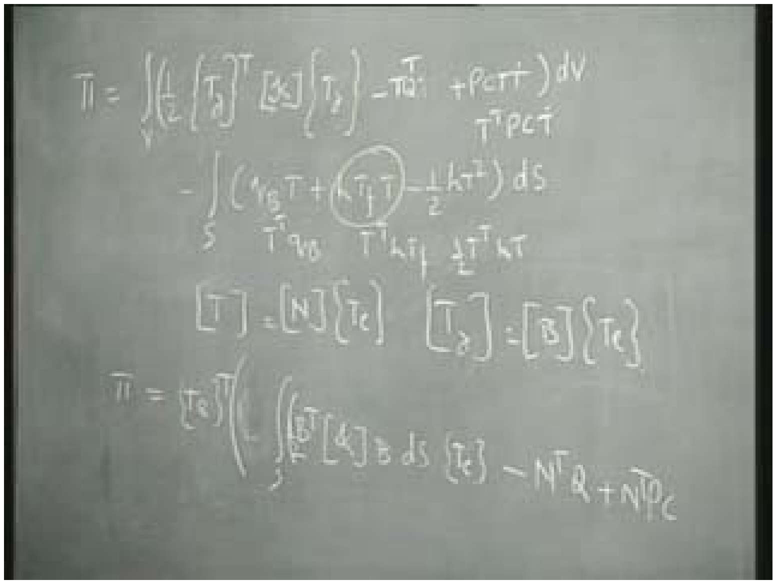

This becomes T_e transpose B T_e. What I will do is I will say that T_e transpose half into integral, say S, B transpose kappa B say dS or you can retain dv itself, into T_e. B is, of course, a matrix, I need not tell that. Then T transpose Q, T becomes minus, it is all inside the integral of course; so, T_e transpose, I think I will put a bracket here. T transpose Q now becomes T_e transpose N transpose. So, this becomes N transpose Q and T transpose rho C T dot, what happens to T transpose? That is T_e transpose N transpose. Is there any confusion? This is, actually the whole thing is integral. I think, I should put, let me put integral separately or if you want it together you can put it. So, all these things are inside the integral. All of them are inside the integral. I think better I shift half here, so that there is no confusion; half is here, so, there is no confusion. Half is only for this term; sorry, half is only for this term. So, I will put everything under the integral; N transpose Q.

How do you handle this term? T transpose is T_e transpose N transpose, so, T_e transpose I have already taken out, so, this becomes plus N transpose, N transpose rho C. What is T dot? Yeah, wait, wait, wait; let us, yeah, this is one thing. That is a very good answer, because that needs some explanation. What is T dot? T is equal to NT. What is T dot?

Dow T by dow small t that is the rate of change of temperature with respect to time. In this N T_e, what is going to change with respect to time?

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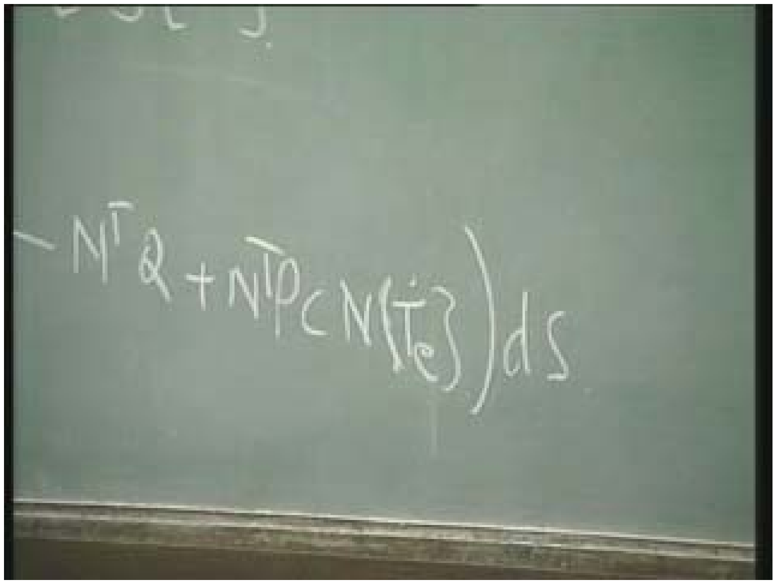

T_e not N; so T dot now becomes N T_e dot, so, rho C N T_e dot. These are the terms which are inside the dv, so, the whole thing or dS because you said that we will consider it as a surface. Is that clear? How about this? Minus, minus, what happens to this term?

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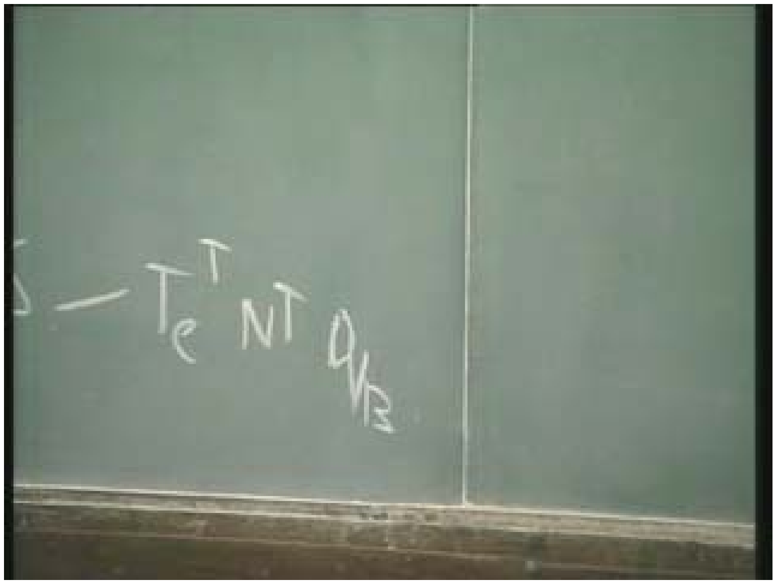

N transpose q_B, T_e transpose N transpose, T_e transpose N transpose q or is it q_B? q_B. What is the next term?

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N transpose T_e transpose, sorry, T_e transpose N transpose h T_f plus minus, so that becomes if I put whole thing into, then this becomes minus here. That is up to the first one. That is up to the first one, so I am just continuing that.

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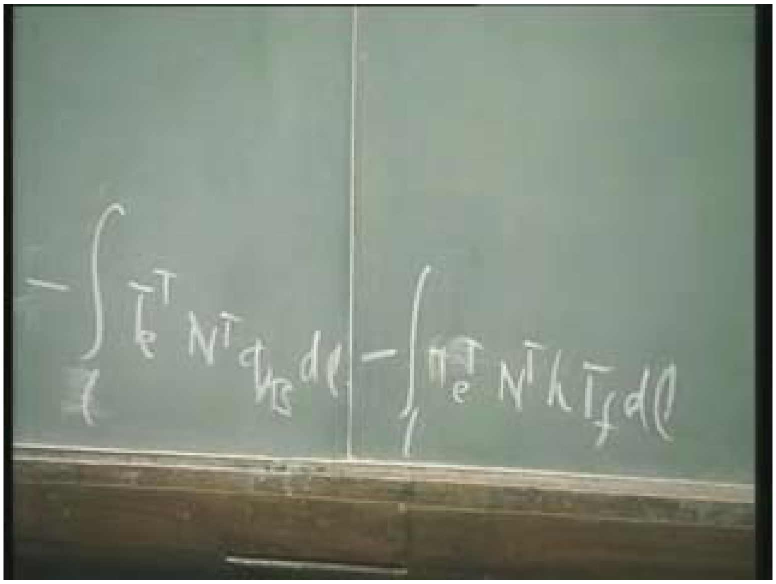

If you want, you can put that separately as minus S T_e transpose N transpose q_B; a line. Say, let us say l T_e transpose N transpose q_B say dl. Next is again T_e transpose N transpose h T_f. I know this is a very, very long expression; it just keeps going, but plus wait, wait. Yeah, minus is outside. So, I am right, minus. Next term only becomes plus; plus add this to N_e transpose (pl check) h N and so on. Just write it down, I think whole expression you can write it down. I will write down. Since all of you know how to do it, I will write down the final expression for this. What is my next step?

My next step is to, what is my next step? Differentiate it with respect to T_e. Pi, you know what is to be done? Pi I had substituted it.

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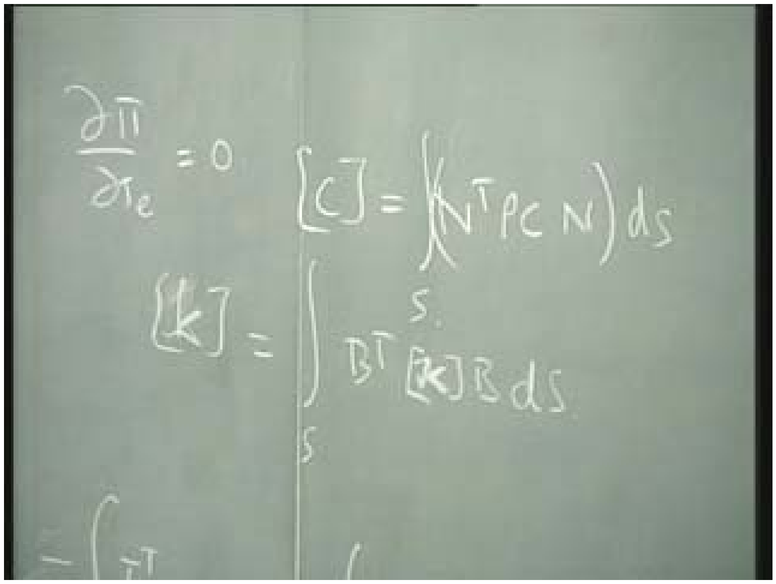

So my next step is dow pi by dow T_e is equal to zero. So, dow pi by dow T_e is equal to zero. Half T_e transpose B transpose k B T_e. Actually, this whole thing is multiplied by T_e. This becomes, you remember the same thing as half k transpose sorry u transpose ku becomes ku. So, the first term becomes B transpose k B T_e. B transpose k B T_e. Let me call a matrix K, let me call this is K is equal to integral S B transpose k B say dS. Yeah, k is this. This is Kappa, sorry, K, Kappa B transpose dS. Is that clear? First thing, let me give another name say C, let me give another name C to N transpose rho C N; N transpose rho C N dS. Another name for rho transpose, sorry, N transpose rho C N dS.

Let me call, let me give one more name. What is this? Anyway, what is this? Is it very similar to stiffness matrix? It is very similar to stiffness matrix. So, let me call as h N transpose h, where is it, T_f; let me call this as h N transpose, no, there is one more h term is here. Yeah, yeah, yeah, this term; this term, this term, now becomes, this, this, one.

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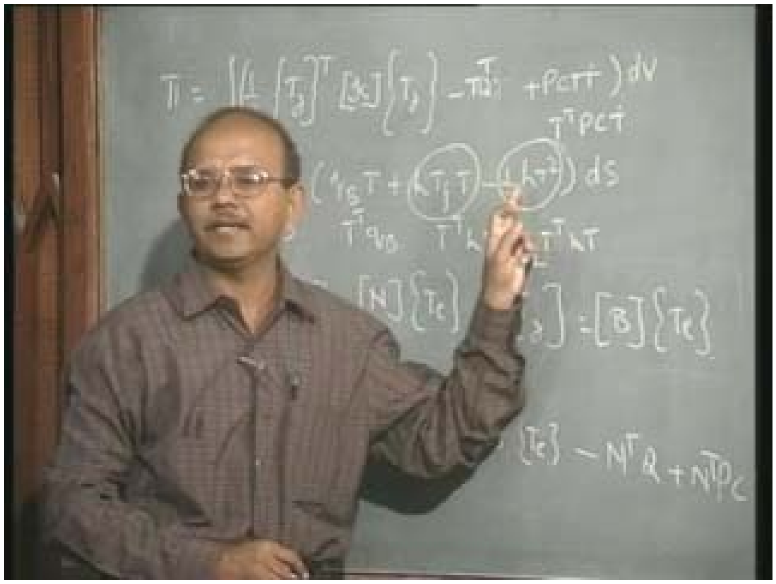

This becomes half T_e transpose N transpose h N T_e.

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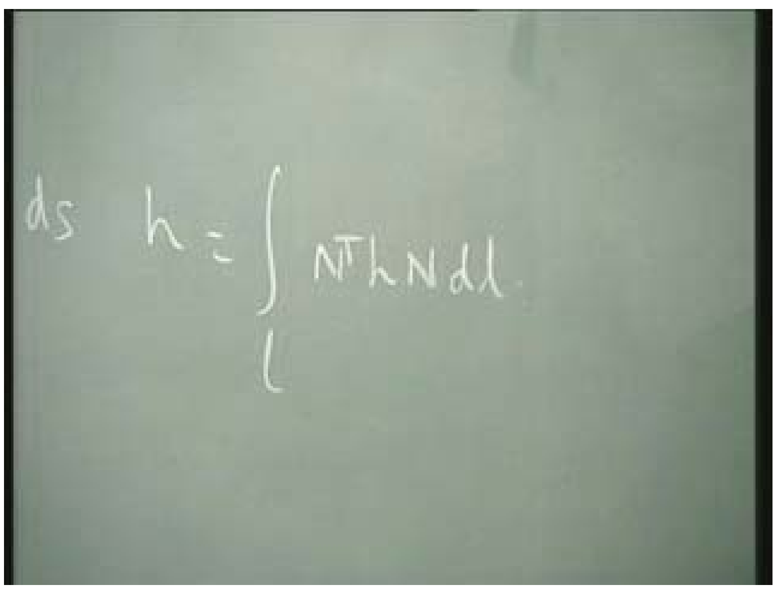

Let me call for that h is equal to integral S N transpose h N dS. No, no, yeah, dl obviously, because it is a line integral. Let me call this as say r_Q.

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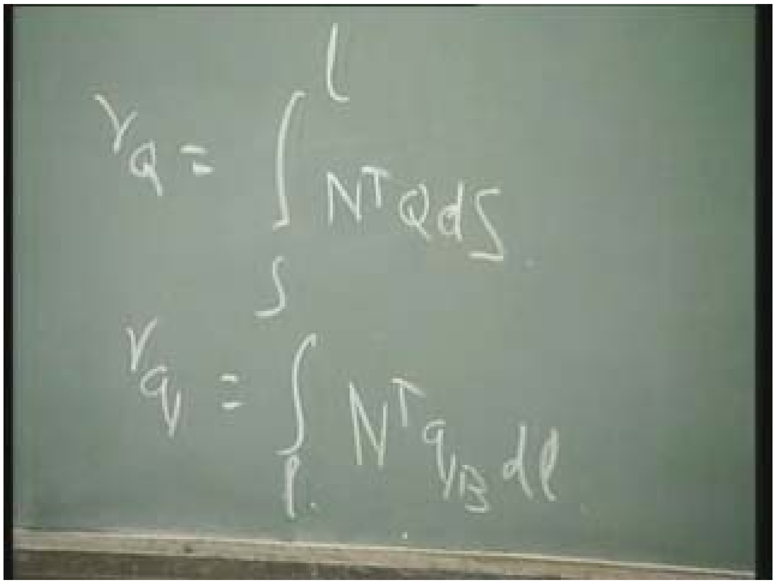

So, in other words, let me call as r_Q is equal to N transpose QdS. Note this each of them carefully. Just let me review this. k is equal to B transpose Kappa B dS, C is equal to N transpose rho C N dS, h is equal to N transpose h N dl. r_Q is equal to N transpose Q dS. Let me call I have to consume so many things; yeah, this also is there. No, no, one minute. Why I am doing this is, because later when there is a problem, whenever there is, something is zero, you can just put it as zero. It is not a problem that is why. Let me call this as say, r_q N transpose q_B dl.

Have I consumed everything? One more term? Which is the term that is there? r_h.

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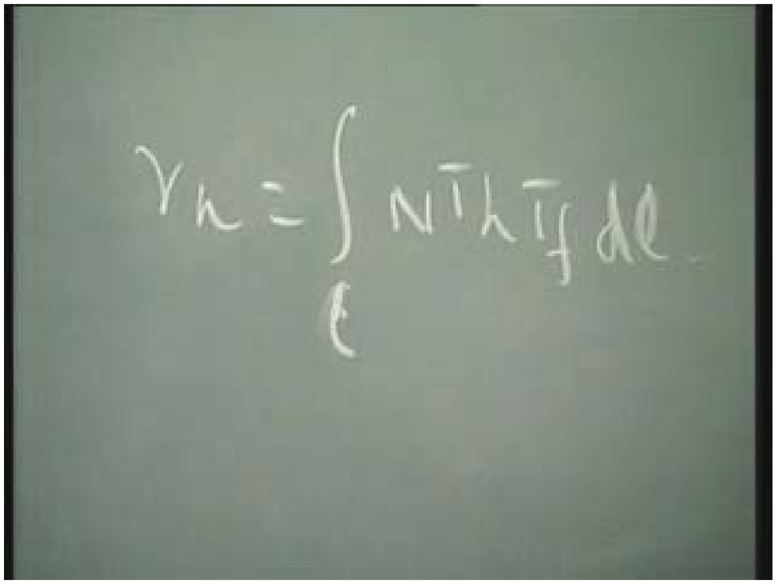

Yes, so, let me call this say r_h, line element, N transpose h T_f dl. Have a look at this for a minute, you know, all these terms, term number 1 2 3 4 5 6. Before we go further, we will interpret each of these terms from our knowledge. Obviously all these terms are going to enter my ultimate equation to solve the final heat transfer problem. In fact, that you can write it straight away. I am going to do that next step, but this matrix is something like the stiffness matrix. This matrix is actually a matrix which is related with …… T dot e and is very similar to, this is what is called as capacitance matrix, first of all note that. This is called as capacitance matrix and is very similar to in a dynamic situation a damping that is or mass matrix, sorry mass matrix that is given. In mass matrix, we write it as mx double dot in dynamics. Here, it is T dot. Second order and first order, there is a difference. That is what is called as the capacitance matrix or C matrix.

This matrix here is very similar to what we have not seen, but very similar to what we call as elastic foundation material and elastic foundation. Let us not worry about that right now. This matrix what is r_Q is a force term, something like a force matrix. It is something like a force matrix and this is the force due to internal heat generation. What can you relate this to? What can you relate this to? What? No, no, no; not internal. This is internal heat generation. I am trying to relate this to what you know already. What is it?

Body forces; very similar to body forces, yeah, gravity which is acting throughout the volume of the body, so, this is due to body forces and this is due to surface forces. This is due to surface forces and this is also very similar to the surface forces. Only thing here it is a new matrix which is for convection. Is this clear, so far this derivation, clear? Look at this once more, all these one by one; you know, term one by one. Try to assimilate these terms. N transpose rho C N dS, B transpose kappa B dS, then N transpose h N dl, N transpose Q dS, N transpose q_B dl and N transpose h T_f dl. The only difference if it is 3D is that dl will be replaced by S dS and this dS will be replaced by dv. That is it. Now, I will give you 2 minutes. You have to now get me the final equation form, how it will look like?

Differentiating this, a very straight forward differentiation and look at these expressions and I want to write down the final equation. Let me give you two minutes to write down. Just note these things and then write down the final form.

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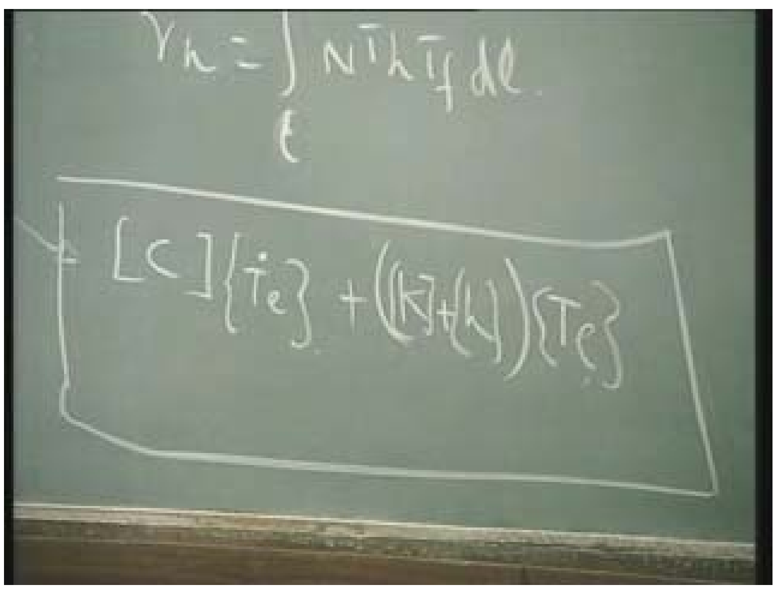

Look at these things. T dot e, let me go one by one. T dot e, so, what will be T dot e? C, will that be there? Will it be there? Yes, this corresponds to this, because in my original equation it was multiplied by T_e transpose T dot. Remember that and hence this one becomes C T dot T_e dot. Look at that dot there. What would be the other things that will be there? K; will only K be with T_e? Is there anything else which will be with T_e? h; so, I would say plus K plus h T_e. Is that right? Then, no, no, no, no, plus you have made a mistake; plus that is right. C T_e dot plus K h T_e, then all other guys will move now to the right hand side. Is that clear or if you want, you can put it left hand side and equal to zero, either way.

Now, tell me correctly with signs what are the terms that you would get from this equation?

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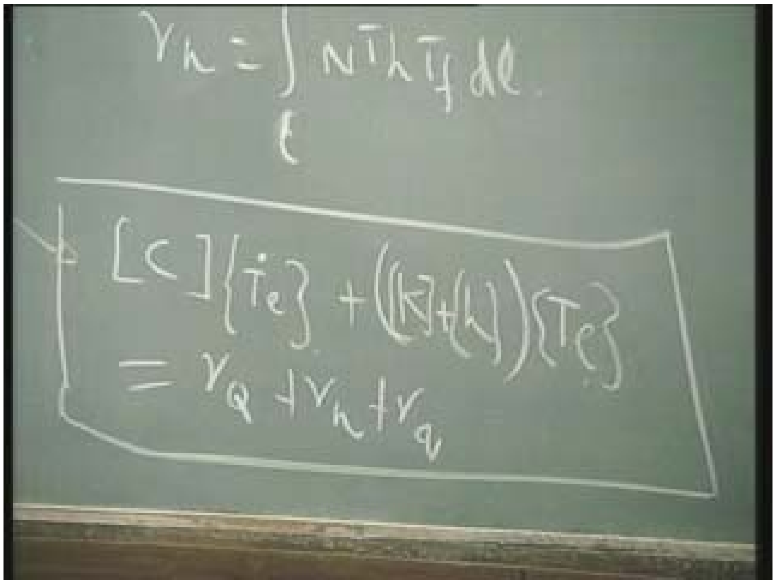

r capital Q, what is the sign that it is there? Minus here, so, when you go there it will become plus. r_Q plus, plus then T_f that term plus r_h then plus small r_q, plus small r_q. Is there any question? Is it clear how we got this equation? We started with the differential equation, we formed the functional, substituted my well known shape functions and other things, differentiated the functional with respect to T_e and ultimately wrote down, nicely collected these terms and wrote down, different matrices and ultimately got an equation of this form. Is that clear, is that clear, any question? Yeah. No, I have brought it to the right hand side. I have brought it to the right hand side. Q is considered to be positive when it flows in. I have brought it to the right hand side and hence I am writing it like this; look at your signs.

Now, the question is that what sort of problem can we solve with this? Can we solve a transient problem with this? Yes, obviously because, T dot e is there. If you want to look at the steady state problem, what do I do? I just remove the T dot e terms. Now, is that clear? How am I going to solve this problem? How am I going to solve a transient problem? Steady state problem is simple. If I remove this, then this becomes something like a K matrix. So, K T_e is equal to some r, simultaneous equation with temperature being specified at some boundary condition. The procedure is exactly the same. Temperature specification is something like a displacement being specified; the procedure of solving the problem is exactly the same as what you would have done for displacement. Is that clear?

How am I going to solve this problem, where I have T dot e? I am going to solve this with an approach called as an incremental approach. I am going to solve them using an incremental approach. What is it? It is very similar to what probably some of you would have learnt as finite difference scheme. There are number of ways of doing it. The simplest one is say T dot is replaced by say T_{n+1} minus T_n divided by delta T. What is n and n+1? Now, I am going to introduce some new terminologies which you may have to know. Is that clear?

Have a look at that expression again and I am going to introduce some things new.

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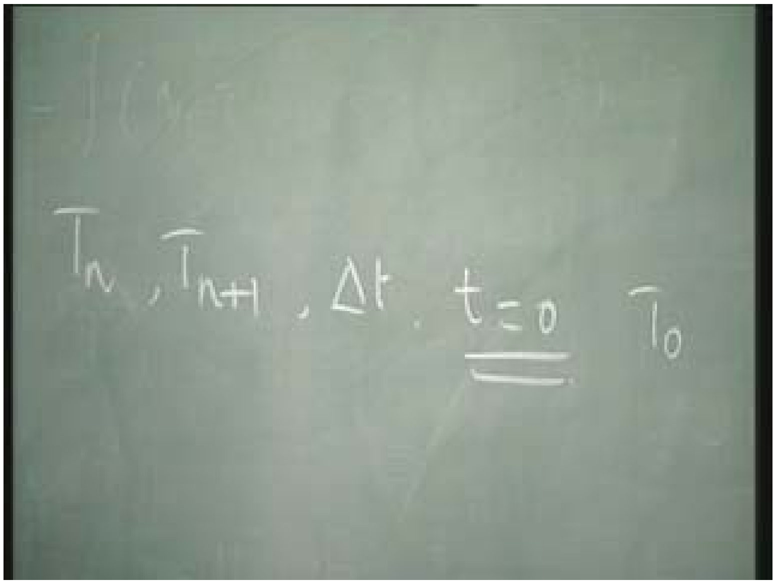

I am going to put some subscripts called T_n and T_{n+1}. I am going to introduce what is called as delta t. These are three new things which I am going to introduce; T_n, T_{n+1}, delta t. How do you solve a transient problem? We solve the problem by starting at time t is equal to zero, at time t is equal to zero and give certain initial conditions or in other words T_0 is known. This is known and that is the initial condition. Is it clear? Then what I do is I increment the time. With the results at t is equal to zero, I calculate t at t plus delta t. The number of increments that I make, that number is written as n. That number that I make is written as n, then n+1 and so on. So n+1 is actually time at t plus delta t. If n is at t, n+1 is at t plus delta t. Is that clear?

Look at this expression. Look at this expression and how do you think we can write down the final form? Have a look at that here. So, T dot e is my culprit there. I have to write it, write T dot e. There are two ways in which I can do it. There are number of ways. We will generalize the way in which we can write down T dot of n+1. If T dot is not there then, obviously, no n+1, no n, because we are in a steady state situation.

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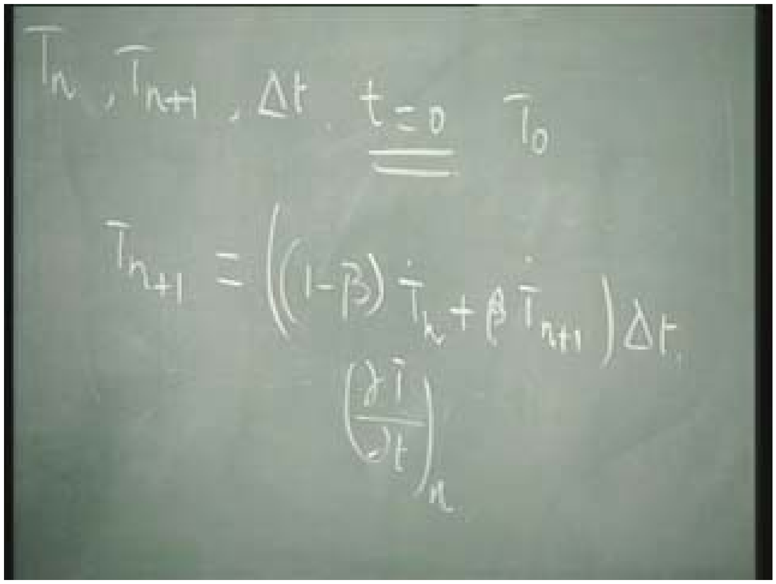

What I am going to do is I am going to write T_{n+1} is equal to 1 minus beta. What is beta? We will come to this in a minute; 1 minus beta into say T dot n plus beta into T dot n+1. The whole thing multiplied by, the whole thing multiplied by delta t. Look at this expression, look at this expression. What happens when beta is equal to zero? We come to a very simple approximation for delta t when beta is equal to zero, sorry, for T_{n+1}, when beta is equal to zero, then T_{n+1} is equal to T dot n delta t. In other words, what is that we do? We use the slope of the temperature. What is after all this T dot n? Dow t by dow small t at n; dow T by dow small t at the point n. We approximate it, the T_{n+1} by knowing the slope at n. Dow T by dow small t at n multiplied by delta T gives me T_{n+1}. In other words, T_{n+1} is explicitly determined. Knowing the previous slope, knowing the slope of the previous problem or previous step, I calculate the current T_{n+1} from that.

What happens when beta is equal to 1? What happens when beta is equal to 1?

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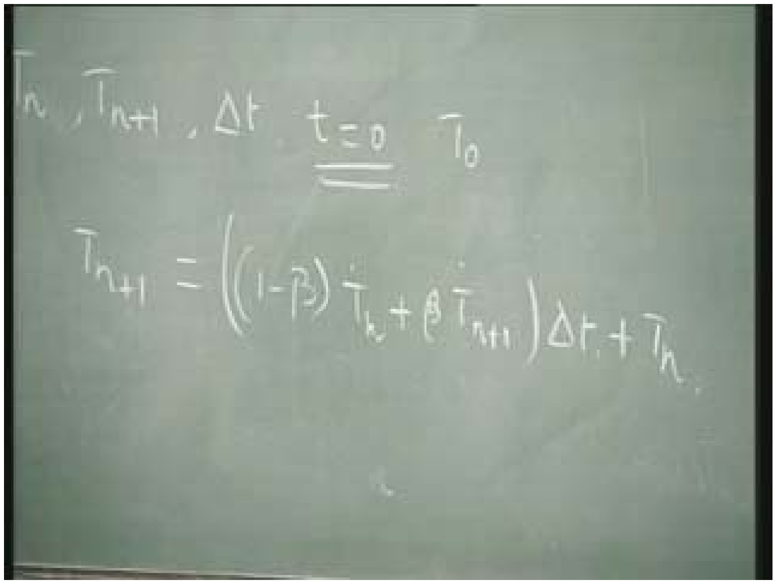

Then, look at this. T_{n+1} is equal to T dot n+1 into delta t; T dot n+1 into delta t. In other words, we calculate the temperature at n+1 using the knowledge of the slope in the current step, which means that we are in the, we have what is called as an implicit scheme, we have what is called as an implicit scheme. The whole point here is to say what slope you use. After all slope into delta t, dow T by dow small t into delta t, will give me my new T_{n+1}. The whole idea here is to find out what is the slope that I have to use.

On the other hand, I can use beta is equal to half, beta is equal is equal to half. What happens, just check up what happens when beta is equal to half? I think I made one small error. I have to add this T_n anyway, sorry. I have to add this T_n here, this because after all, yeah, T_n and that is understood because delta t by delta t into dt. So, what happens when beta is equal to half? When beta is equal to half, you take the slope to be the average between the n and n+1. Is it clear?

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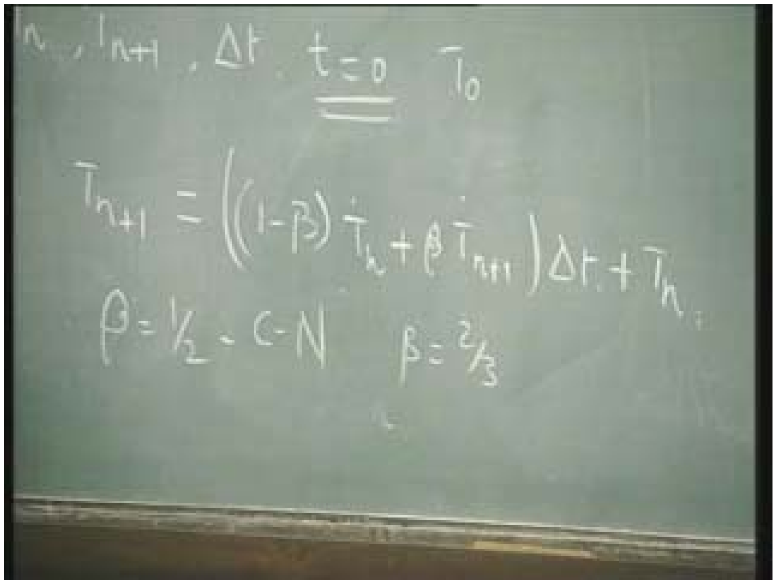

Such a scheme, where you take beta is equal to half is called as the Crank Nicolson scheme or sometimes people call this as midpoint rule because you are taking the midpoint and is the most popular among the schemes that are used to write down or approximate T_{n+1}, so that when beta is equal to half, T_{n+1} is equal to of course T_n plus half into T dot n plus half into T dot n+1. You take an average of the slope between n and n+1. Mathematical analysis has been done for these three conditions, more than three conditions. Infact, you can have beta to be 2 by 3, the weightage given can vary. This is called Galerkin scheme. When beta is equal to 2 by 3, it is called as the Galerkin scheme.

People have done lot of numerical analysis or worked on this problem and they said or they found out that the first scheme, the explicit schemes are conditionally stable. They said that these schemes are conditionally stable. All of you have heard about the terms stable, stability, conditionally stable and so on. No? Let me explain. The question that comes to our mind is that look you have defined delta t as a step which takes us through the complete solution. We can start at T is equal to zero and go up to say T is equal to 5 seconds or 10 seconds or 3 hours, whatever it is. What is the step I should take? Can I take whatever step I want? That is the question that you will have in mind? Why is that step becomes important? Please note that at every step, I have to solve the problem, solve this equation, I have to solve this equation. At every step, I have to reconvert this, solve this equation, get it into simultaneous equation and keep going.

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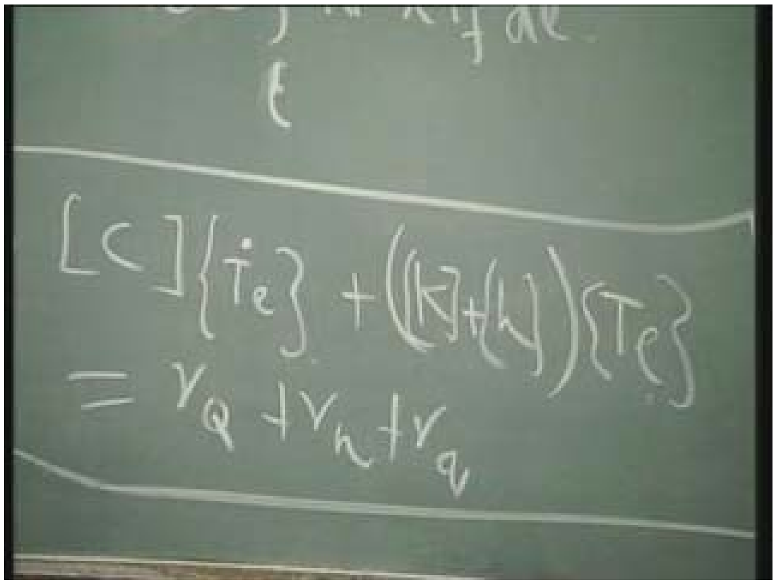

That means larger that number of steps I have, more the time I am going to solve this problem. Very clear; so, the first question that comes to your mind is why should I take 100 steps, why not I finish it in 10 steps? Immediately another intuition tells you that look, if you are going anywhere take only 10 steps. Then what suffers? Accuracy is sure to suffer. I mean it is very clear. But, apart from accuracy there is also what is called as stability, there is what is also called as stability. There is lot of theory into stability.

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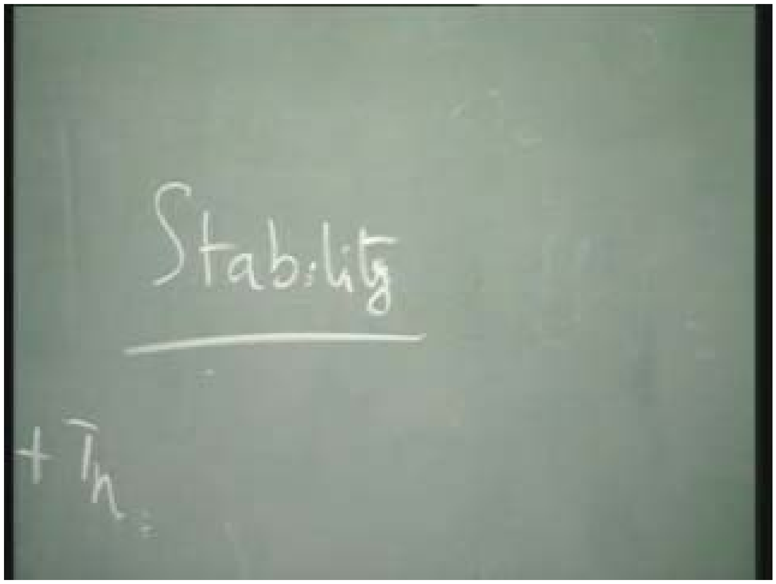

Again, we may not have time to go into this stability, but stability means unbounded oscillations; unbounded oscillations. What does it mean? What is meant by unbounded oscillations?

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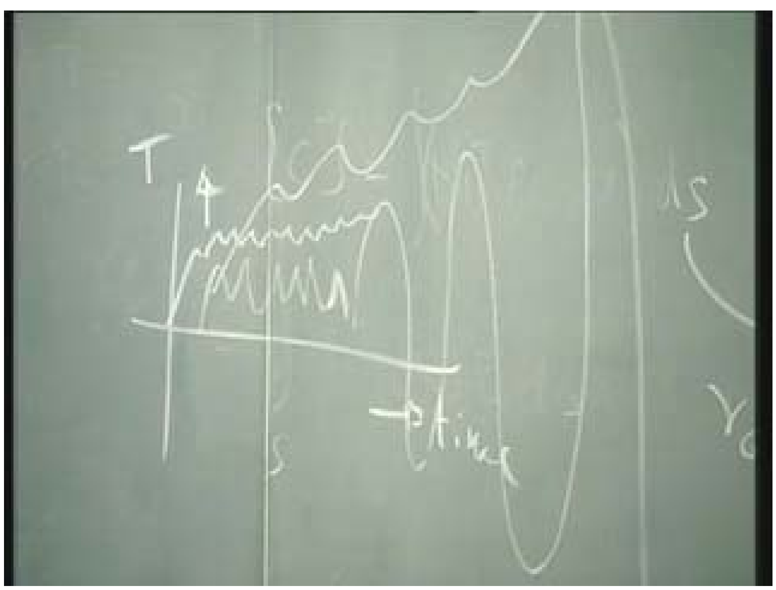

Suppose, after the solution or as I go along with respect to time I plot the temperature. I start plotting the temperature with respect to time. There can be say, some variation like this. This is what is called as oscillation. The oscillation may now grow and they become unbounded, may become unbounded or it need not actually oscillate like this, but it can oscillate like this and so on. Suppose I take one point, let us say one point in the domain of interest and start following this point with respect to time, then the solution becomes unbounded. We say at this point of time that the solution is not stable, the solution is not stable.

Please note that oscillations are different from unbounded oscillations, stability oscillations are different from stability. The solution may oscillate. If I say like that what is means is the temperature may oscillate a bit with respect to time and it can stabilize, whatever be the result. It can oscillate a bit from one step to the other step, something like this. Say for example, it can oscillate like this, as long as it does not go unbounded, does not go unbounded. In other words, what I mean by unbounded is in 10 steps it may go to infinity, 10 to the power of 32 and so on. As long as it is not unbounded, we say that the solution oscillates, but the solution is stable. Is it clear?

On the other hand, it can, the oscillations can lead to complete instability. It has been found that there is a limit for delta t; that we will discuss in the next class, what the limit is. There is a limit for delta t, when beta is less than half, when beta is less than half, for these solutions or in other words when beta is less than half, the solutions are conditionally stable; when beta is less than half the solutions are conditionally stable. In other words, I have to put a restriction on delta t. For beta greater than equal to half, note what is that beta? Beta is here. When beta is greater than equal to half, then the solution is unconditionally stable; unconditionally stable. I want you to distinguish this clearly, because students get confused between the two. These two are different from accuracy. They are different from accuracy. It has been found that see, one of the, before even I go further, one of the things that immediately you would conclude is that, then I will take all the time beta is equal to 1.

I am away from that line that you have drawn on beta is equal to half. I will go to beta is equal to 1. Why should I go to beta is equal to half? Infact, you told that beta is equal to half is very popular and called this as a Crank Nicolson scheme. Why should I go there? Why not I go to beta is equal to 1? But, when you go to beta is equal to 1, does anyone know about this, probably you studied in numerical analysis? No? I am giving you only the results because this is a topic by itself. When beta is equal to 1, the scheme is only first order accurate. It is accurate only up to delta T, first order accurate. That is both beta is equal to zero and beta is equal to 1, these schemes are only first order accurate. That means that the accuracy suffers. Though you can get unstable, I mean, solution without problems of stability, though the stability problem can be solved, accuracy is a different story altogether and the solution suffers and hence the solution is not accurate. On the other hand, when beta is equal to half, you get a second order accurate solution.

I will summarize this again in the next class, but I want you to note this down. Look at beta is equal to half. It is, one, unconditionally stable; whatever be the delta t, you are going to get a nice result. The accuracy increases, but look at that but. But, oscillations are not eliminated, oscillations are not eliminated. In other words, especially when the transients are large, Crank Nicolson schemes start oscillating. This has lot of practical significance. For example, many people who do casting simulation, simulation of a casting process, after all what is the simulation of casting process? It is heat transfer, transient heat transfer problem. How do we deal with it? We will look at it after I finish this thing. They use Crank Nicolson scheme, but when they use Crank Nicolson scheme, they will initially find that, especially initially they will find that, the temperature of the mould which they would have probably put as 35 degrees would go to 28 degrees. They will be wondering whether they made a mistake. Immediately when the metal is poured initial condition t is equal to zero, metal has very high temperature, start the stimulation; you will suddenly find that the 35 degree mould goes to 28 degrees. Then you will immediately think that there is some drastic problem that you have done, I mean, drastic mistake you have committed, there is a problem and so on.

No; infact, I know a student who wasted nearly 3 weeks to 4 weeks looking at the result all the time, looking at the code, finding out if there is some mistake that he has done and not finding out any error in his code, breaking his head. Then, I asked him what the problem is and he said my temperature in the mould goes down. Then I said this is an oscillation. He said Crank Nicolson scheme is the best scheme I studied, so, why is that giving, why is it that this scheme gives rise to oscillations? Please note that this is the problem, because people get confused with stability, with oscillation, with accuracy. All of them are different things. Crank Nicolson scheme, especially initially when you start would result in oscillations, but will give you a very nice result, accurate result if you just smoothen that out, dampen that out, then you will get a nice result, but there will be an oscillation.

We will now continue. What we have to do is very simple. We have to substitute this into my original equation and develop a scheme to carry on with the solution. We will do that in the next class.