Introduction to Finite Element Method
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Lecture - 23

Before we proceed with our quadrilateral, quadratic elements, let us just go and look at what we did before that. I had just left it vague, this triangular element, the 6 noded triangular element. I had left it vague deliberately, because I wanted you to look at it and see whether you can develop it and now I know that many of you have not done it, because you have to put in some effort and may be still things have not crystallized. So, let us go back and have a look at as to how to develop this D_N matrix. I know that the problem is in the development of D_N, because once you develop D_N, I am sure that the rest of it follows quite easily. Is that clear? I just want to go back and tell you as to how this can be solved.

Let us see what the difficulties are. Initially we had, I hope all of you are now clear about psi_1 psi_2 psi_3 coordinate system. Is it clear? Is there any doubts? Yes, is there any question on that? You know psi_1 psi_2 psi_3 coordinate system.

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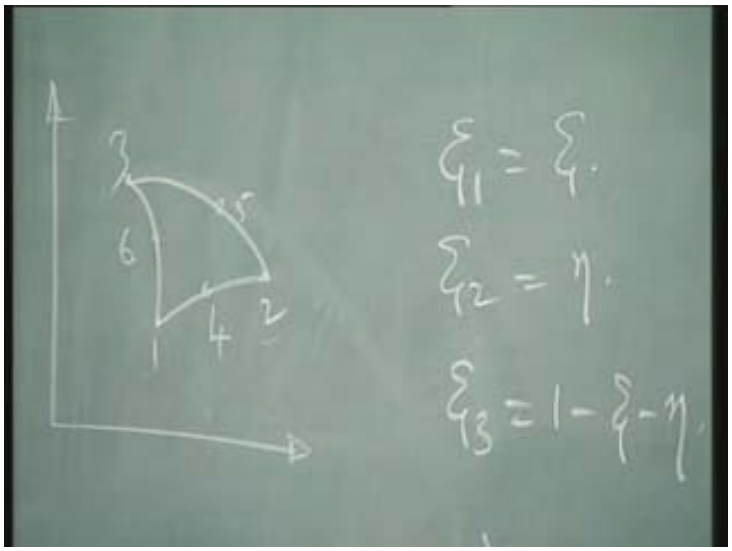

That is the first thing I did and then this was followed by, I writing down, N_1 N_2 N_3 so on in terms of psi_1 psi_2 psi_3 and so on.

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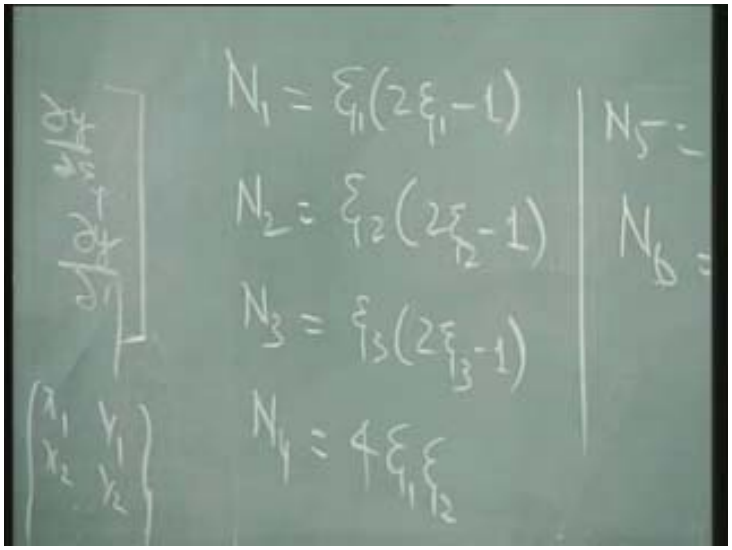

They are quadratic in nature. I have written down in such a fashion that our shape functions are satisfied and so on, I mean properties are satisfied and so on.

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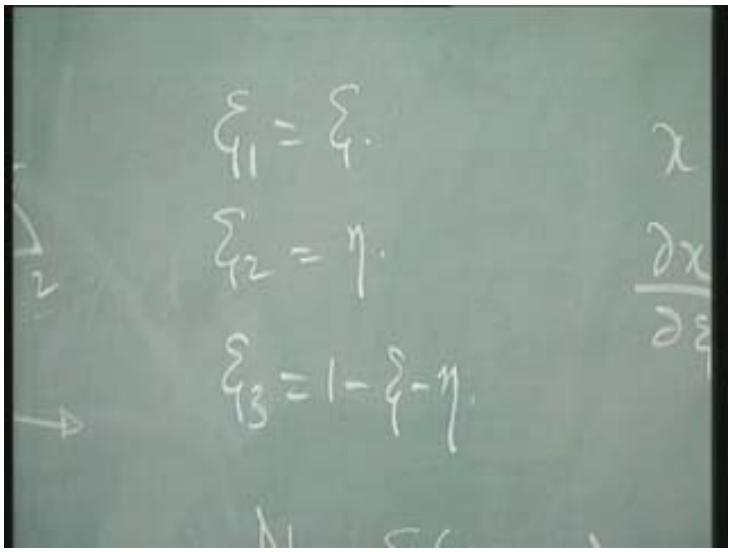

The next thing I did was to define psi eta with which we are familiar with, which is easier to do with the natural coordinate system, because we also pointed out that psi_1 psi_2 psi_3 are not independent and psi_1 plus psi_2 plus psi_3 is equal to 1 and hence we said that let me call psi_1 as psi, psi_2 as eta and psi_3 is equal to, obviously, 1 minus psi minus eta. You can retain psi_1 psi_2 and 1 minus psi_1 minus psi_2 and call psi_1 psi_2 as the

psi eta. I did this because psi eta are the two coordinate system, so, I wanted to develop it in the same language. That is all, nothing more than that. Because, I have to introduce this they are not independent, they are dependent; that is what I did. After doing this, I wanted to calculate the J matrix.

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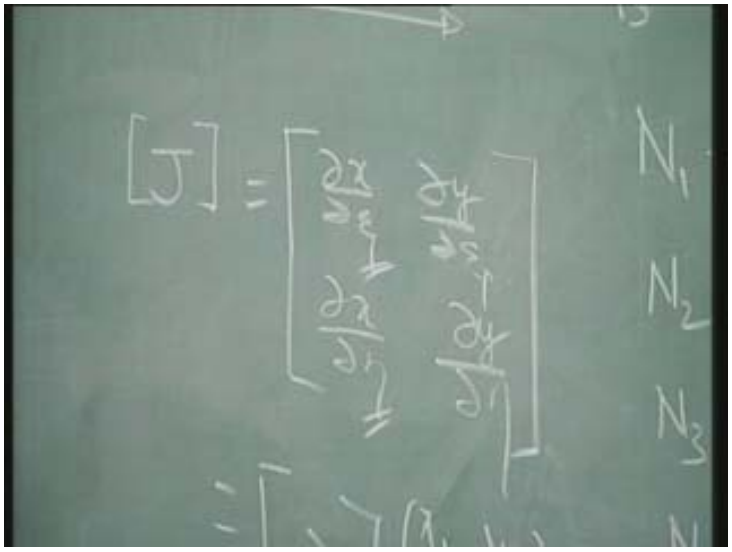

What is that I need? I need to substitute for x in this equation. What is x? $N_i x_i$; is that clear? Now, look at these things. I have put psi and eta. I think the major difficultly, if I understand from you, is how to calculate it in psi and eta, because I have written all the N‘s in terms of psi_1 psi_2 and psi_3. This is a very, very simple thing to do. Is that clear? Is that the question? It is very clear because psi_1 is a function of psi.

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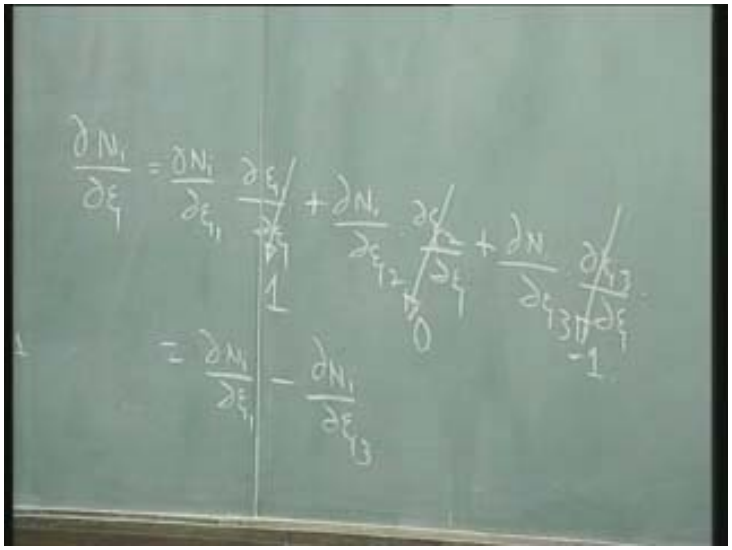

If I want to write down, say, dow N_i by dow psi, what I do is very simple. This is equal to dow N_i by dow psi_1 into dow psi_1 by dow psi plus dow N_i by dow psi_2 into dow psi_2 by dow psi plus dow N_i by dow psi_3 into dow psi_3 by dow psi. Is that clear? That is it that you have. Now, how do I calculate psi_1 by dow psi_1 by dow psi. I have the relationships here. Dow psi_1 by dow psi, this becomes 1, dow psi_2 by dow psi becomes zero and dow psi_3 by dow psi becomes minus 1.

Is it clear? Essentially, dow N_i by dow psi becomes now, dow N_i by dow psi_1 minus dow N_i by dow psi_3. Is it clear? Once I have this, my entries in D_N become very straight forward. There are two ways in which D_N can be written, again you know, I am not confusing, but I am trying to say that these are very, very logical things. There is nothing very difficult about it. Say for example, I can write down the same J as D_N; after all what are these?

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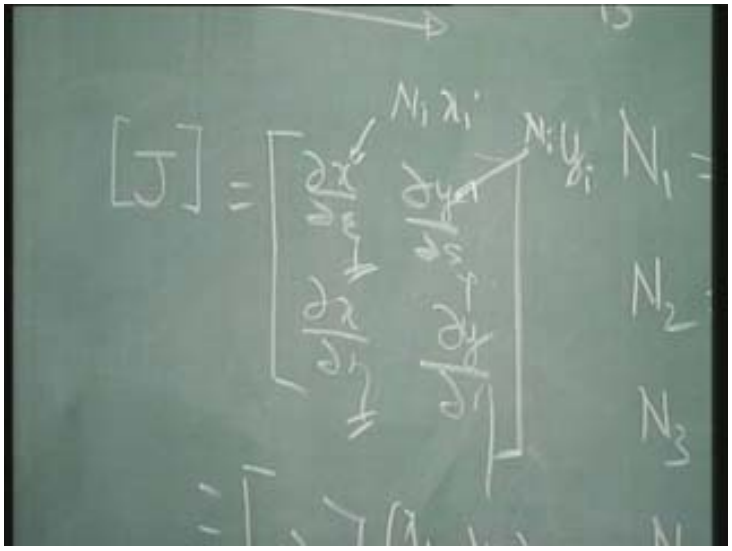

This is, I am going to substitute here for $N_i x_i$, here $N_i y_i$, I am going to substitute these two things there $N_i x_i$ and $N_i y_i$. I can look at this x_i's and y_i's. What are they? x_1 x_2 x_3, y_1 y_2 y_3, as a matrix, I mean, x_1 x_2 x_3 x_4 x_5 x_6, if you want to complete it, as a 6 by 2 matrix as well. I can write it down like that and then what is the size of D_N?

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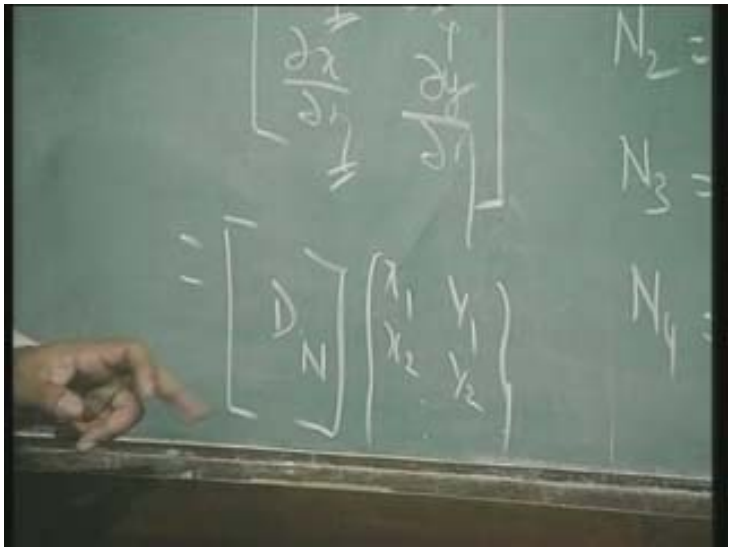

2, no; 2 by 6, so, 2 by 6 into 6 by 2 will get the 2 by 2 matrix. See, these are all the ways. It is very important that you know how you can manipulate with it. Especially, when you want to develop a code, this kind of things is very useful. You can write

this down as 12 entries. x_1y_1 x_2y_2 x_3y_3 x_4y_4 and so on, as a vector you can write down. Then D_N will correspondingly change. That will become a confusion; that would not be very easy, whereas here it is easier to write. Is that clear? So, 2 by 6, 6 by 2 will give you a neat ...., then you have to calculate differently, separately, then add it up and all that kind of things you have to do.

Having done this, I want you to tell me what D_N is? Have this in mind and let us see how many of you are able to calculate D_N. Can someone calculate D_N for me and tell me? Have these things, now calculate D_N. Is there any question? Let me stop here for a minute, because I think one or two of you said that we need some clarification. Is there any other question, which is there? Yeah, any question? Is it clear till now, whatever I am doing? Then in that case, you should be able to calculate D_N. Let us see whether you are able to calculate D_N. Yeah, have a look at these things, all these things should be in your notes, so just calculate them.

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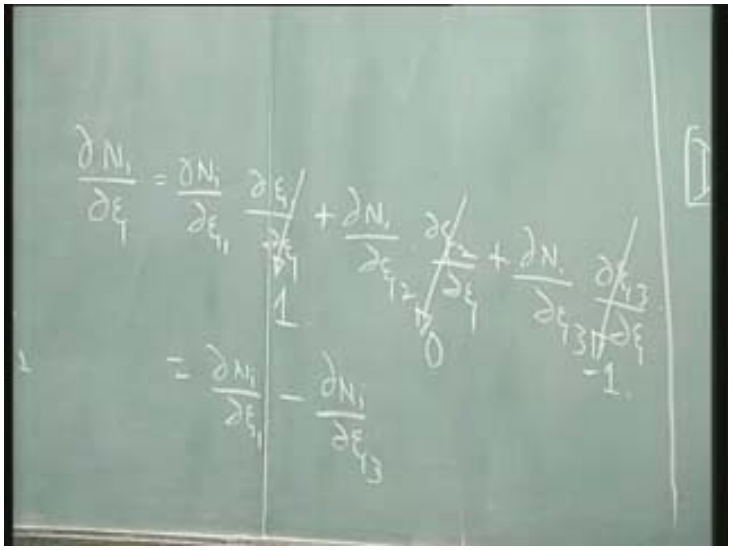

What is the first entry? Dow N_1 by dow psi; so, what is dow N_1 by dow psi? Look at this; one minute, one minute. DowN_1 by dow psi is dow N_1 by dow psi_1 minus dow N_1 by dow psi_3. The second term goes off, I mean for this N_1. So, that is same as this.

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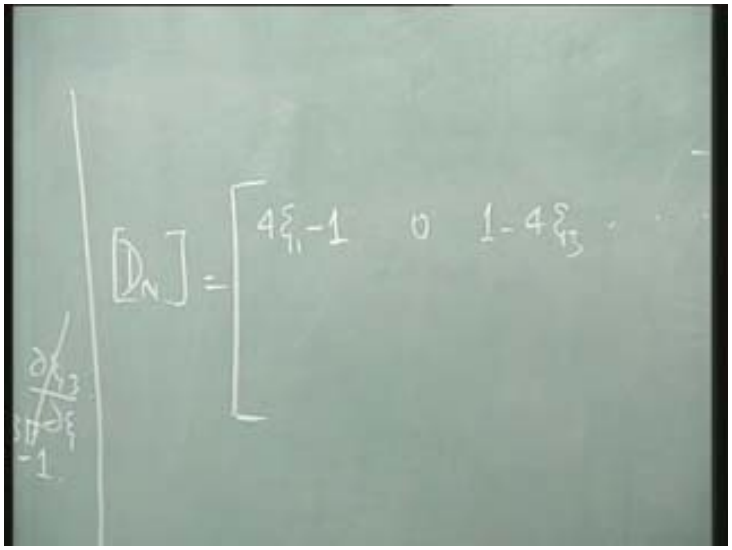

So, that is equal to 4 psi_1 minus 1. That is all, very simple. Now, what is your next entry? Zero; Fantastic. What is your third entry? 1 minus 4 psi_3, 1 minus 4 psi_3 and so on. I am sure, now you will be able to fill it up. Here again zero and so on and here again you have to put zero 4 psi to just fill it up. Once you do this D_N, then the procedure is exactly the same. You know, the procedure of doing it is exactly the same as what we did before. Is it clear? In other words, what is that we do? Now, we find out psi_1 psi_2 psi. What happens here? Now, this becomes a function of psi_1 psi_2 psi_3 and so on. What is that you do? You substitute this in k and then do the integration, area integral straight away. When you do that area integral, the procedure is exactly similar to what we did before. We have some sampling points, we have some weights. At these points, we have some weights. So calculate this B transpose D B J, what we have there, at those sampling points, multiply by weights.

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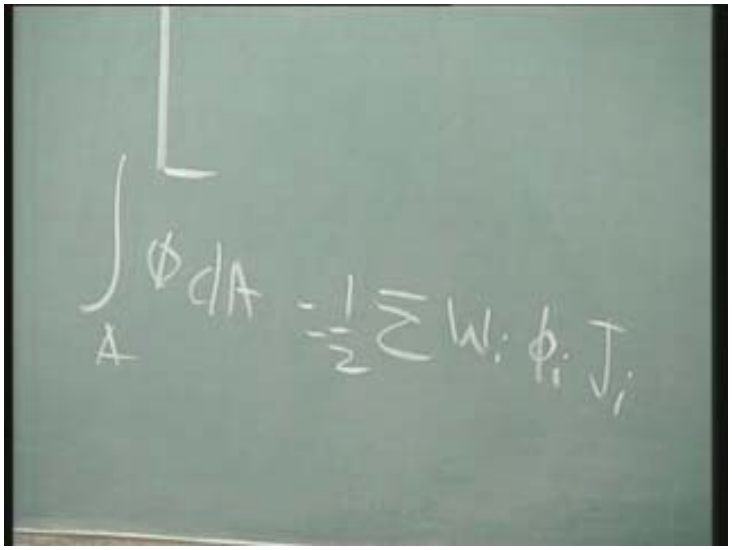

Suppose I write that, say for example, phi d A that is equal to sigma W_i phi_i J_i. Many text books, what they do is to put half due to certain reasons. Let us not get into that and then give the weightages. That is because of the area calculation in the natural coordinates, but nevertheless, this is how many textbooks also write. But, if this half is not there, then it should be in the weights. This half, many people put this half and write it like this. You calculate whatever is there in this phi, like what we did for the quadrilateral elements same fashion, calculate it get the stiffnesses. It is not, Please note the difference here; there for a two dimensional element, we did two summations i and j, 2 into 2, but in this case the quadrature points are very well given and as I told you refer to any of the standard text books on these issues. That is an exercise. That is one of the assignments that I give you. Please go and have a look at the chart of a book, then you will know how this is calculated. Is this clear now, any question, how all the steps are done?

If it is clear, let us get back to our higher order elements with which we started. I am going to go into details of certain very interesting things today, very practical things as well as very important things. Before we proceed, these practical and important things are based on the integration rules. Remember, we had already talked about integration rule, just touched upon it; there are lots of very interesting and important things that happen in the integration rule. But before we go further, let us look at that shape functions once more. Let us look at the shape functions once more.

As someone pointed out in the last class, see, after I put down, it became simpler for you to write it down. But before you put down, you are hard-pressed to write down the equation. That was the very nice comment which was given in last class, but now, I will tell you that you can write down yourself. How do you write down? Let us see.

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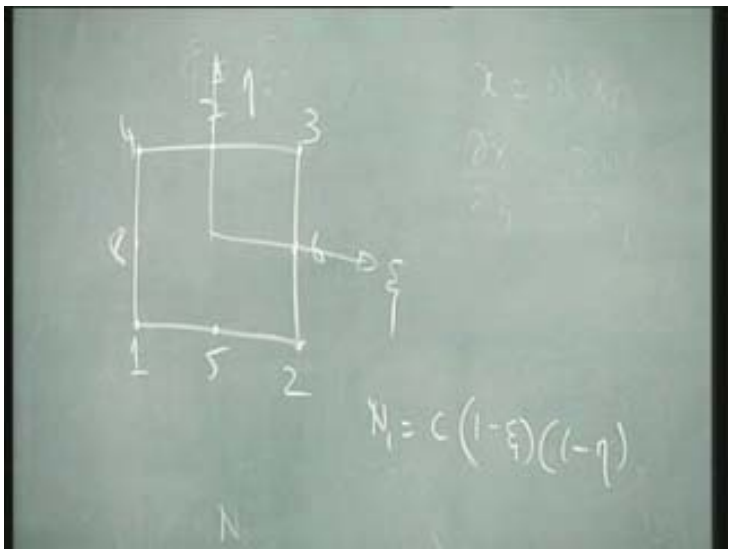

I am sure many of you would write down now the shape functions for an 8 noded element. Why am I asking you to write down, because the whole idea is once you write down you become very familiar with the subject as to how things are written. In fact you can develop a code. That is the ultimate in finite element analysis. Let us now look at 8 noded elements. 1 2 3 4, look at the numbering. It is always done like this; psi eta coordinate system. Remember that we developed the shape functions for our 4 noded element by looking at what? There are 3 methods which we did. Inspection, then the Lagrange interpolation and we did one more thing also. Line; very good, by looking at the equation of the line and seeing whether that particular point falls on that line or not.

Get back to that, just have a look at it. For example, we said that Forget for a moment 5 6 7 8. Just say that there is only 1 2 3 4. We said that I have to, when I write the N_1 here, I have to eliminate those two lines here. What did we do? We said that N_1 should be a function of some C into 1 minus psi into 1 minus eta, because 1 psi is equal to 1 and eta is equal to 1 are the equations of these two lines, Once I put it,

because this node does not lie in these two lines. So, once I put it, automatically when I substitute psi's and eta's for these nodes, since they involve the equation of the line in which they sit, obviously, it will go to zero. …. I say that I hope all of you understood it? Very nice.

Can you use the same method to develop shape functions for this element? I will give you one minute to think about it. Fantastic. I have eliminated by writing this, by writing it like this, first of all I have, suppose I write it like this. I have eliminated the nodes. What are the nodes that I have eliminated or in a sense, I have taken care of, within brackets taken care of, or within quotes; 2 6 3 7 4, I have taken care of them, but I have not taken care of 5 and 8. I should multiply a line, an equation of a line which passes through 5 and 8. Is that clear?

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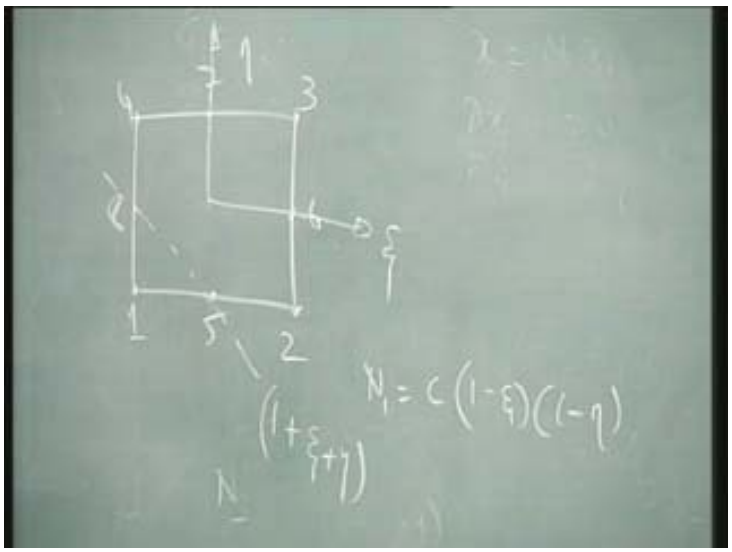

Just have a look at that and tell me how do you now write that term, first equation of this line? Yeah, correct in terms of psi and eta; how do you Psi is equal to 1, I have brought this, this side. So, psi where do they cut? Your A is what? Minus 1, B is minus 1 so, x by A plus y by B is equal to 1. What does that equation become? 1 plus psi plus eta; fantastic, so, 1 plus psi plus eta. What do I do now, with this?

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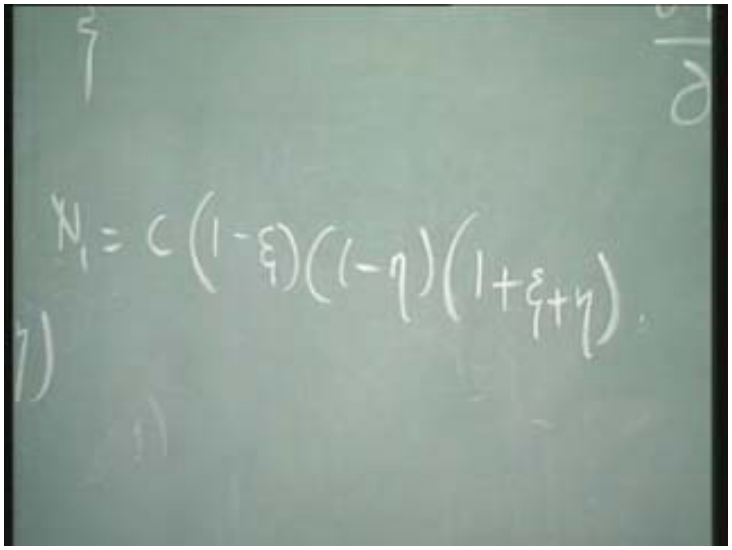

I multiply it that here, 1 plus psi plus eta; very good. Now tell me, by using our regular properties of shape function how or what would be the value of C?

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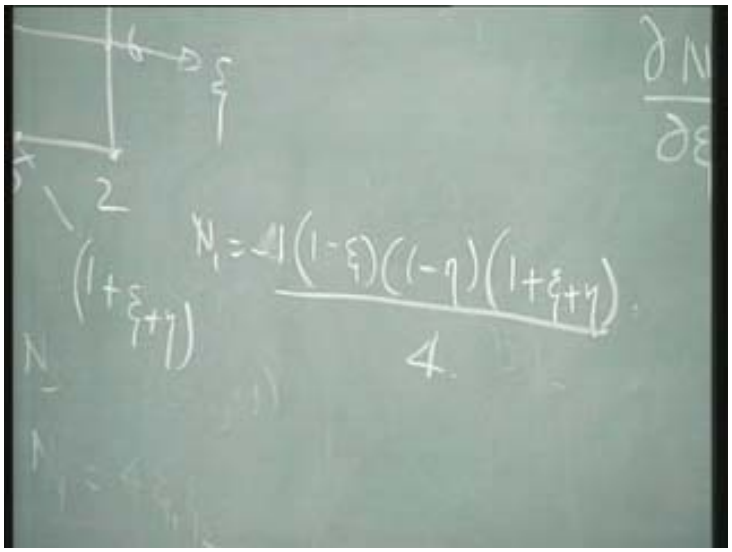

Minus 1 by 4; beautiful. Minus 1 by 4, you would notice when you multiply that this is exactly similar to what I did in the previous class. You can do that in very simple fashion. You write down now N_1. Is it clear, any question? What I did was to write down all the equations of all the lines through, I mean, for other nodes; fine and so

calculate and then see. I can do the same thing with N_2 N_3 N_4. The only thing I should know is equation of this line, that line, that line.

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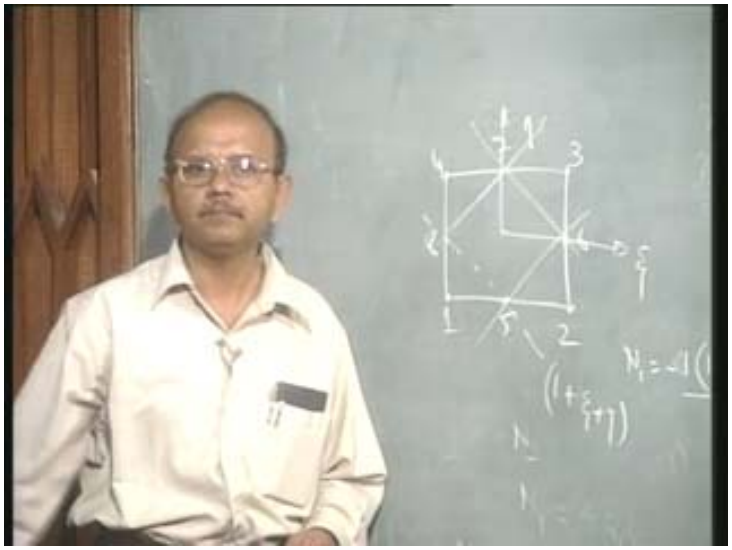

Once I know equations of these three, other three lines I can calculate N's; N_2 N_3 and N_4. Let us see how I am going to write down N_5. Let us see now, any question? 2 3 4 is clear N_5, how am I going now to write down N_5?

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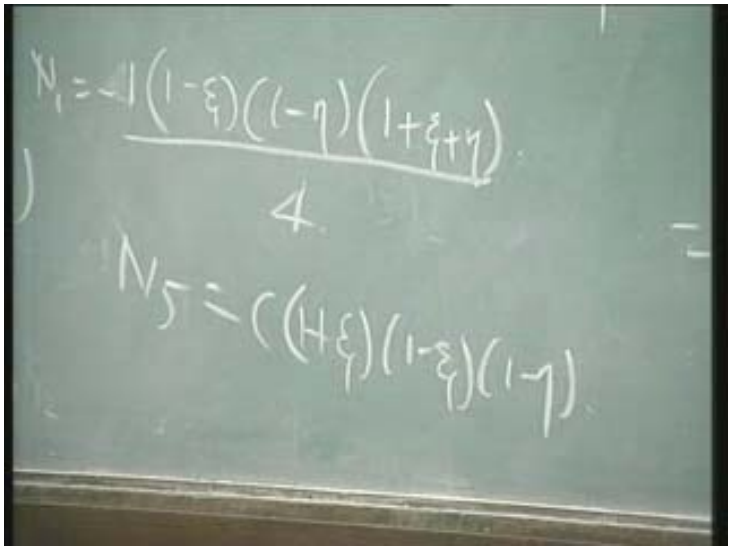

Yeah, note this N_5. Come again, enter three lines; beautiful. So, that is C into what are the equations of those lines? 1 plus psi into 1 minus psi 1 minus eta.

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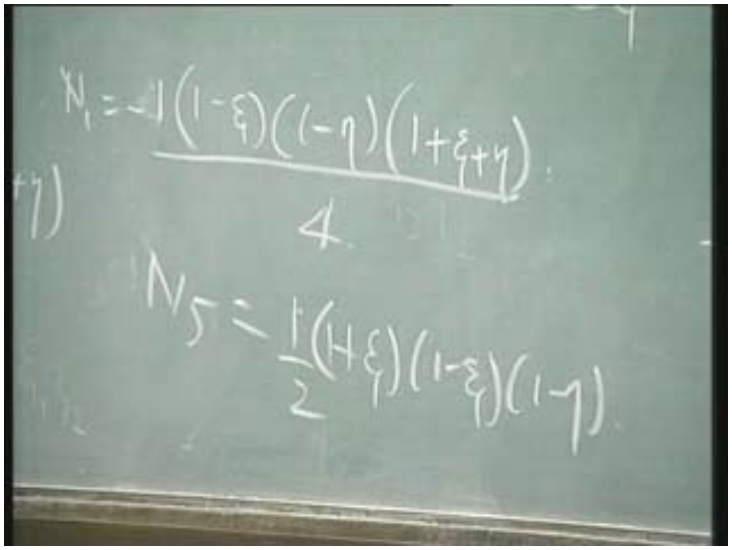

Remember, what is C? Half. Get back and see what we did in the previous class for N_5? You will see that that is what you get. This is a simple method to understand also how to develop shape functions. So, N_1 to ……. Now, I am leaving the rest of it for this higher order element, as an exercise, calculation of D_N and so on. We can later in a class, may be in a tutorial, we will do that, but I would like you to complete the picture. I would like you to complete it. If there is any question, you are not able to do it, tell me in the next class or after that.

Now, we will go, we will move to an important topic of numerical integration. I hope all of you remember numerical integration, what we did before and one of the important rules, numerical integration had existed, you know independent of finite element. We call this as Gauss quadrature. We also said that there are number of ways in which numerical integration can be done. Newton codes is one such another formula and in fact Newton codes is used in certain shell elements, because of certain advantages. But extensively Gauss quadrature rules are used in finite element analysis. We saw that and also we said that this is a rule to integrate, for example, a polynomial and it is possible to arrive at exact integration rule. What is it? 2n minus 1 integration points would integrate a polynomial of degree n.

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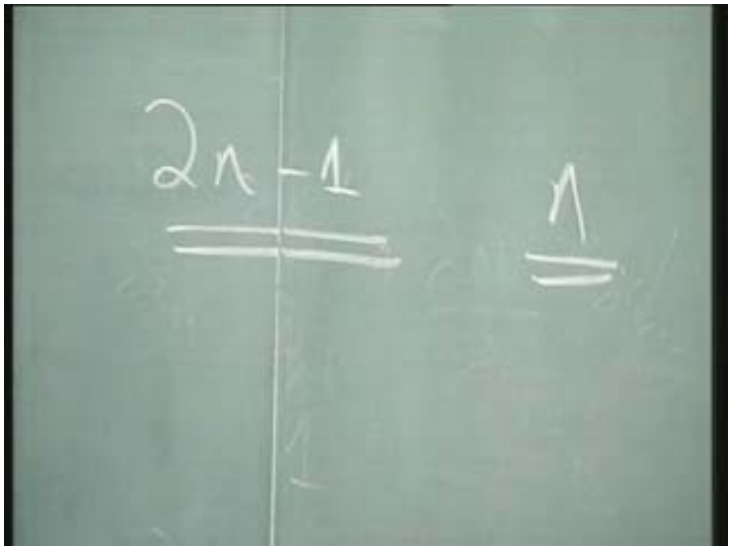

This, may be you know it already that 2n minus 1 integration points would integrate a polynomial of n. This is called, this is what we call as, exact integration. This is what is called exact integration, but there lots of peculiarities to it or lot of things are plane.

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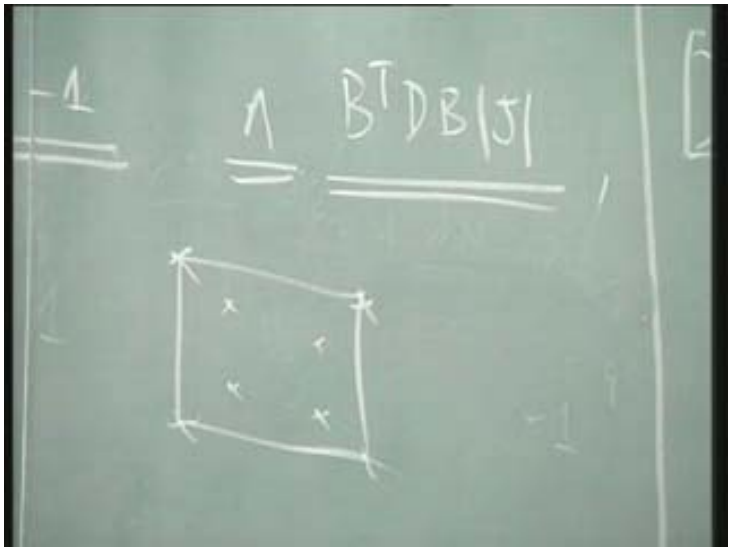

For example, if you look at a nice element like this, remember that the powers, the polynomials of psi and eta may be present in, there are, what are the terms that are involved in k? B transpose DB J d psi d eta; B transpose D B det J, determinant of J. These are the terms that are involved. Psi and eta can sit in these guys. Certain times,

J may become a constant throughout the element. Some of the nice things that nice elements, it may become constant, but certain times it will not be a constant. You have to know what exists in each of them, so that you will know what the order of the polynomial is and then use the exact or the correct integration rules.

For example, this element I can tell you that the usual 2 by 2 rule is used. The rule that I am going to use is a 2 by 2 rule. On the other hand, if I use an 8 noded element, usually a 3 by 3 rule is used. There are 3 Gauss points on one direction, three on the other side, so, 3 by 3. Totally there are how many Gauss points? 3 by 3; 2 by 2 is 4, 3 by 3, 9 Gauss points are there. We will look at two things. One is what happens when we do not use 2 by 2 Gauss points, say for example, this problem? What happens or in other words, let us say that the recommended integration rule is 2 by 2 and we use what is called as reduced integration and we say for example, use an integration rule only at the center, 1 by 1 integration rule.

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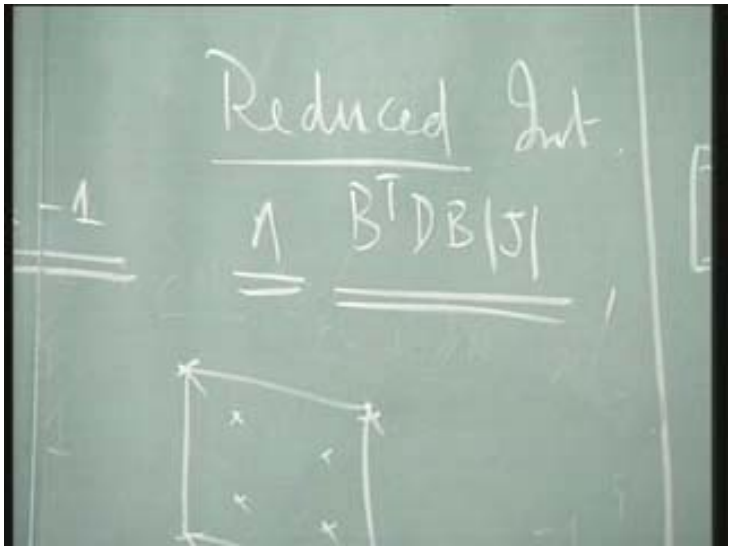

Let us say that we are using 1 by 1 rule instated of 2 by 2 rule. When I use this kind of rule from 2 by 2 and when I reduce it to 1 by 1, then I am indulging in what is called as reduced integration procedure, reduced integration. When I use reduced integration, what is your comment immediately? What can be your comment? You are doing something totally wrong; you are not following the procedure and if you are doing something totally wrong. But there are very interesting things that happen.

What are the interesting things that happen? When I use a reduced integration point, the first thing I immediately gain, forget about anything else, what is that I gain immediately or why am I looking at first of all one integration point instead of 4 integration points?

One thing I immediately gain. Computer time, cost, period, first thing, because I need to now calculate it only at 1 point. Number 2, what is the other thing that happens? That is a difficult question to answer. For that we have to understand finite element, whatever I have done so far, in a proper prospective. After all in finite element what is that we did? We broke down an infinite degrees freedom into a finite set of degree of freedom. In that process what did we do? We stiffened the structure. We said that look, you cannot have all your infinite degrees of freedom, you can have only finite degrees of freedom. The structure was stiffened.

Interestingly, when I go down from 2 by 2 integration point to a single integration point, the structure or the stiffness softens; the stiffness softens. On one hand it is stiffened by discretization, on the other hand when I go to 1 by 1 rule that error is on one direction. I introduce another error in another direction, the structure softens. It so happens that this error, though I know it is an error, tries to compensate for another error which I have introduced due to discretization and hence ironically gives better results many times, with a warning; with a warning ironically gives better results.

In other words, what is that you are doing? What is that you are gaining? What is it that is happening when I go to 1 by 1 rule? One, I am reducing the cost; two, I am introducing an error in integration and three, I am getting better finite element results. Basically this is because, certain higher order terms are not integrated at lower or they go to zero at lower integration points. Some complex deformations that are introduced by higher order terms are not present when I integrate it at lower integration point, because these guys are meant to integrate lower polynomials.

If you now look at most of the softwares, they use reduced integration to impress you, because times are very nice, very good. But, do you always get good results? Let us look at what happens, small output? What happens when I use reduced integration?

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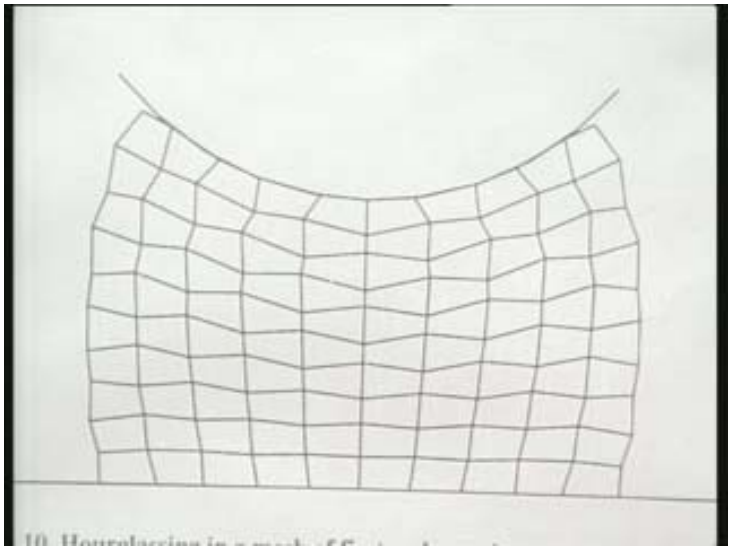

Let us look at this chart and see how exactly we get the result. Look at the result. This is a problem, where you have say elements that are defined here and a contact that is the contact with a rigid body. This problem is non linear problem with contact and so on. Let us forget about that problem for a minute about the contact and other niceties in this, that we will come to later, but look at the mesh now, after deformation. What is that you see in that mesh? The mesh is very jagged. This problem has been done using what we call as reduced integration and what you get here, this jagged mesh, is what is called as hourglassing. I will explain hourglassing in a minute, but I want you to look at the result and recognize hourglassing. In fact, many students have come and told me, sir, I am getting result like this, some jagged thing. What is happening?

First off all, I want to recognize that if you get a jagged front like this, your mesh is undergoing what is called as hourglassing. I will explain that in a minute. Let us look at a proper result in the next chart.

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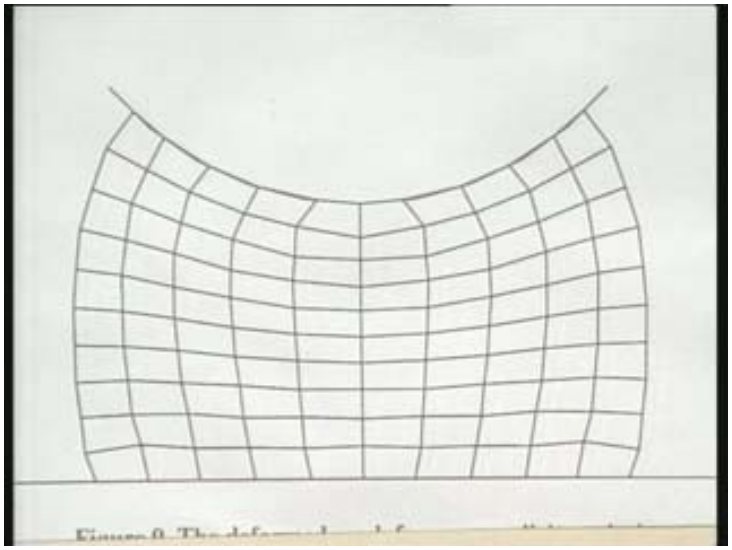

That is actually the proper result. The mesh is smoother with a proper integration and hence, we also land up in trouble by using reduced integration ones. In other words. there are two issues here. One is that, we have good results, better results with one point integration, but we also introduce problems of hourglassing, problems of what we call as hourglassing.

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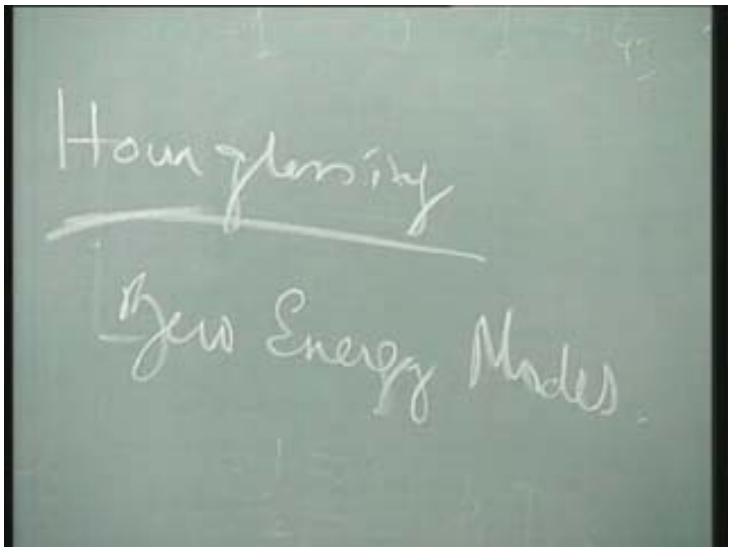

The hourglassing, it is actually one word. I should not write it as two words. Hourglassing is known by different names. Sometimes people call this as spurious

energy modes or zero energy modes, people sometimes call them as rank deficiency and so on. What exactly happens or why do we call it by these names? Why do we call this as zero energy mode? We call this as first off all hourglassing, basically because if you had seen that chart, you will see that the mesh, say which was like this, sometimes become, as if it becomes something like this.

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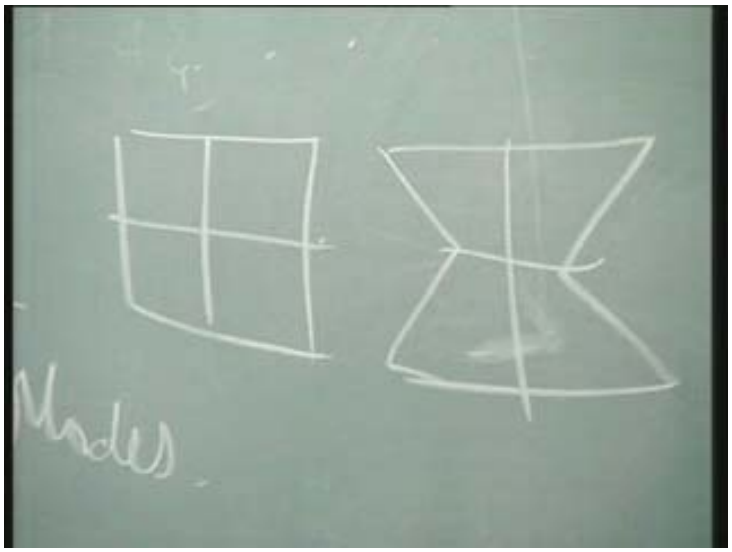

The jagged front gives a, gives some sort of impression that an hourglassing is produced and hence we call this as an hourglass mode. What do we mean by zero energy mode? Please understand that they are very important. Let me just recapitulate what I am trying to say? What I am trying to say is that there is a 2 by 2 rule or 3 by 3 rule. Depending upon the element, which is the correct rule that has to be applied? We will again come back to this correct rule in, I mean, later, but before that we should understand that though there is a correct integration rule, it is possible to follow what is called reduced integration.

People do reduced integration; people do reduced integration, because it saves cost. Your software can really rip through, run very fast, number one. Number two, I am introducing an error which is in the opposite direction of the error which I have introduced because of discretization. So, my final results are better, nice things that I have and hence I attempt reduced integration. But things are not as rosy as that because, when I do reduced integration, I also land up in certain other troubles called

hourglassing or what are called as zero energy modes or what are called as rank deficiency; all of them mean the same thing, rank deficiency and so on. The name hourglassing comes because of this jagged front that you get. Is that clear?

Before we go to a much more scientific or mathematical explanation, let us understand what we mean by, what we call as zero energy modes. What do we mean by that? All of us know the expressions for strain energy. What is it? 3808

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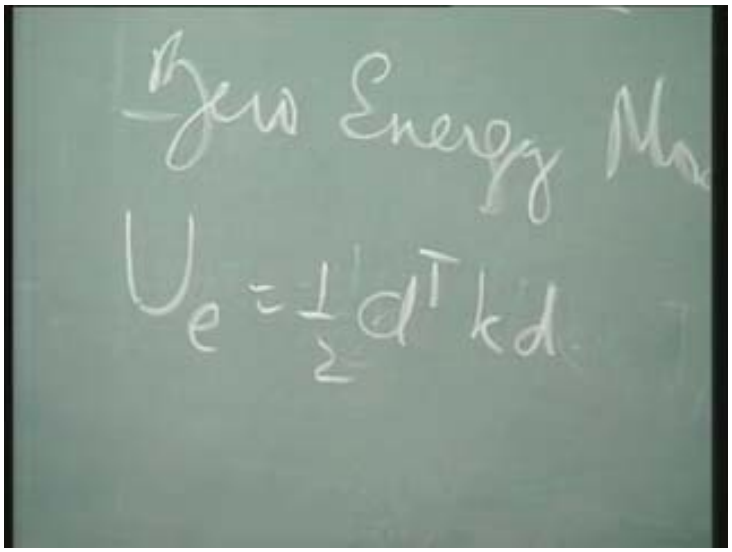

I can write this down, strain energy as half k transpose, sorry D k, sorry, small d k. We can write it down as half d transpose, sorry, what am I writing? d transpose k d, sorry, sorry, d transpose kd; sorry about that, d transpose k d. That is half k d square or half k u square, so, half d transpose k d. In other words, strain energy has to be present when there is displacement or deformation, rather and can be absent only when, only when d is equal to zero or deformation is equal to zero. Strain energy can also be obviously expressed in terms of our epsilon as well. That is very simple.

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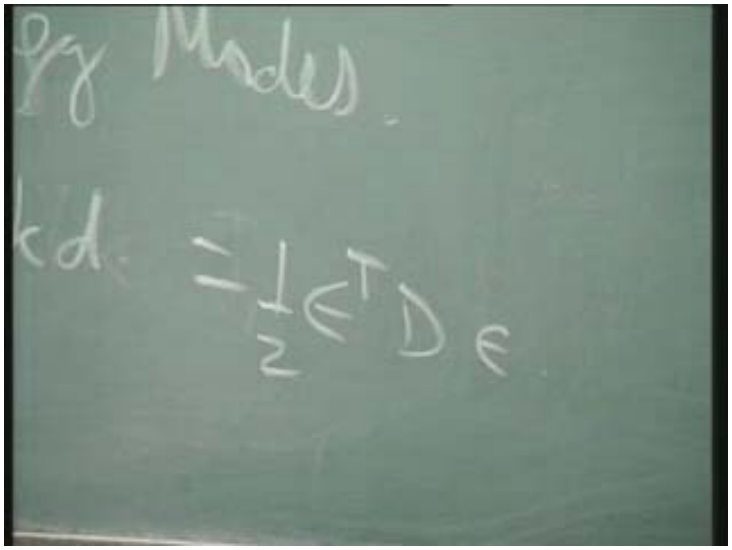

That is half E epsilon squared. Hence, I can write this as half epsilon transpose D epsilon as well. So, strain energy should be present when strains are present, which are present when deformations are present, obviously. When strains are present, strain energy has to be present. They go to zero, only when epsilon is equal to zero.

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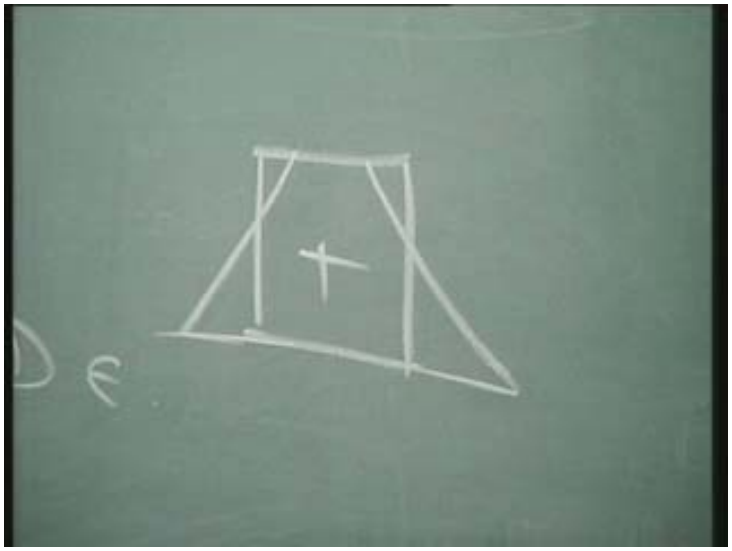

Let us look at one element, just ….. that. Let us look at this element just for understanding. It is not very rigorous, but we will do it more rigorously later. But, let us say that this element deforms like this. There are other issues in it, but let us not

worry about that. Let us say that this element bends and deforms something like this. The elements bend and deforms like this. There is say compression on the top and tension at the bottom and so this element becomes like this. Let us say that I am using a 1 by 1 rule. What am I doing here? I am doing reduced integration. What is that I am doing? I am trying to find out the strains at the center. Now what happens, what happens to the strains at the center? Zero, because what am I trying to do with this integrations? I am trying to concentrate on certain select locations; clear, on certain select locations.

One of the select location is my Gauss point, I mean center of the element, because I am using that select location to be the 1 by 1 Gauss point. Though there is strain in this element, though there is strain in this element, my concentration being at this point gives me epsilon to be zero. So, what happens, so, what happens? Substitute it back into this expression here, so, strain energy goes to zero; strain energy goes to zero. Hence the name zero energy modes. There is another mathematical explanation to it. We will not go into that for rank deficiency. We will just stop here the explanation, because there are lot more things I have to talk about. If there is time, we will come back to this term, rank deficiency.

By the way, I want to go further in this. But by the way, look at this and is there anything else that is bothering here? What I am saying is that it is bending. Actually this kind of a, this kind of deformation exist because of bending.

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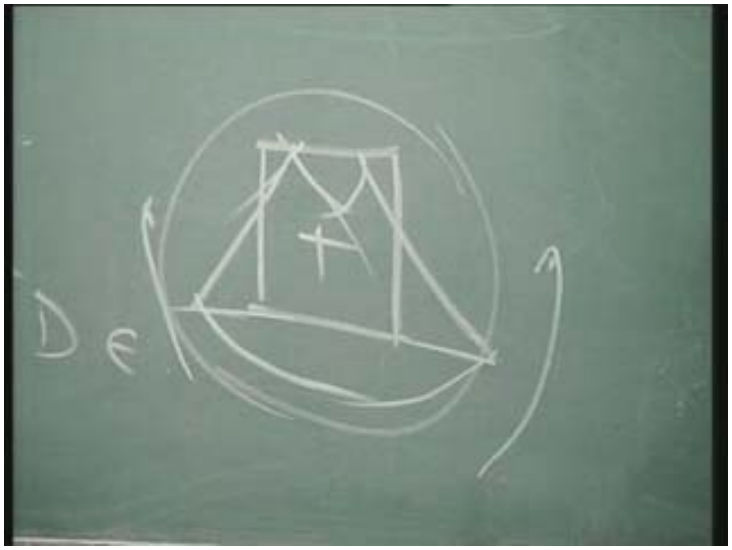

It is trying to bend and you get this. Is there anything else that is bothering you in this? Go back to your, dig up your memory on strength of materials and tell me whether you are bothered by something else in this? Yeah, just look at this; just look at this. Yeah, yeah, common, that is it. No, look at this figure here. Yeah, yeah I hear murmurs, but you can be bold. What happens to the shear? What are our famous rules which we used before in our earlier strength of material classes? Plane sections remain plane. Why did we used that rule? Because, Plane sections we want it to remain plane, because we do not want any shear, shear deformations. In other words, in actuality, this bending would be like this, so that this angle would have remained 90 degrees and plane sections would have remained plane.

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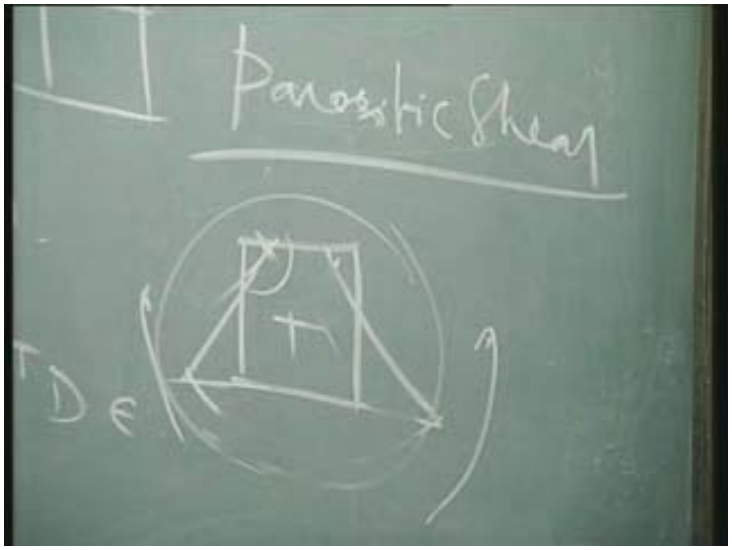

But, this kind of deformation is not possible and hence we get something like this. This is another issue, because I wanted to point out this; this is another issue, nevertheless it is also very important. Now, what happens because of this? You are introducing shear, because this angle is no more our famous 90 degrees. So, people call this also with different names and this is one of the draw backs, what they call as parasitic shear. Please note this is different from rank deficiency or zero energy modes. Why I pointed out is, because many people will get confused between the two. This is what we call as parasitic shear. It looks like a, you know, parasite which does not move; you know, does not go off and sits there and introduces a shear and hence it is called as parasitic shear. On the other hand rank deficiency or zero energy modes come out because we are in the neutral axis. At the point where it is zero, epsilon is equal to zero we go and integrate it; at that point we get into trouble.

Now, the question that you would ask is, if you go and look at many packages, like for example, if you look at a very, very good package like abacus, go and do explicit codes with that. You know, few of them may be working on explicit codes with abacus. You will see that those people have done or use only reduced integration. Then, is it that their results are always like this or do they do something else? For every problem, there is always a remedy. So, guys do not want to leave this single point integration, for example. We will expand this list later, but let us first understand what classical integration is or correct integration or reduced integration.

What they do is, look, the results or the benefits are enormous with reduced integration, so, I will do reduced integration, but they also do a repair job by doing what is called as hourglass control.

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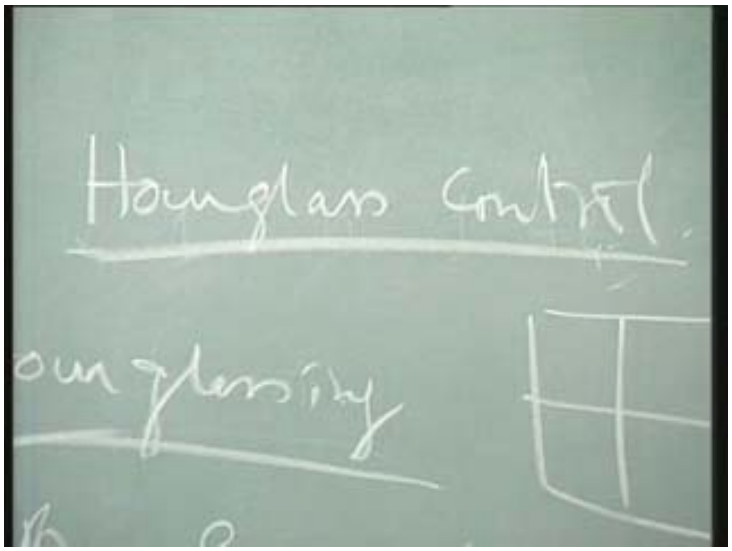

They also do a repair job or they do, I mean, they have another algorithm which also takes into account or which also controls this hourglass modes and call them as hourglass modes, hourglass control. But hourglass control does not work all the time, because of certain limitations. We will see that if time permits in the next class or at least I want to get you to this term, hourglass control, what hourglass control is. They control this hourglassing modes by still using this kind of a reduced integration with hourglass control, but it does not work because theoretically though it is correct, works only for some nice elements. One of the things that is recommended is, look, use reduced integration, but take more number of elements that will to a certain extent sort out the problem.

What are the effects of hourglassing? Now you know how to recognize? I hope you now have a big picture. You know what a big picture is? There is a correct integration rule; there is what is called reduced integration rule. There are advantages of reduced integration rule, but it results in hourglassing and how the hourglassing looks like was also shown to you and lastly to add to this big picture we also have what is called as hourglass control. Is that clear?

With this we will stop today, because we have to now look at, expand other things like I want to talk few more issues on reduced integration, if possible introduce hourglassing control. That we will take it up in the next class, but if there are, I am sure that I mean all of you are able to understand, because I have introduced lot of concepts in this class. But there is lot of theory into it; may be if time permits we will cover it in the next class and I hope you understand at least the complete big picture. Is there any question? We will meet in the next class.