Introduction to Finite Element Method
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Lecture - 21

Is there any question on what we did in the portions which we covered in the last class? No. I hope now all of you know how to deal with isoparametric elements, but we have to continue a bit because we have not completed isoparametric elements. Quite a bit of things have to be done before we close this chapter; may be next two to three classes we have to do and this is very important. As I have been telling you that this is a very important element; isoparametric element is an important element, because this is a concept. Isoparametric element is a concept. It can be used for a variety of elements. In fact, after this, may be next two classes I will derive an element for heat transfer which I gave as a problem in the test; this will also be derived. In other words, we will look at those problems as we go along; it will be solved as we go along.

But, what I want to emphasize is the fact that it is very important that we understand isoparametric elements. We will look at, when we even interpret the results, then also we would be interested in the theory of isoparametric elements. In other words from further on, from now on, the course is going to go something like this. I am going to teach isoparametric element and finish them, next three classes. Then I will deal with beam elements. I am not going to go into the details of plates and shells because it requires lot more theory and we do not have the time to cover in a first course. Nevertheless, I am just going to indicate the type of the shell element and axisymmetric shell, what it means, what are the types of, say, degrees of freedom and so on. I am going to just talk about that for a short time.

I will not be in a position to derive them, because that requires lot of background and also time. Once we complete that, then we will take a break to look at how to interpret results. I am not going to cover in this course solution techniques. Finite element is a very vast field. In other words, kd is equal to r or f; that is only one part of it. Then, I have to solve, ……, apply boundary conditions and solve them. All of you have been solving this boundary, this kind of problems, small things which I had given you by using Gauss elimination. That is the technique many people still use, but there are lot more advanced techniques which make things faster, what are called as preconditioned conjugate gradient methods, relaxation techniques and so on. Many things are there, but we are not going to cover all these things in this course. Again that requires a big background in linear algebra and so on, so we are not going to cover that. But nevertheless, if you know Gauss elimination which you would have studied in eighth standard that is enough to understand that you have to solve this problem and enough to go further, because if at all you are going to use a package, you are not going to solve this. But, if at all you want to write a package and then with this knowledge you can always pickup as to how to solve problem. They are all readily available today.

It is a specialized area. Again, I am not going to cover that, but what am I going to do, after looking at the niceties of results and niceties of interpretation and so on, we are going to go to an introduction to nonlinear finite element analysis. How exactly nonlinear finite element analysis is done that is what I am going to talk or in other words algorithms that are used. We are going to talk specifically about what are called as increments, iteration and so on. In that respect, I am going to bring in the formulations that are used for solving manufacturing problems. There are number of formulations that are used. It is again not possible to cover all these formulations. We need three courses like this, if you want to cover all those things. Again, it requires lot more basic understanding of what we call as continuum mechanics.

Hence, again we are going to look at one of the formulations which is very important, which is extensively used, for example, to solve problems in a metal forming, what we will call as fluid formulations or flow formulation rather. This flow formulation is what I am going to emphasize as a next step. So, after beam elements some results and then flow formulations. If you have time further, we will talk about contact formulations as well. Many of the principles that are used in metal forming simulations I will indicate it and we will look at it with some practical examples. Again applications of finite element to say metal forming, it is a huge area, so, we are not going to talk about again sheet metal, because that requires knowledge of shells and so on. But, we are going to talk about application of finite element to bulk-forming processes and where many of the techniques that we have used will be of use. Why I am saying this is because, this topic is extremely important. Once you understand this, the application of this topic to many other fields or many other fields of application of finite element becomes rather routine. Tomorrow you want to go and do heat transfer problem or determine temperatures, the issues are very similar to what we have done now. Hence, you please follow whatever I am saying.

We stopped this problem at this place. We said that one of the major, I would say, issues here is to write down the stiffness matrix of a triangular element. We also gave a reason, why triangular elements are important till recently and tetrahedron elements its big brother, its three dimensional brother is very important even today. Basically, because most of the softwares, most of the softwares, generate triangular element and now, yes, there has been some change, the technology has moved. Now, they are capable of generating a four sided quadrilateral element. But even today, most software, 99% of the softwares, unless you make efforts, lot of efforts, when you say automatic machine that means you do not make efforts, the software does only tetrahedron elements. Unless you make an effort by what are called as mapped meshing, which owes its origin to Professor Zinkevich (Pl check the spelling) in the early 70's, then only do you get an element which we call as a hexahedron element, which has 6 sides or else you still have the big brother of triangular elements that is the tetrahedron elements.

In other words, what I mean to say is, whatever we are talking about for triangular elements can be, many of the properties can be, extended to this kind of tetrahedron elements. So, once you understand the basis of these elements, it is very easy to extend them. There are two ways in which we can develop the stiffness matrix of a triangular element; there are two ways. One is to collapse a node which is probably the easiest to understand and many softwares do it in this way, as well as in another way which I am going to discuss later.

One way is to collapse. For example, look at this element.

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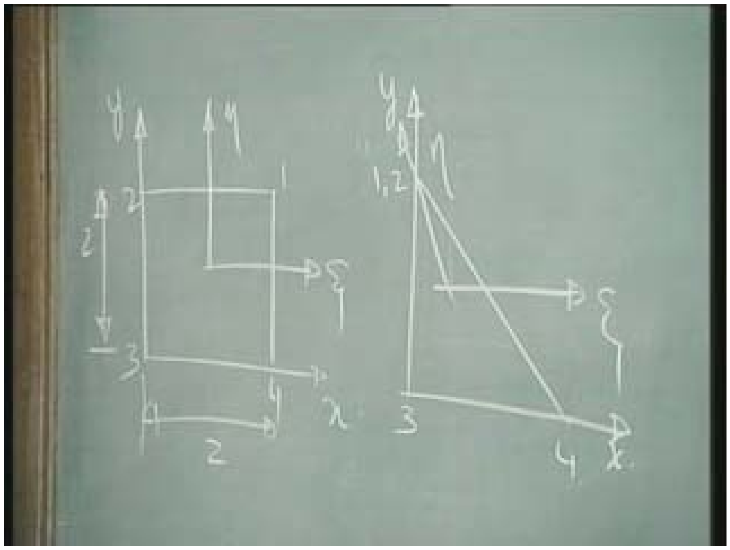

This, I have given specific dimensions in order that we can work out a problem and you get a feel of what is happening. I do not want to be vague and hence I have given some dimensions to it. Say for example, if I collapse this 1 onto 2 and get an element, collapse means make this coincide or make x_1 is equal to y_1 and x_2, sorry x_1 is equal to x_2 and y_1 is equal to y2, when you make like that then the resulting element becomes a triangular element. Is that clear?

Now, what I am going to do is to proceed in this fashion and workout a B matrix to it. I want to show some peculiarity with the B matrix, what happens to the B matrix? The procedure can be written as a code and so on or you can follow a second technique for the generation of stiffness matrix of a triangular element which people call as area method.

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We will keep that, we will just reserve this area method for the next step and let us now look at this, because you get some nice results with this, so your understanding will also go up looking at this. I need your help in working out. I am sure all of you are familiar with this problem, because last time also I left it at that stage. Let us see how to work out this problem. These are isoparametric elements and hence I can write down say x by means of our well known formula N_i x_i.

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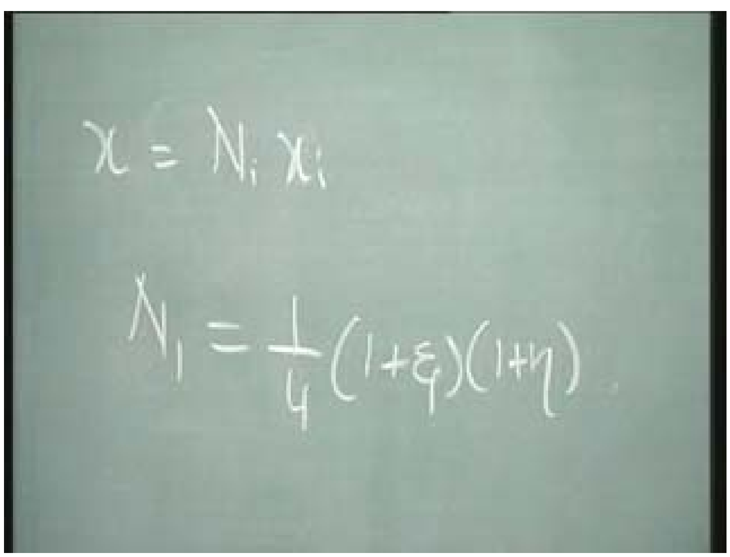

By the way what is N_1 in this case? I think there has been mistake in this as well. Yes, look at the way I have numbered. I have deliberately done it. Look at the way I have numbered these things. See where is 1, where is 2 and so on. It is important that you understand that you cannot blindly put N_1 N_2 N_3 and N_4 and hence I say N_1 is equal to 1 by 4 into 1 plus psi into 1 plus eta. Is that clear?

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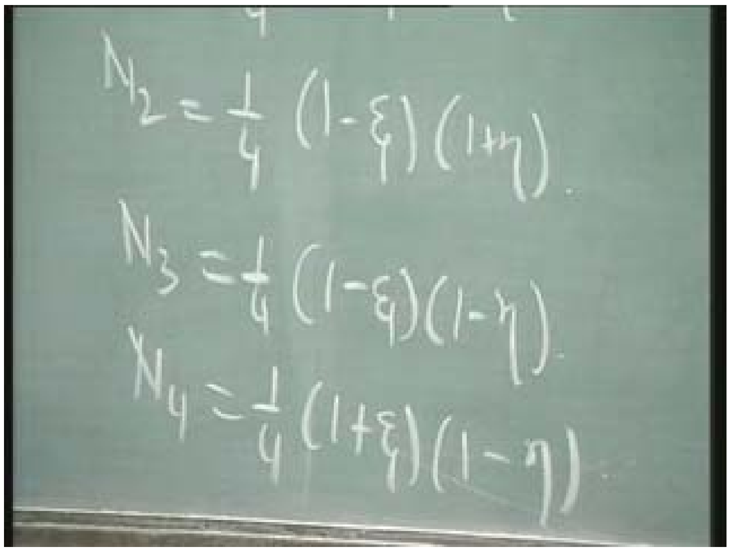

What is N_2 now? 1 by 4 into 1 minus psi into 1 plus eta; N_2's position is 2 here and three, N_3 is equal to 1 by 4 into1 minus psi into 1 minus eta and N_4 is equal to 1 by 4 into 1 plus psi into 1 minus eta. It is very simple to just check this up, substitute them in these equations, I mean, sorry, in this triangular coordinates you will see that familiar properties are satisfied. Is that clear?

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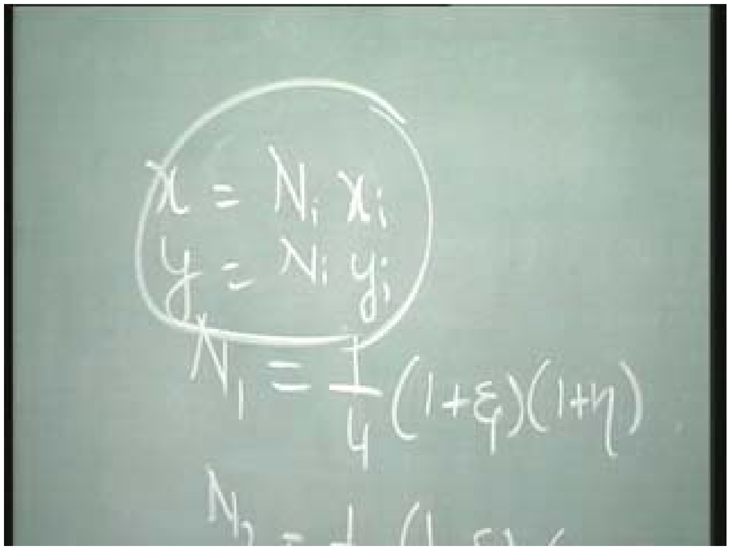

My first step is to substitute this here and as well as y is equal to N_i y_i. Please note what am I doing by writing it like this? I am invoking my isoparametric principle, the basic principles of natural coordinate system. Is that clear? My first step is to substitute these equations into this equation and then substitute or put x_2 is equal to x_1 as well, so that now I can write down. What do I do ultimately? I am reducing the coordinates so that I can write down, for example, x in terms of x_2 x_3 and x_4. Let us see how I write it.

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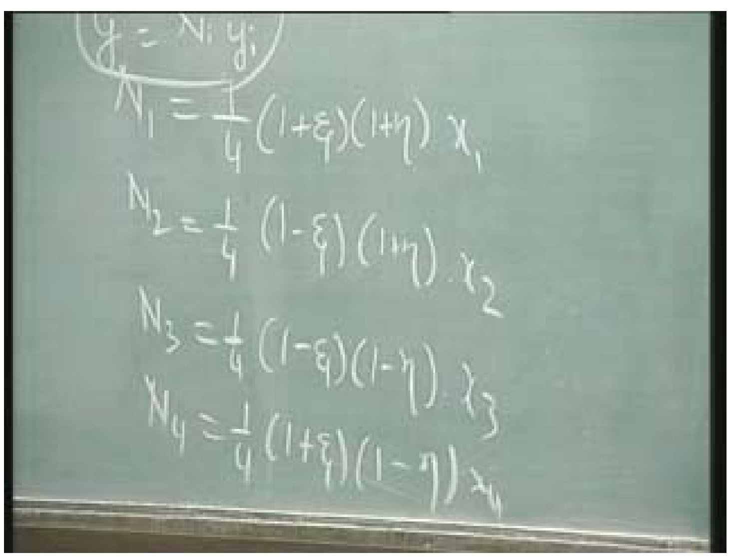

x is equal to so one by This becomes I will just write it here, so that you can just follow this easily.

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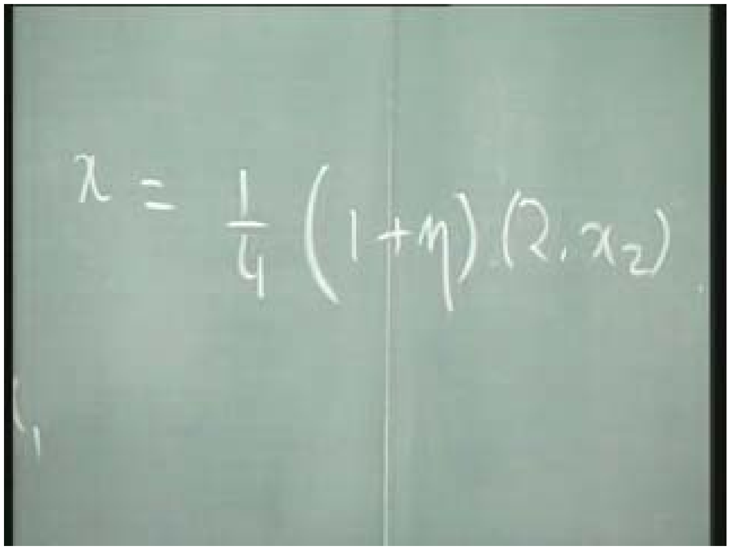

Now, x_1 and x_2 becomes the same. What happens to x? 1 by 4 into what? 1 plus eta into, no, no; first let us write it down in terms of x_2 x_3 and so on into 2, x_2, so that that and this goes off and so 2 into x_2 plus N_3 x_3 is present for x_4.

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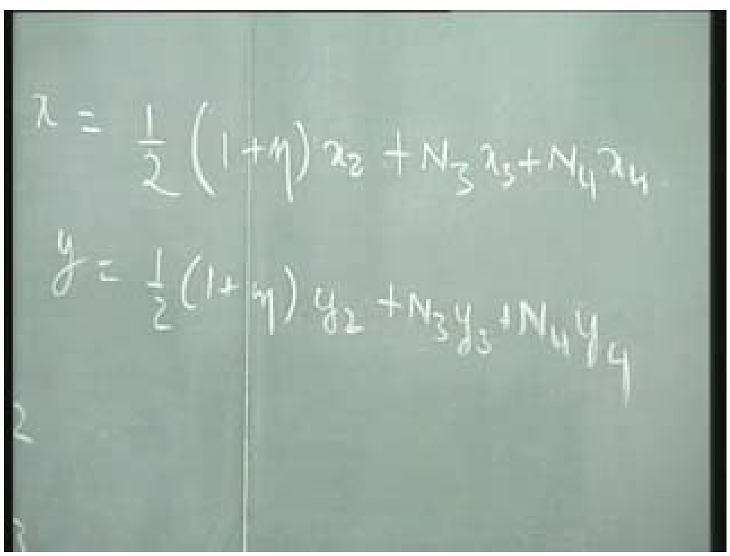

Similarly, you can write down y as well. Is that clear? My next step is let me substitute; let me substitute, the coordinates x_2 x_3 and x_4, from this figure.

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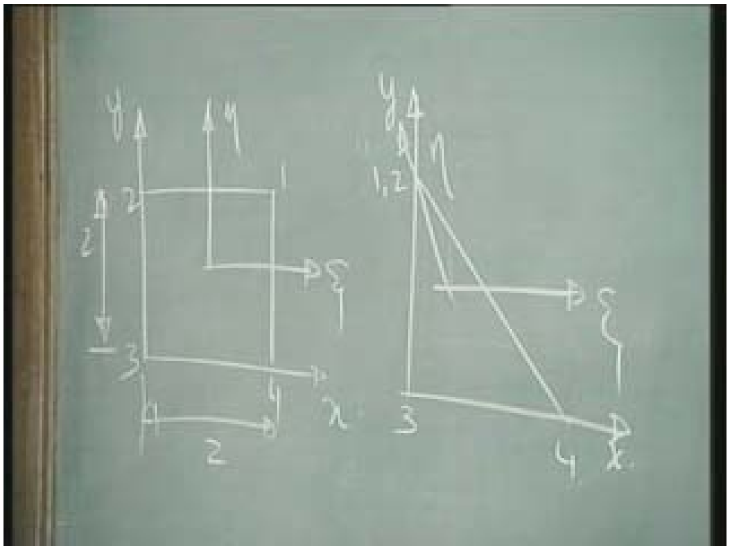

Let me substitute it from this figure onto my equations, so that I can write down now x and y.

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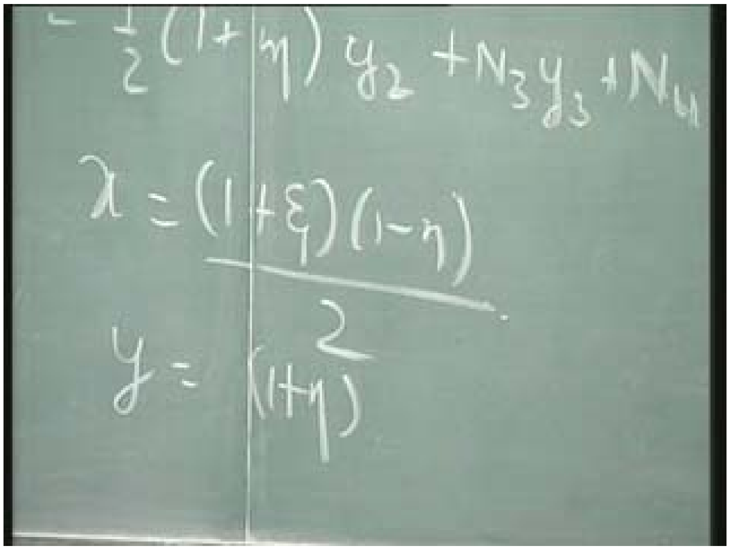

x is equal to, can someone help me? Take a minute to look at that and then substitute this. What happens to x_2? x_2 is zero, x_3 is zero. What happens to x_4? 2; x_4 is here, 2, so, 1 plus psi into 1 minus eta by 2. Is that clear? How many of you did this problem? How many of you did this problem? Not even one? How many came up to this step? Very good; let us see. Just think with me, let us see whether you are able to do it. But, none completed it. But then, you will be surprised when I do it that this is the one of the simplest problem, because I am just going to follow exactly what I did for the previous case; nothing more than that.

What is y? Substitute now for y_2 and y_3 and so on. Yeah, yeah! 1 plus eta into, no, when you, whether you write it in terms of x_1 and a x_2, you can you are saying that you will write it in terms of x_1 x_3 and x_4. No; x_1 is equal to x_2; x_1 is equal to zero. No, no, I do not follow your question. You can, if you want you can, eliminate x_2 and write it in terms of x_1 x_3 x_4. x_1 is equal to x_2, so what does it matter? Is that clear? So, what is y? 1 plus eta.

Now, why I did this is because, I can keep it as it is. I can keep it in terms of x_1 x_2 x_3 and continue, but then my algebra becomes very big. I am solving a particular problem that is why I substitute it or else I need not substitute it. I can keep it as it is so that I will get a much more general picture, but then since we are only illustrating one problem, I have just substituted. Is that clear? I need not have substituted, if that is your doubt? I need not have substituted it. But then I will again, I mean fill up the board with lot numbers, then it becomes very confusing and you may miss the central result of this problem and that is the reason why I wrote that. Is that clear?

What is my next step? What are we trying to do? I am trying to solve this isoparametric problem. I mean, I want you to listen, because none of you have done this problem. It is very important that you understand it. What is that I am trying to do? What do you think is the next ultimate step, not the immediate step? What do you think is the next ultimate step? Jacobian; right. Why not I calculate the Jacobian? Is that clear, any question? Now, calculate the Jacobian.

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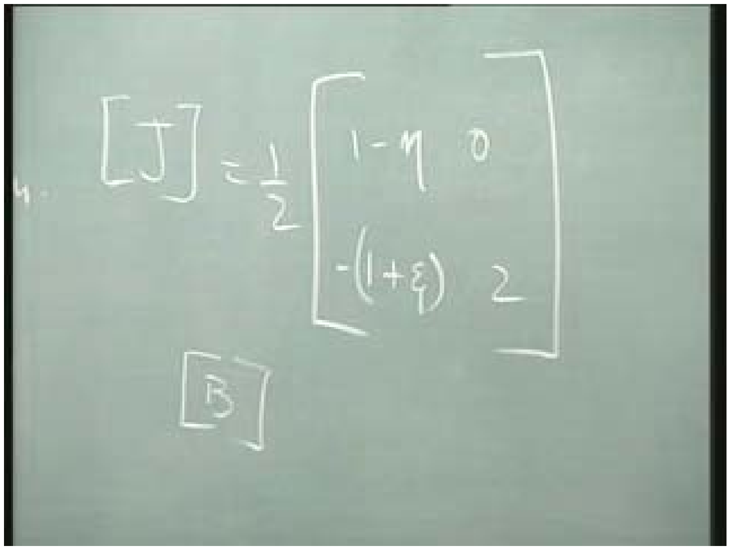

I will give you 2 minutes, just do that. Just look at these things, look at this and then calculate and write down Jacobian. What is it? dow x by dow psi dow y by dow psi dow x by dow eta dow y by dow eta. So, substitute them and check what you get? Who is going to give me the answer? 1 minus eta zero minus of 1 plus psi 2; beautiful, simple; dow x by dow psi and so on. That is one part of the story. Now, you have not yet answered, this is the second step Jacobian; you have not yet answered what is it that I am interested in? What is the problem? We are going to look at B matrix. We are going to look at B matrix and see how B matrix is going to look like? My whole idea is to go to B matrix. Is that clear?

What is it that I should do in order that I go to B matrix? D_N, J inverse; yes, of course, I have to calculate J inverse. What is J inverse? Let us do that straight away, because J is available here.

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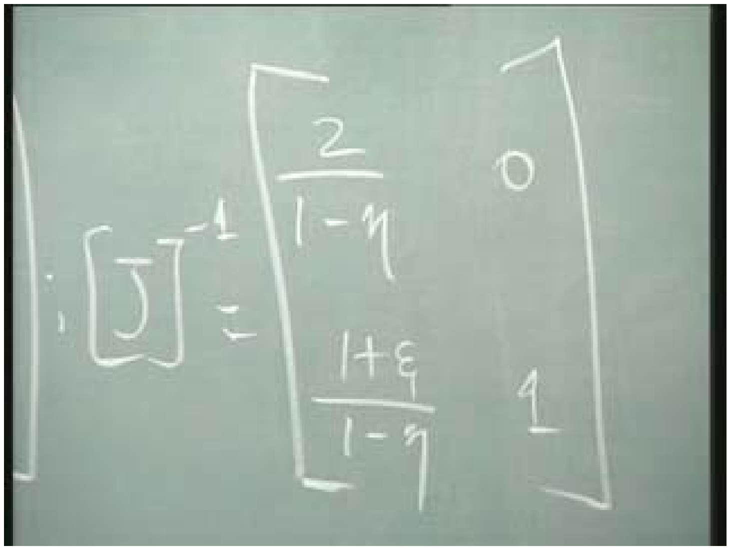

What is J inverse? Just calculate that and let me know. Yes, no. 2 upon 1 minus eta zero 1 plus psi 1 minus eta, beautiful, 1; lovely. See, the only way to learn this subject quite thoroughly is to do these things, so that you will get familiar with each step. Now, we have J inverse. Now, what is my next step? What is the other principle that I am going to invoke? What is the other principle that I am going to invoke? Let me remove this. I think you know all these N's and you have it in your note books. What is the other one? I am going to now write, what is it that I am going to write? u in terms of N_i u_i and v, N_i v_i.

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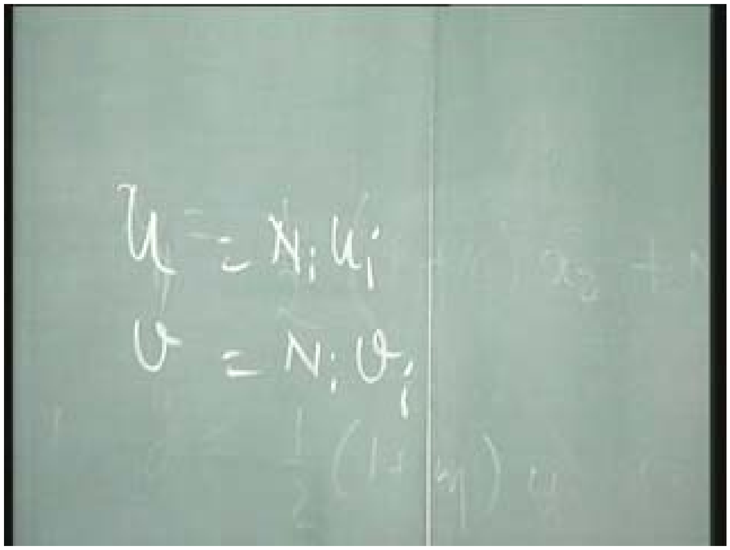

Sigma is implied because, u_i I am just repeating and sorry u, v is equal to N_i v_i. What am I going to do here, when I collapse? When I collapse, what is that I have done? u_1 is equal to u_2. That is all; u_1 is equal to u_2. So, my degrees of freedom become u_2 u_3 u_4, v_2 v_3 v_4, period. Get that, so that you can now write it in terms of say N_1 or u_1.

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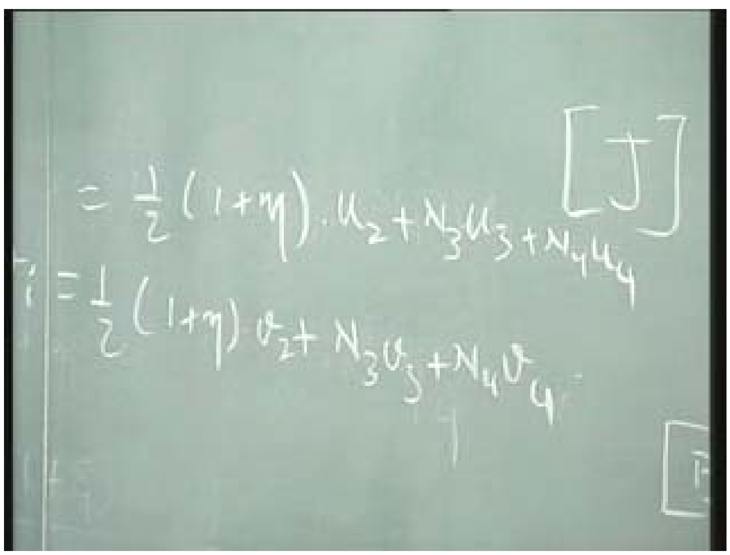

I will not write that 1 plus; the same thing what we wrote in the previous case, half into 1 plus, half into 1 plus eta into u_2 plus N_3u_3 plus N_4u_4 and that is equal to half into 1 plus eta into v_2 plus N_3v_3 plus N_4v_4. We have invoked two principles which we used for isoparametric elements. What is my next step? What is that I should calculate? D_N or in other words I should calculate dow u by dow psi. Why, because ultimately, what is that I want? dow u by dow x; very good. So, I want dow u by dow x dow u by dow y and so on, because in order to calculate B, I need them. Remember, I had that 1's and zeros matrix multiplied by our u comma x and so on. My next step is to calculate dow u by dow psi. What is that equation, if you remember, just kept that in your mind?

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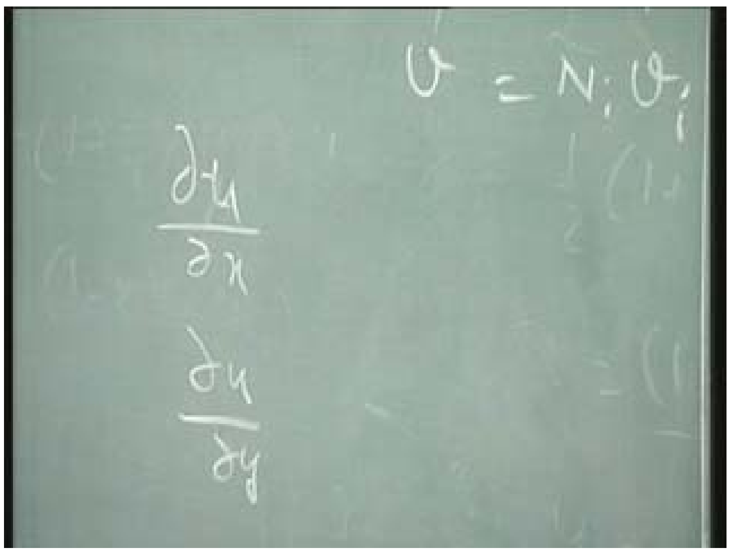

That is dow x by say dow u by dow x and say dow u by dow y and so on. What is it? What does it involve? It involves J inverse. We will go to that step as a next step, but before that what is that I should calculate?

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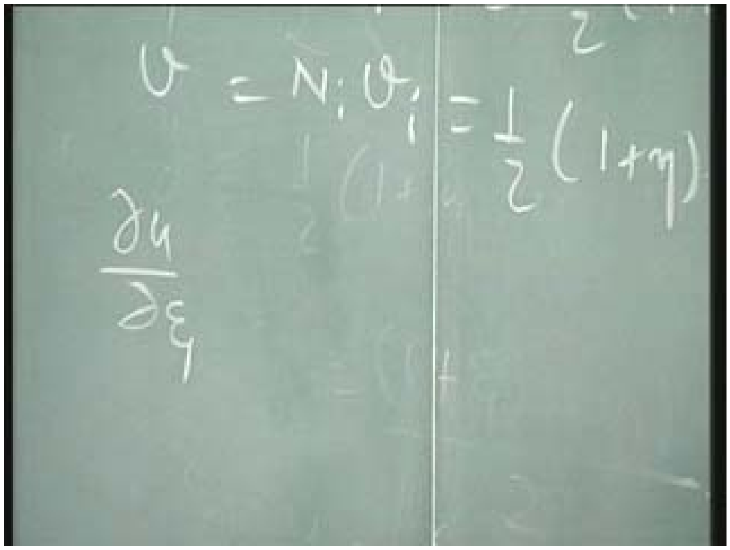

D_N or D_N consists of, I have to get dow u by dow psi. That is what my next step is. I have to calculate dow u by dow psi dow v by dow psi dow u by dow eta and dow v by dow eta. In other words, you can split this up into D_N into, what is that you will get? D_N into d. What is d in this case?

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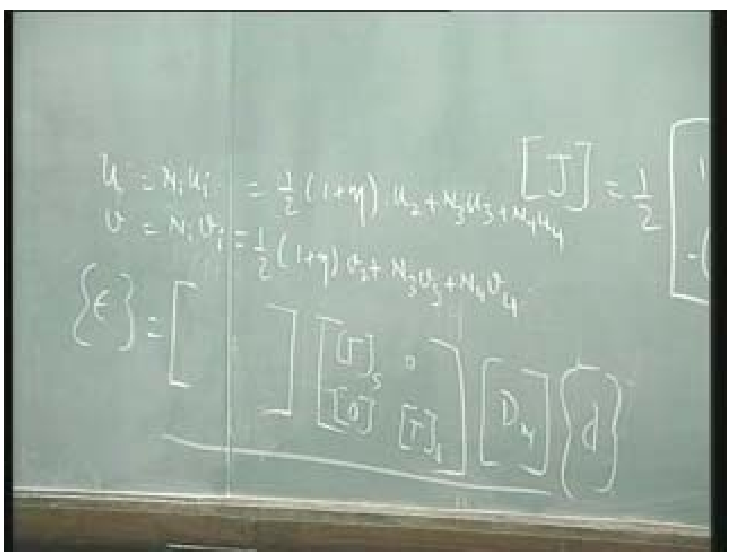

You can write down, say in other words, just to preempt what we are going to do, in other words B is equal to those zeros and 1 matrix which you have, refer to your book, multiplied by what? What did we write it as? Yeah, gamma; gamma s we wrote remember? Zero gamma s into d matrix, small d; small d. What are they? They are the degrees of freedom. Sorry, D_N into d; remove this B. That is actually, no, no; sorry, sorry, sorry and so this whole thing becomes B. Is that clear? How many of you had come to this step? There is nothing that we have done; same procedure, whatever we have done.

Has anyone come to this step in the exam? What exactly was the problem? You know, let me just go back and try why you have not done this? Is there any difficulty in understanding this? What is the confusion? Whatever step I have written is known to you. In fact B matrix, I have told you so many times that this is B matrix. The only difference now is this d is now a compressed d. So, 1 2 3 4, we have only 2 3 4; that is it. Now, I will give you two minutes, I want you to calculate, say D_N. Just calculate D_N and tell me what D_N is? Just do that; since you have not done it before, I would like you to do that and tell me what D_N is?

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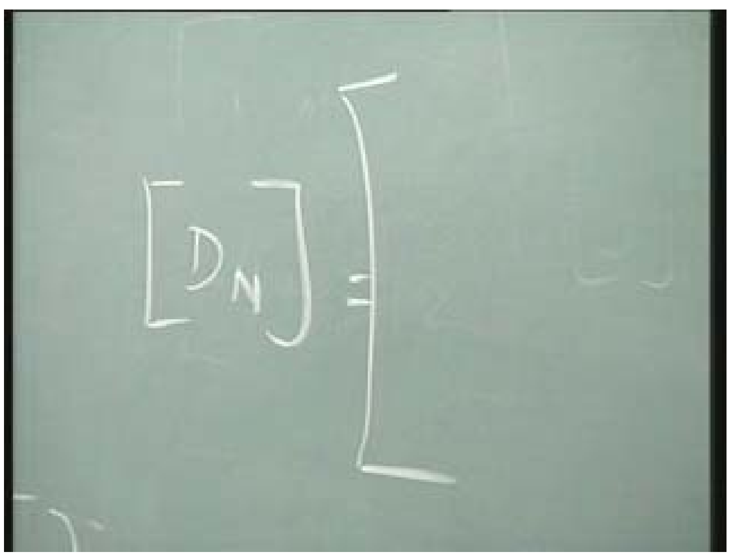

Any question? Because you have not done, is there any confusion? Anyone has not understood the steps? Yes; now, it is clear, everything is clear. It always happens that after the exam everything is clear and in the exam you do not work out usually and that is a problem with all of us. Even when I was a student it was the same problem. Let us see now. Let us see how many of you get this problem correctly. Let us just stick to only two terms, so that we will not get all the things. I mean, because it is a very big matrix, we will just stick to only …… In other words, let me make this problem simpler. Just calculate dow u by dow x and dow u by dow y, we will not do all the things.

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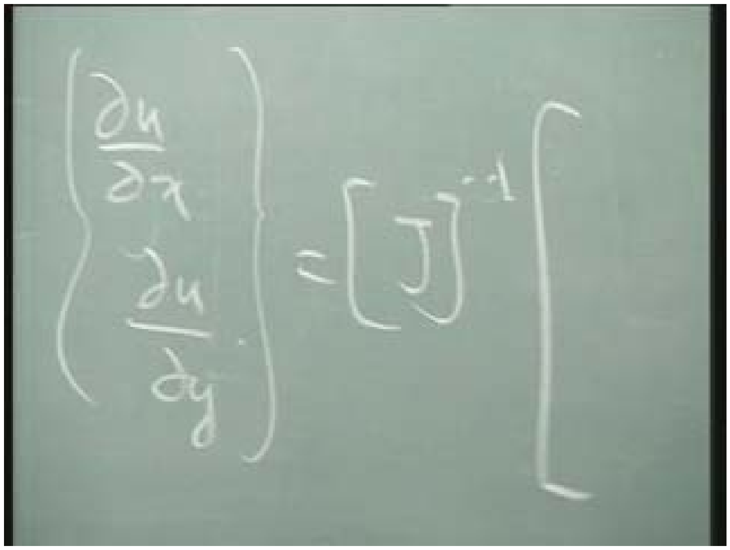

We will just stick to the first one, dow u by dow x, like only one variable; dow u by dow x and dow u by dow y. Similarly you have to do dow v by dow x and dow v by dow y; just dow u by dow x and dow v by dow y; they will have, it will just be J inverse, straight away. J inverse and then we will have another D_N term followed by small d's, followed by u_1 and so on, u_2 and so on; just these terms. Yeah; correct, correct. That is what I want you to workout; that is what I want you to workout. Correct, substitute; that is what I am asking you. What is it? What is this value? You can directly get it. You have, what is that you have here?

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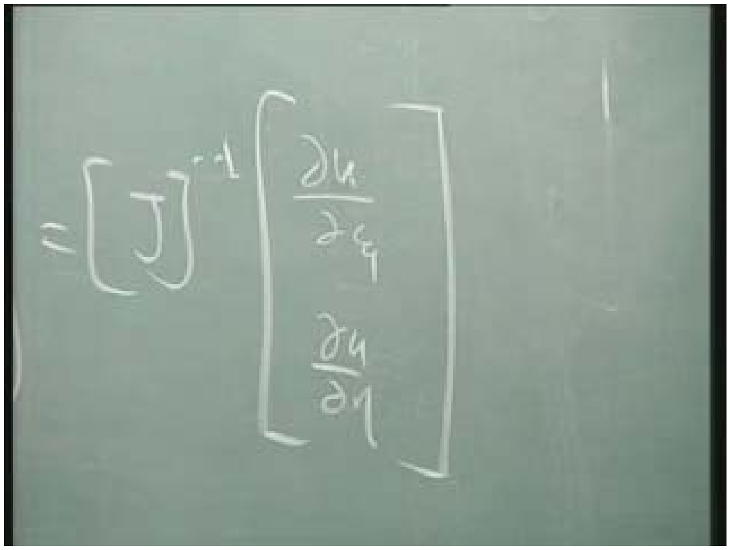

Dow u by, can you fill this equation? Dow u by dow psi and here dow u by dow eta. What I am trying to ask you is, what is this? What is this?

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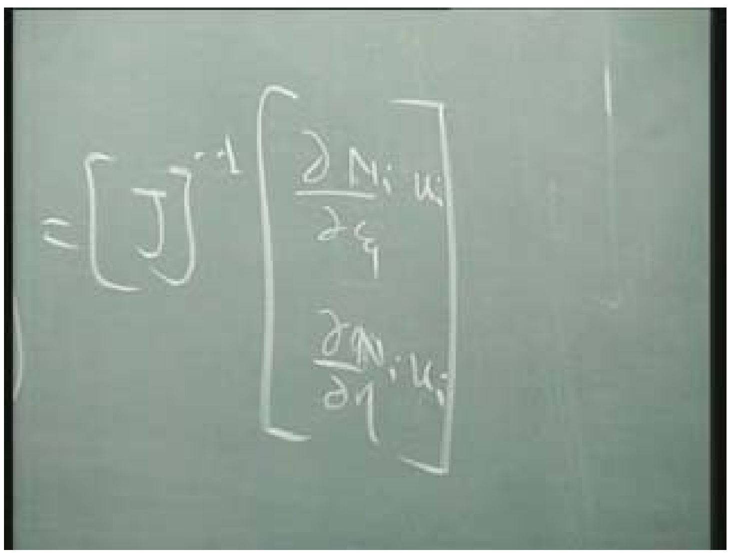

This can be written as N_iu_i; same way, here also N_iu_i. Is that clear? How do I write this? How do I write this thing? Dow N_i by dow psi; dow N_1 by dow psi dow N_2 by dow psi or dow N_3. There are three terms u_1 or u_1 is equal to u_2; so u_1 u_3 u_4. Clear?

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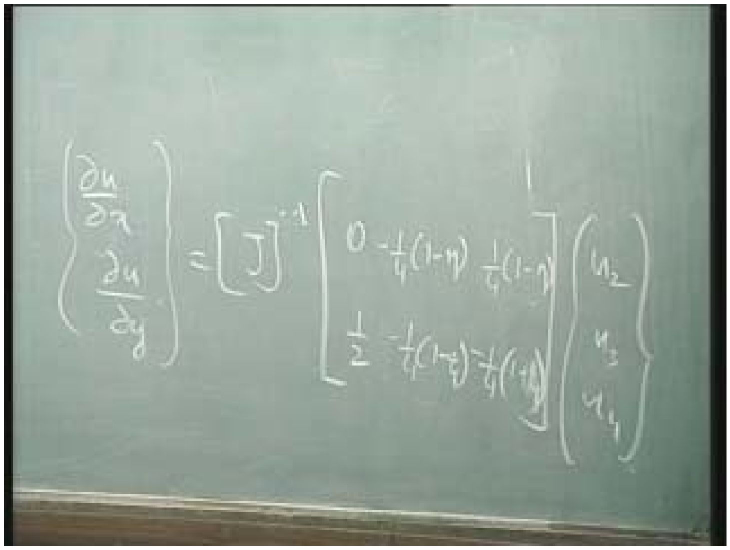

You will have some term u_1 u_3 u_4, just for illustration. Then, you can write the same thing for dow v by dow x dow v by dow y, combine them together; that is all, appropriately place it. What will be the first term here? Yes, what will be the first term? No, no. What will be the first term? Dow N_1 by dow psi. You remember that, whatever we had written. It is not actually N_1 but a combination; half of 1 plus eta. That would be the first one because that multiplied by u_1. Is that clear? Because, N_1 and N_2 we have combined them together. What is that? How did I write? u is equal to half into 1 plus eta into u_1 plus N_2 into u_2 plus N_3 into u_3. That is how I wrote. The first term here would be that dow of dow psi into half of 1 plus eta. Is that clear? What, yes or no? So, first term is zero. What had happened to my second term?

Why is that it is zero? No; look at the way I have written it. To just make it simple I have just written only in terms of u's. If you want, you can write it also in terms of u as well as v. Just to make things simple right now, I have written it in a simple fashion. So, minus 1 by 4 into 1 minus eta; minus 1 by four into 1 minus eta plus 1 by four into 1 minus eta; is that clear? What would be the other things? Here it is dow u by or that one half of 1 plus eta, that has to be differentiated with respect to eta. So, what will be the first term here? Half, very good; see, so simple nothing there. Then you have to differentiate the second term. Second term, when you differentiate it minus 1 by 4 into 1 minus psi; very good.

What will be the third term there? Is it minus 1 by 4 or plus 1 by 4? 1 minus, sorry, 1 plus eta, very good. Now, I want also dow v by dow x dow v by dow y. I can write the same thing in a similar fashion. Sorry, same I say; whether u_1 or u_2, u_1 is equal to u_2, so, u_2 u_3 u_4. Is that clear? What is my question? My question is, how is this B going to appear? That is my whole question. How does that B matrix appear? I can look at it even here to tell how B is going to appear?

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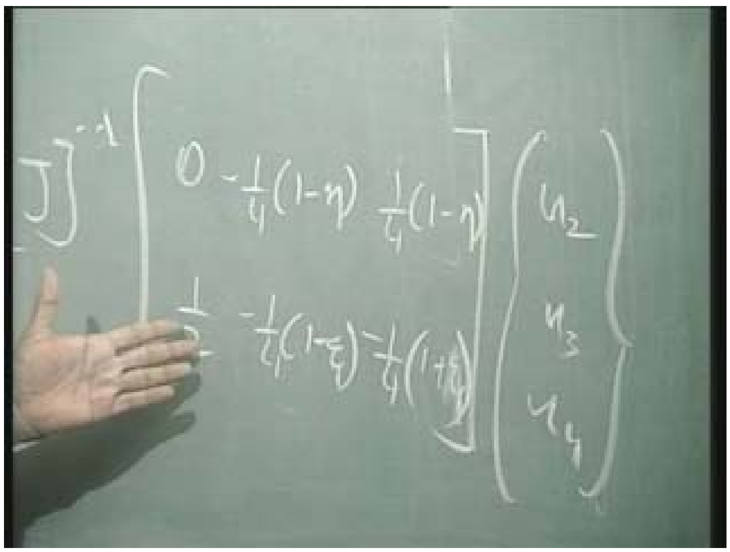

What is that I am going to do? Substitute for J inverse; I already have it, multiply it with this equation or with this matrix and tell me what is that you will get, because that will get repeated. I have to only readjust, nothing great in it. In other words what I want you to do is to substitute for J inverse, multiply here, multiply it and tell me how dow u by dow x and dow u by dow y will appear.

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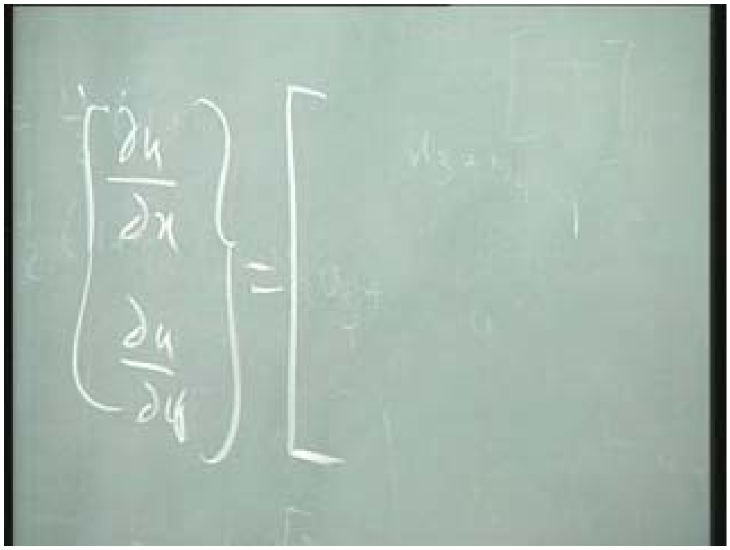

I will just write it here itself; dow u by dow x and dow u by dow y. How it will appear to me? You had J inverse; what is J inverse? Yeah, 2 by 1 minus eta zero 1 plus psi into or by 1 minus eta 1; period, that is it. So, substitute it there and multiply it, simple multiplications. That is the reason why I just stopped with that and just tell me how it will appear? Now you know how to calculate D_N. Our regular D_N is what? Just expand this; expand this to u_2v_2 u_3v_3 u_4 v_4. Similarly, put zero here, zero here, keep on expanding that. Is that clear? Very simple, no problem; that is done. So, now multiply J inverse and this and tell me what is that you get out of it and this result is very important.

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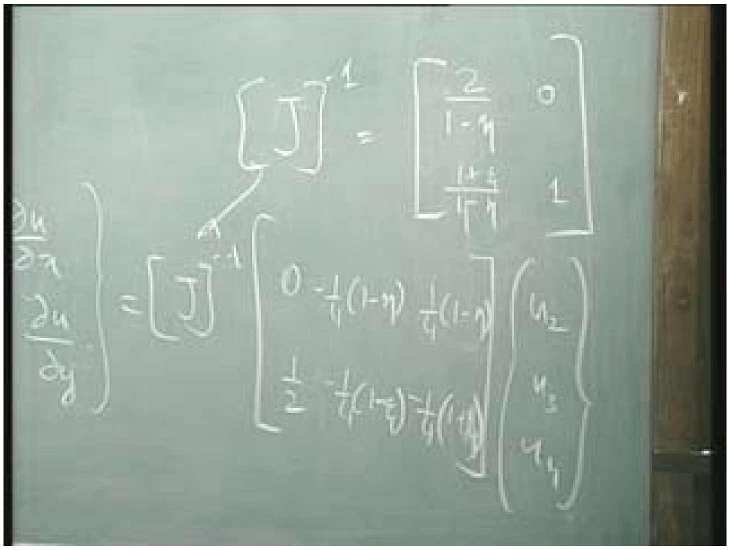

If you want for your … what is …. J inverse is 2 by 1 minus eta zero 1 plus psi into 1 minus eta and 1. Am I right? Yeah; just for your reference and so just substitute that here, multiply it and tell me what is that you are going to get?

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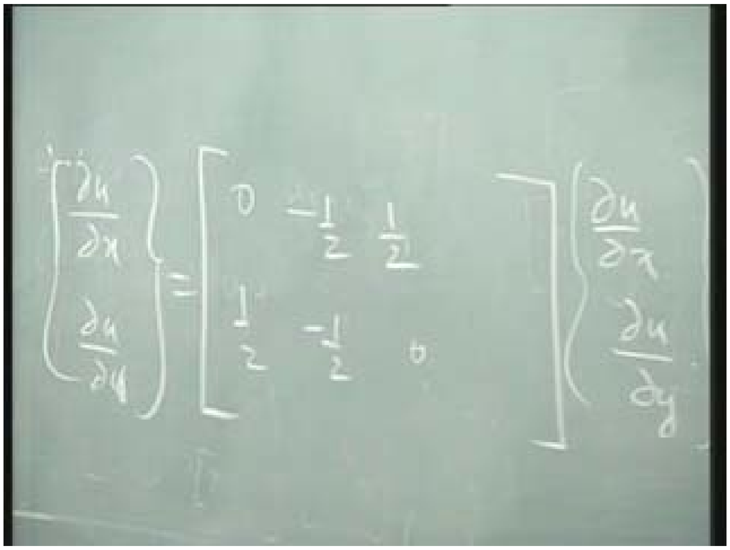

First one is half; wait, wait, wait; slowly tell me. Let everyone complete it. So, first term is zero minus half minus half. Next, next one half minus half zero. What does this tell you? So, my B matrix can be, once I know this structure, I know how B matrix is going to look like. Is it clear or not? Is there any confusion on this? I have taken a part of B matrix and worked out because I do not want to write, again fill the board; that is why. In other words, I can write down dow v by dow x dow v by dow y; it will have the same form, but will be multiplied by what? v_2 v_3 v_4; period. I can now combine them together to write my big B matrix. What does this tell you? It is constant throughout the triangle. B is a constant, the whole of the triangle. You go back and look at what we did with the other element, our quads; quads were not like this.

What does this give you from using a triangular element? Suppose there is a place where strain changes. Then obviously, the strain inside a triangular element is going to be a constant; strain inside the triangular element is going to be a constant and hence I will be hard put to save myself if I use a triangular element in that area, a big triangular element. But, there is one big but. If I use many number of elements, then the problem is obviously to a great extent alleviated, because I can fit any curve by means of, when I take more and more elements by means of, small steps.

There are other peculiarities to this element. I am not going to answer that now. I want you to keep this in mind that the constant strain state, I will answer it in the next class when we look at, what is it that we are going to get out of finite elements? So, lesson number 1, major lesson, if you want to use triangular element or as I told you, as I called it as a big brother, that tetrahedron element, then you have to be careful in meshing. Because especially when there is a strain change, for example, in bending strain changes linearly, then if you use this kind of elements you will be in trouble and that is what I showed you in one of the exercises, which we did in one of the classes. Is that clear?

This exercise was not very difficult. I am sure now, you all agree that it is not very difficult; quite straight forward, but it cannot be that easily extended to higher order elements or in other words you may look at it and say that it is a simple thing, but can I have a much more general technique of doing a triangular element in the same fashion as you did our quadrilateral element? That is our question. Is it possible to have an element? Why is it that I am looking for a much more general issue? I will come to that in a minute. Is it clear? Can I go to the next issue? Fine.

What we are looking at so far, are what are called as only a linear element. Next class, I will illustrate that we can move over to a more quadratic elements.

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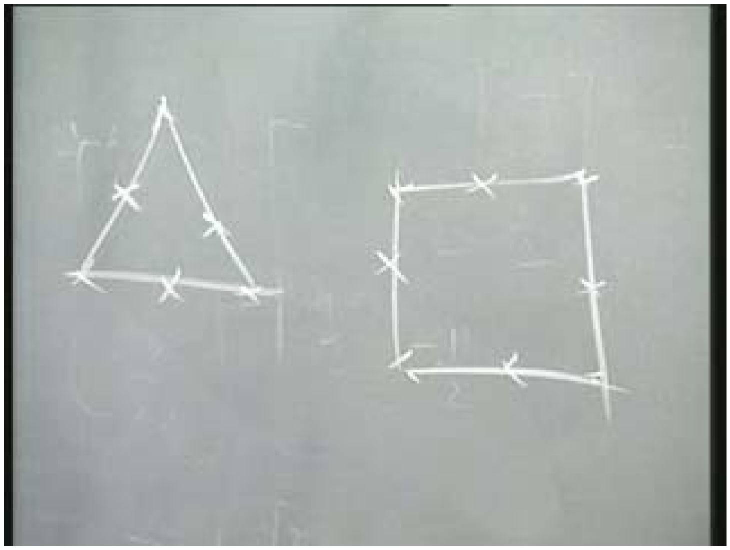

For example, I can say that a triangular element can have more than 3 nodes, like I have, say, quadrilateral element can have more than 4 nodes. Say for example, in a triangular element apart from these three chaps, I can put nodes at other places. Why am I doing this is because I am stung by the criticism that that B is a constant. Why did B become constant, because of the simple fact which I illustrated about 4 or 5 classes back, that u is written in terms of x and y; a_0 plus a_1x plus a_2y. But all those things are camouflaged in whatever I did. That is why I did that much more elaborately. But if I can use not a linear interpolation, but a quadratic interpolation, I can to great extent solve this criticism.

In fact, today for many of the very large deformation cases, elements which have been developed, which are triangular elements with 6 nodes and perform extremely well under much more difficult conditions. The major advantage is with good mesh generators, triangular meshes are very easy to generate and I can use those generators. So, I have a need to develop a much more general technique for a triangular element, which can accommodate not only just a linear triangle or constant strain triangle, as it is called, but higher order triangular elements like this which has, say, 6 nodes. Once we do that, we will get back to this element which has apart from these 4 guys here, 4 nodes there, I will have 4 more at these places. Is that clear? In order to develop this isoparametric or this element from an isoparametric fashion, I have to, I have to, now generate what is called as a natural coordinate system for a triangle. I will leave it with this small indication and then we will continue it in the next class, because it is a very lovely idea. What is that we did in the last instant?

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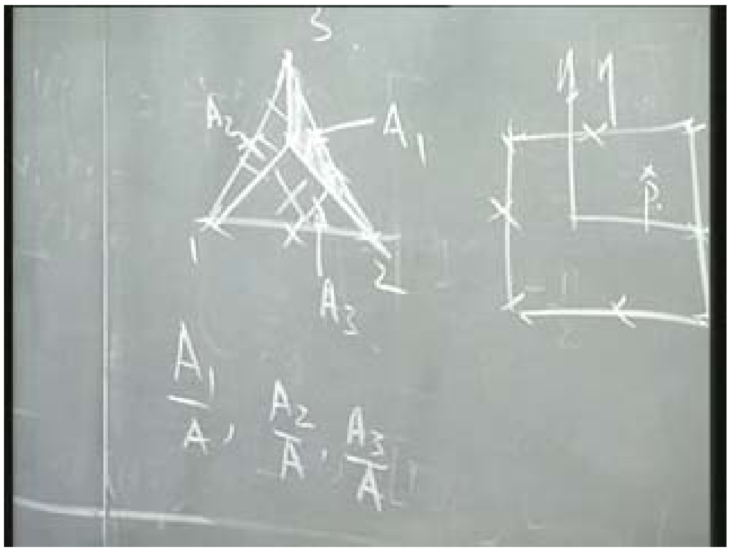

I had, say, a point here, a point say P. We expressed it in terms of psi eta x y and so on. In other words, we had a, we mapped; we mapped or got psi eta in a fashion such that we can get psi and eta, by joining. How did we get it? By joining midpoints on either side we got psi and eta, mapped it. Here we are going to have another approach, say for example, if this is the point P, then I can join this P with the coordinates say 1 2 and 3; 1 2 and 3 and call this area, this area here as A_1 and this area as A_2 and this area as A_3 and use A_1 by A, A_2 by A and A_3 by A as a natural coordinate system.

We will see that or we will expand this in the next class. How we are going to do it, we will see in the next class.