Introduction to Finite Element Method

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Lecture - 16

In this class, let us review what all we did in the last class, because there are some questions about certain things, which we did and clarify it completely. The first thing what we have done is to re write some of the equations.

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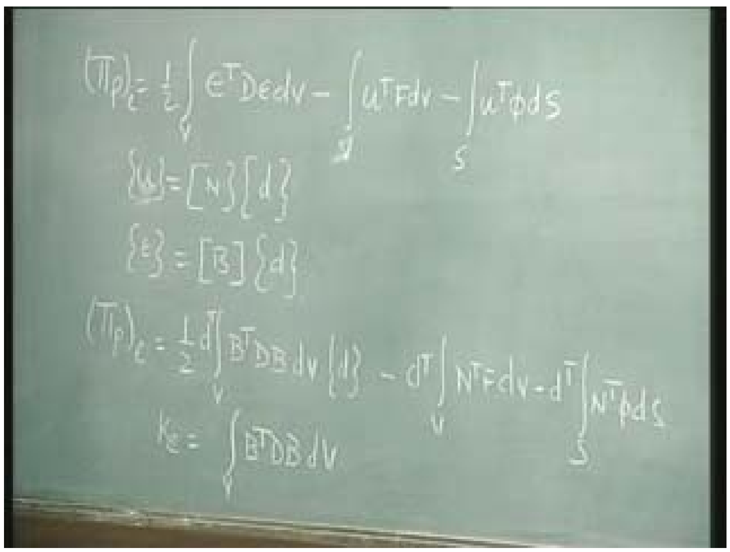

Only thing is I have removed the two terms which were there, the initial strain and initial stress which was causing some problems. Yes, that is bit difficult or bit confusing to understand from this point of view. In fact even what I had written there, instead of minus sign there should be a plus sign. But nevertheless I am not going to deal with it in this way. After completing it, I will deal with it again, the initial strain and initial stress in a slightly different fashion. So, as we were proceeding that day let us forget about the initial strain and initial stress and let us look at this equation.

What we did was we started with, say for example, let us write it down for an element. This is what we did. Then we wrote down the relationship between u. u is a function of x and d which is the displacement at the nodes and then what we did was to write down the strain displacement relationship epsilon is equal to Bd. This is what we did. Then what did we do? We substituted these two into my original pi_P equation. That is the next step we did and wrote down pi_P for an element. I am just calling it pi_P of an element, because so that you will not get confused about it; pi_P of element. I have deliberately removed the last term, which I symbolically added for the time being, the last term indicating concentrated loads.

Finite element especially has concentrated loads. Usually pi_P does not include it, because either we prescribe a surface forces or prescribe displacements in a classical sense. As I told you earlier in one of my classes that there is no such thing as concentrated loads. Is that clear? When we talk about usually loads, we talk about surface forces and body forces. Surface forces are like pressure. In other words what we mean to say is that there is no such thing as concentrated load. Even if you look at it as a concentrated load, it is an engineering approximation and that it acts actually over small area. But, in finite element we have to apply loads at the nodes and hence concentrated load also becomes or becomes initial or it is present in finite element terminology.

We will come to that when we come to the next equation, but before that let us have a look, without looking at the concentrated load. Now, look at this equation. What is that I have essentially done? I have substituted the previous cases or previous u epsilon and so on into this expression. Is that clear? Have a look at it. Till that it is clear. Now, what I have done is to write it down in a fashion with d, with small d, which indicates the local displacement or degrees of freedom and now I have defined what is called as k_e which is B transpose DB dV. Is that clear?

Now, the next step is what the confusion is. I would just say that again. I think after the class some people asked me, so, just look at this.

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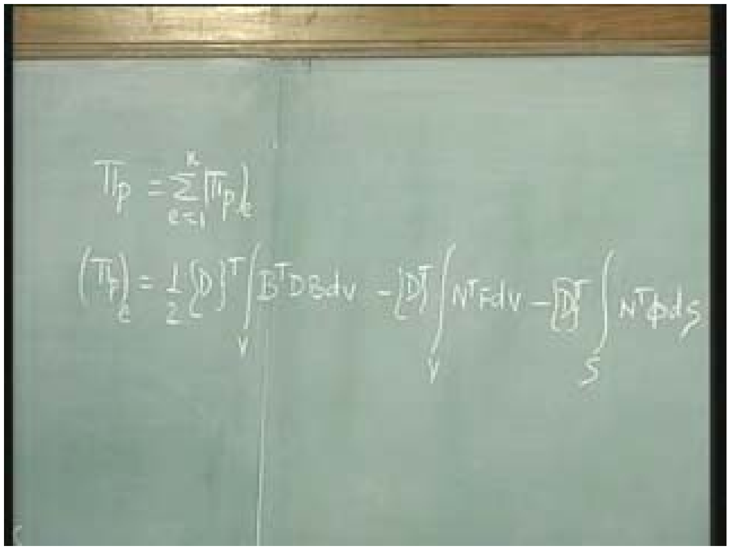

The confusion is what is the different between this D and this D? Please note that this D is a vector.

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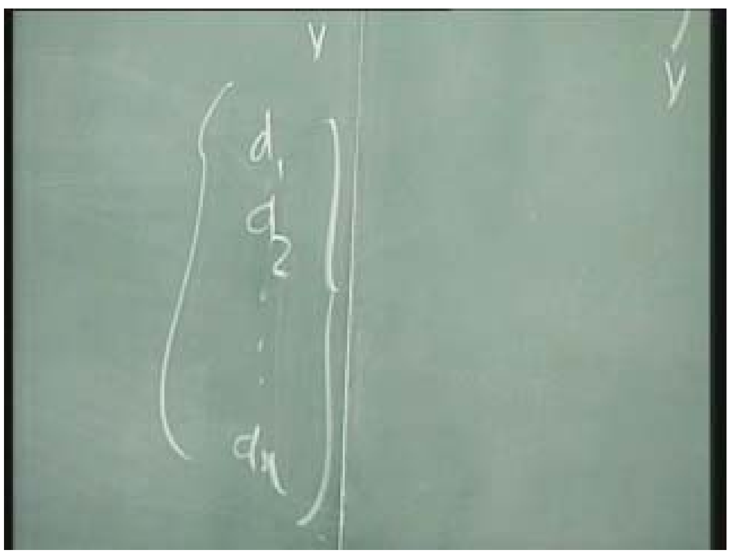

It consists of all the displacement that is say d_1 d_2 up to d_n, where n is the number of degrees of freedom of the body. If it is confusing, I will replace that d by another term, say for example, I will say U. But, now you should not confuse this U with strain energy, because this is a vector. I think the position will indicate to you what it is? But, many people get confused on it, because at that place it is obviously a vector.

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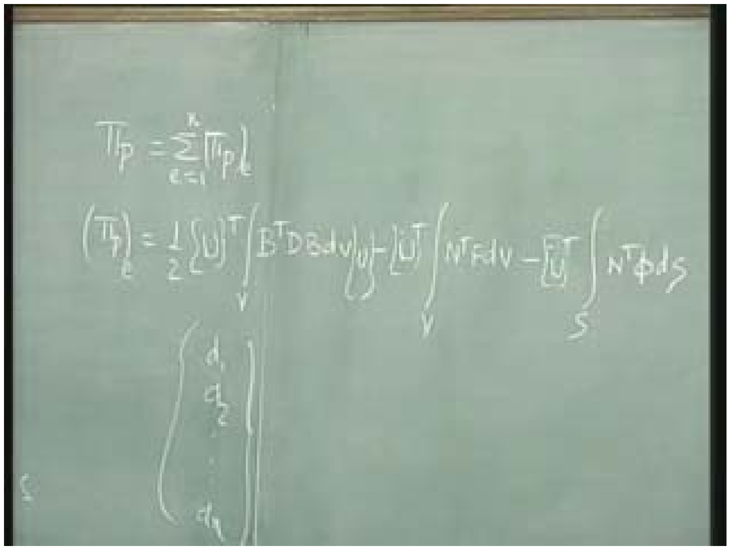

If you want I can replace this by means of U, that vector by U so that, that confusion is solved; here U, U and so on. So, all of them are now U. Now I hope your confusion is solved. Is there any questions? Is that clear? Now, what is that we do? We substitute this equation into this equation. So, substitute it into that equation. You will get now capital K. What is capital K? Sigma of k_e; sigma of k_e.

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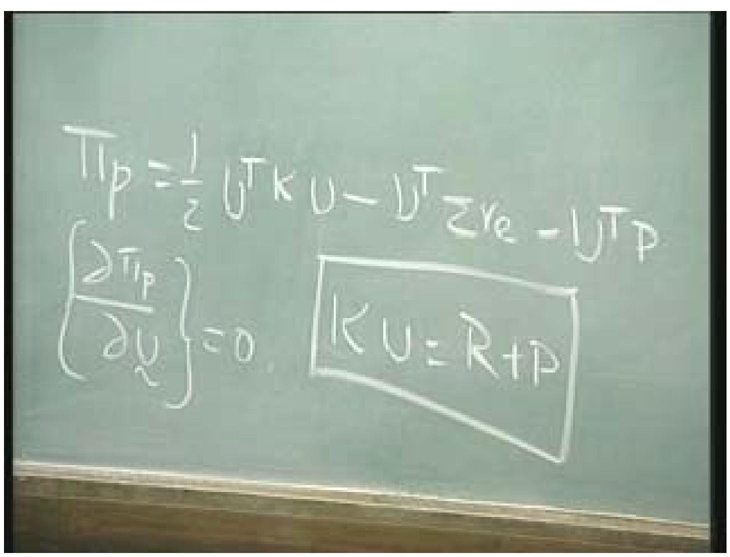

I can write down that pi_P, the total pi_P, to be equal to half D transpose K. Sorry, again, because I am so used to that I think; we just now changed and it is U transpose KU. It is U transpose KU minus U transpose. What is the next term? What we saw? Minus U transpose sigma of r_e's. r_e is the remaining term there; that is N transpose Fds minus N transpose sorry N transpose FdV minus N transpose phi ds. These are the two terms. Is that clear? Now, I will introduce my concentrated loads. How do I introduce concentrated loads? Minus U transpose, say, P. Is that clear? When I differentiate this dow pi_P now, I am differentiating with respect to U; that is equal to zero.

When I differentiate that matrix half U transpose KU that results in KU. I am not going to expand it and write it down, but may be, you can in your spare time just try that out. You can just expand it; you will see that half U transpose KU when you differentiate it, it will become KU. Is that clear? That means that KU is equal to R plus P. K is a matrix, U is a vector, R is a vector and P is a vector. Is it now clear, any questions on it? It is actually, differentiation is not with respect to a matrix. What it means is actually what I am doing is dow pi_P by dow U_1 that results in my first equation, dow pi_P by dow U_2 that results my second equation, dow pi_P by dow U_3 that results in the third equation. Is that clear? That is what I mean by this statement. Look at the statement here. What it means is that we are differentiating it with respect to each of the U's, U_1 U_2 U_3 and so on.

When I differentiate it with respect to U, each of them should be equal to zero. How? Please note in our earlier classes we had said delta pi_P; del pi_P. The del operator, del pi_P is equal to what? Dow pi_P by dow U_1 into delta U_1 plus dow pi_P by dow U_2 into del U_2 and so on. Is that right? So, each of these terms, dow pi_P by dow U_1 should be equal to zero, dow pi_P by dow U_2 should be equal to zero and so on. Any questions, is it clear now? So that is what we mean. When we differentiate like that ultimately, I will get KU.

What I suggest is take just 2 by 2 matrix for K. Just write it down and just differentiate; you will see that ultimately you can write it down in KU. That is what we mean. The simplest way of looking at it is to just expand it and differentiate it with say, with respect to U_1, write down the first equation; differentiate it with respect to U_2, write down the second equation and so on. Is that clear? KU is equal to RP, R plus P. What is that we have done? We have reduced; ultimately, we have reduced a very complicated differential equation into a simple algebraic equation. These forces that we have developed are what are called as consistent nodal forces. Please note, though it was a pressure term, though it was a volume, though it was the body forces were acting over a volume, all those things were there. Ultimately, when I reduced it to this equation, what is it that I have done? I have reduced them to equivalent nodal loads. But I have been very consistent, in the sense that I have used the same shape function in order to reduce them.

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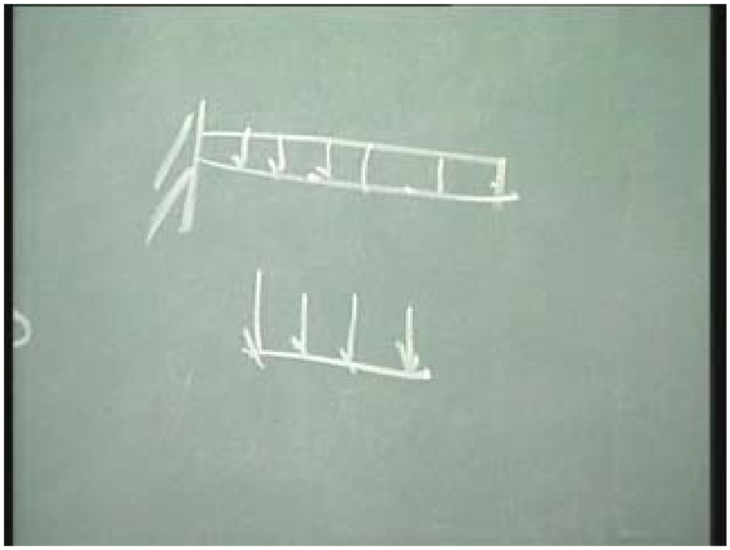

It is not that. Suppose there was a beam and I have elements like this and I had forces acting here like this. It is not that I calculated the total force and divided it and put it on the nodes. No; that is called as lumped nodal forces. You just calculate the total load that acts. Suppose you take one element, calculate the total load that is acting, just divide it by 2 and put it. These are called as lumped nodal forces. They are not the correct way of doing it, because they are not work equivalent of the original loads.

What I have done is very consistent with the theory. Hence it is called as consistent nodal forces. Is that clear? This kind of lumped thing works in one or two situations. Yes, it will work, but for example, in beam it will not work. In higher order elements, I already defined what higher order element is, this does not work and so on. You will be in for surprises, if you just lump the loads on to the nodes. We will work out a problem later in the course, but right now, say that these are all consistent loads. Is it clear, any questions? They are work equivalents. That means the work done is the same whether by the distributed load or by this concentrated loads. It is very obvious from what we have done, these are all work done. The same thing can also be arrived at, what all we have been doing can also be arrived at, from virtual work principle. The same, whatever we have done, can be arrived at from virtual work principle.

The principle of virtual work becomes important because, what is the reason? Why is that it is important from one point of view? What is that? What is the difference? Yes, that is it. In other words, here the functional has to exist; in other words, more mathematically, functional has to exist. For example, functional we interpreted it as potential energy. As I you told you in earlier classes, it is just giving some camouflage to mathematics. You can interpret it as potential energy, because we have to define strain energy and strain energy is defined only for certain classes of materials. What is usually said is, if the differential equation has say 2m, 2m as the order then usually there is a functional that is defined. If it is 2 m plus 1, then you will not get rise to these kinds of very nice equations and so on.

Before we go into that kind of theory, let us see how simple or how difficult it is to get these equations from virtual work principle. Is it clear? The first step that you have to do is to write down whatever we have done in terms of virtual work. Let us see, how you do that in terms of virtual work principle. Is this clear, as to expansion of this term, u term? Is it clear? Can you start deriving it from virtual work principle? What is virtual work principle? Yeah, external virtual work is equal to internal virtual work. So, start doing that. Let us see, I will give you some time. I will also write it down, but it is nice to look at it yourself, so that some of the doubts that you have may get clarified.

Go back and refer to what is the virtual work principle and then just write it down. What will be the internal virtual work?

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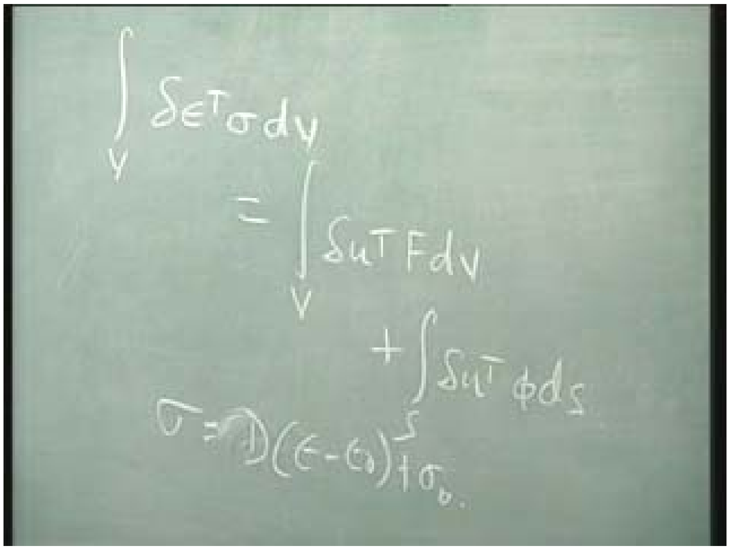

Is that okay? I write this in terms of delta epsilon transpose sigma dV. What is external virtual work, what is external virtual work? How do you write it? What are the terms? There are body forces delta u transpose B or sorry F dV and then what is the next term? Delta u transpose phi ds. You write it down for an element and then we will expand it. Now, substitute for all the terms that you know. On top of it, do a small juggling, which I am going to say. Let us define sigma is equal to some, sorry, yeah, D into epsilon minus epsilon_0 plus sigma_0. Let us define sigma to be D into epsilon minus epsilon_0 plus sigma_0 where sigma_0, let us define it as an initial stress that exists.

Let us now define epsilon_0 as an initial strain. I am going to write it in this term. When I write it like this, you have to be careful when you evaluate what is sigma_0 and what is epsilon_0? Let me write down sigma is equal to D epsilon minus epsilon 0 plus sigma_0. It is much easier to interpret this in virtual work principle, hence one of the reasons why I have removed it from yesterday. Now, substitute this into this expression, concentrate on it and I will give you one minute to develop this equations and tell me whether you get any difference with this and how does the terms appear. What is that you have to do? You have to substitute for all those terms delta u transpose, delta epsilon transpose and so on. Now you have to tell me, what is that you have to tell me? You have to tell me what is r_e? See whether K, there is a difference? What is r_e now, having introduced this term. That is what I want you to tell.

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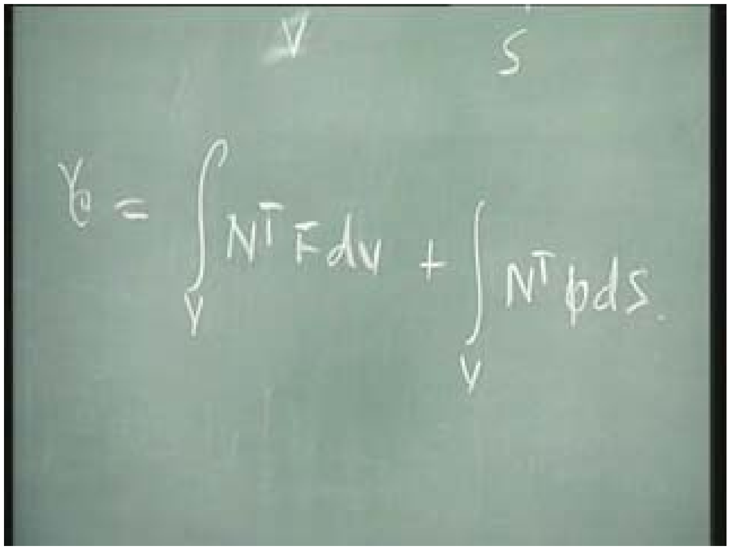

r_e in my previous case, without introduction of that is given by N transpose FdV plus N transpose phi ds. Is that clear? See whether you have got the same thing plus any additional term because of the introduction of sigma. What happens to now, that epsilon$_0$, T epsilon$_0$? See, what is that we had initially? If you do not have, it is just D epsilon. Now, what is that you have to do here?

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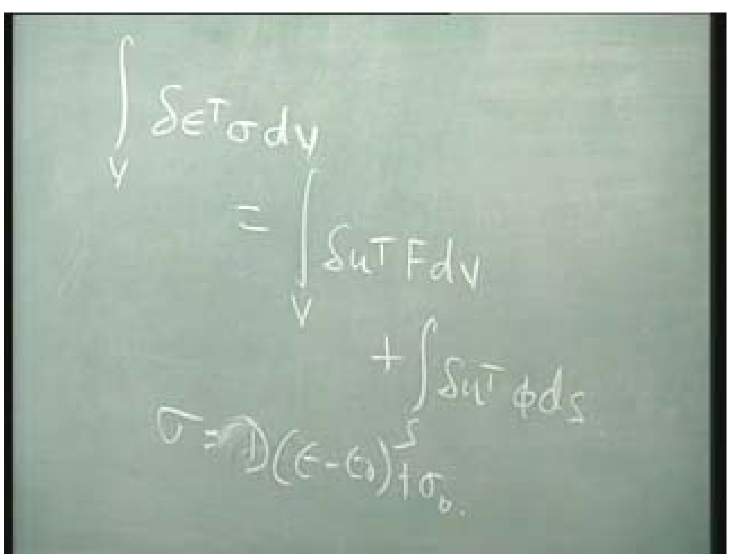

You have to substitute for sigma from this equation. That is what you have to do. That is substitute for sigma in this expression from this equation. What is that you get out of it? You get two additional terms. Is there any confusion? I thought it is quite simple to derive, because this is a very good practice, because you can later formulate your own finite element formulations without much issues. Once you know this, how to do from this point of view, you know you have a small practice, then you will be very confident to develop your own finite element. There will not be any problems. That is why I am insisting that you also look at virtual work principle and then do it. Any confusion or just you need time?

What is that you do now? Sigma; sigma you replace it by means of D epsilon minus $epsilon_0$ plus $sigma_0$. Forget about what you did as far as this initial stress strain in the last class, because this gives you a much better picture and please note the way we have written this equation. That is how usually normally books write. There is an initial stress which is added, initial strain which is subtracted from the current strain. So, any physical situation you have to interpret it in these terms, in that terms. Please note all of them are tensors or rather corresponding units are there; if sigma can be written as a vector, it depends upon- ? I mean, I can help you out. Sigma is what? D, say for example, time being write down, I mean forget about those two terms $epsilon_0$ and $sigma_0$. Sigma is equal to D epsilon; epsilon is equal to BU and delta epsilon transpose is equal to D transpose or delta D transpose B transpose. So, what is that you get for K? Where is delta? What is that you will get for K?

Yeah, there is no minimization here. I mean this is just internal virtual work is equal to external virtual work that is all. What is that you get? Delta D The equation will be delta D transpose B transpose DB d small d. B transpose DB is what? You know, you should just get it into your system B transpose DB, what is that? K; that is it. So, you get the stiffness matrix straight away from the first term; any confusion? What is the second term here? Delta u transpose F. What is delta u? N delta D; so, transpose of it, delta D transpose N transpose FdV. This term becomes now, delta D transpose N transpose FdV. Is that clear?

What is the last term? Delta D transpose N transpose phi ds; period, very simple. Now look at that equation and see whether what you get is the same as what you got from the well known or what we derived initially; I should not say well known, but what we derived initially? Is there any difference? There cannot be. Yeah; now substitute for $epsilon_0$ and tell me, substitute for $epsilon_0$ and let me know. What will happen to those terms? What is that you get for $epsilon_0$? You substitute $epsilon_0$, so you will get B transpose D $epsilon_0$. so that will be when I will take it to the right hand side so that will be plus. So, there will be a plus term for the initial strain. Is that clear?

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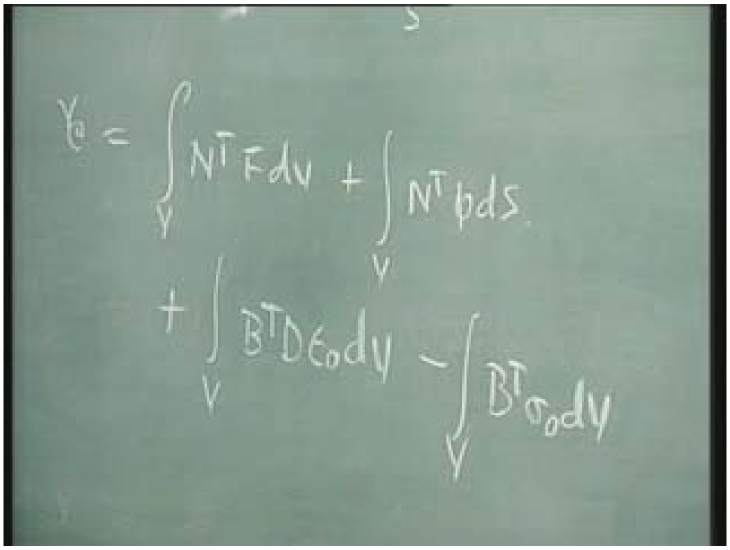

So, I will get plus B transpose D $epsilon_0$ dV. This will be for the initial strain. What will be for initial stress? Look at that and tell me. It will be just B transpose $sigma_0$ dV. That when I take it to the right hand side will become negative. So, minus integral V B transpose $sigma_0$ dV. Is that clear? These two are added for r_e. These are all on the element level. When I want to expand it, you go through the same thing; I am not going to repeat it. Only thing is delta D transpose, when I have to have the virtual work for the complete body, then I will introduce delta u sorry delta u transpose P for concentrated loads. Is it clear? So, I will introduce delta u transpose P as well, so that also should be taken into account. In other words, what we have got is exactly the same as that. It is exactly the same as that of the potential energy, but one more thing you would notice is it is much easier to derive. It is very simple, very straight forward to derive and so on. Is it clear?

Having solved all the problems, now you know how finite element is obtained. The only thing you may be worried is, Oh God! You have lot of integrals; volume integrals, surface integrals, you do not worry about it. The things are made much simpler in later classes, when we deal with some class of finite elements. The first thing people get worried is looking at all those integrals there. You need not worry about that. Things are made much, much simpler though you have to do some integration, simple integrations. So, you need not bother about it and now what I am going to do is to ask you to do exactly the same problem as we did before, as we did before in terms of or in terms of finite element method. What is the problem we did? We applied uniform load, Cx. That problem I want you to do it. Start attempting that problem. It is a very interesting result that you are going to get, so, that much I can assure you.

Let us see how you do this problem. But I want you to, what is that I want you to do? I also want you to derive this stiffness matrix yourself. Please do not use the previous stiffness matrix. See, what we arrived at last time was by using direct stiffness matrix method, direct stiffness matrix method. But, now what I want you to do is to arrive at the stiffness matrix from this potential energy or for that matter virtual work principle. It is much easier to write it in another term. How do I write it? Please note that U can be written as half d transpose Kd. So, K can be recognized directly if you write out u. That is my first clue. My first clue is that you can write down u in terms of half d transpose Kd.

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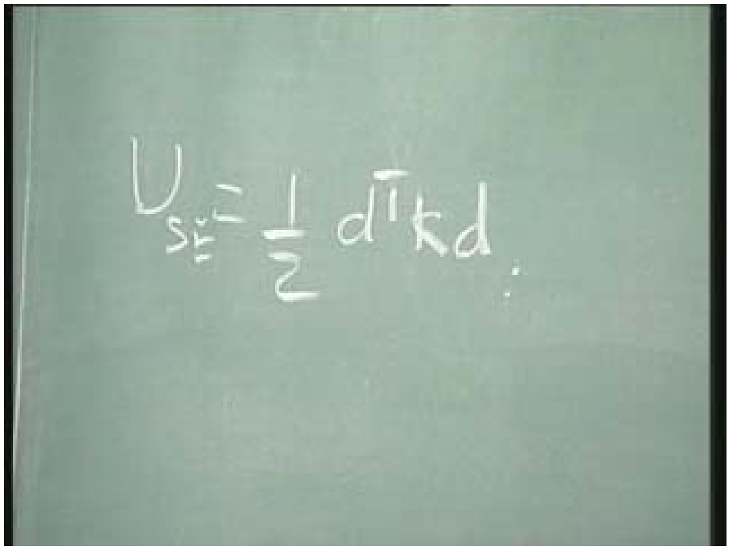

So, develop k_e; U? You tell me another symbol, we can use it. Can I use w for this, I mean? U_{SE}, you want to use like that. Yeah, but please note the major difference. What is the major of difference? What is energy? Yeah, so, it is simple. You know it is scalar and the other one is a vector. Any way, I take your word. I will write this down as strain energy. Usually after some time you will get so used to it, you will not get confused because you will know what comes at what place; small d capital D, yeah I know, I understand, because I have used a to z, capital as well as small. So, that way I know you are getting bit confused, but fine, I will write it like that for your convenience and later you will not have any problems.

Now, I want you start with the strain energy definition for a bar and then arrive at K. Strain energy dentition of a bar, I think we have already written it in the last class. So, you can write it down as half epsilon transpose, now, you can write it same way. Epsilon x transpose, I mean though it is half epsilon, the E epsilon squared you can write it as epsilon_x transpose.

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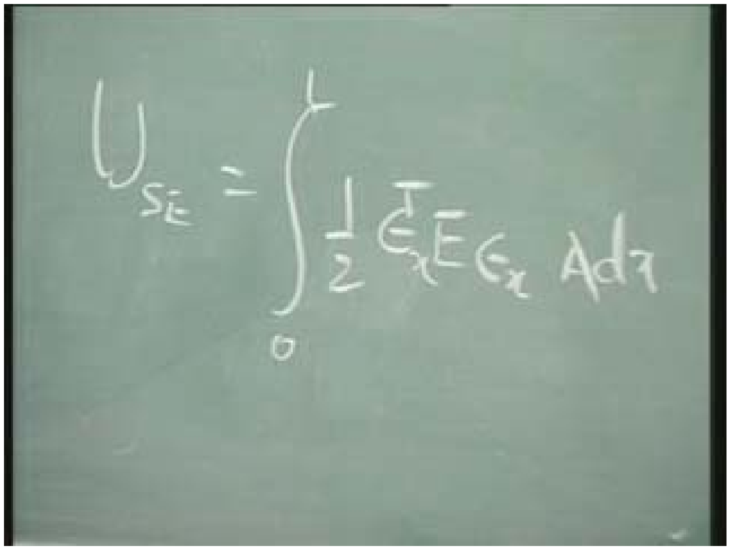

Just for symbolically like the same way we write E epsilon A dx; zero to some L, L is the length, half epsilon; epsilon, let me call this as epsilon transpose A dx. Yeah, any doubts, any questions? What is now, in this case, B matrix or D matrix, within quotes? What is D matrix? AE, because dV we had written. So, AE will be the D matrix.

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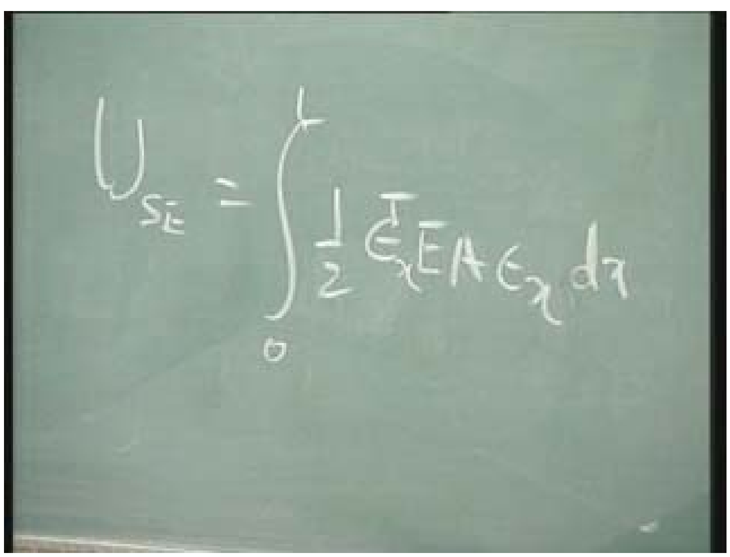

So this can be written as, AE epsilon$_x$. That is why I want you to develop it, so that it will be very clear, you will not just copy it down; you will have ….. That is U. This can be written as, how can you write k? This is so logical and simple. I think this is the most important part in finite element and that is why I am spending so much time so that you get it into your system. Once you do it for one element, it is similar to all the elements. So, I can write k to be AE. What happened to AE, what happened to AE?

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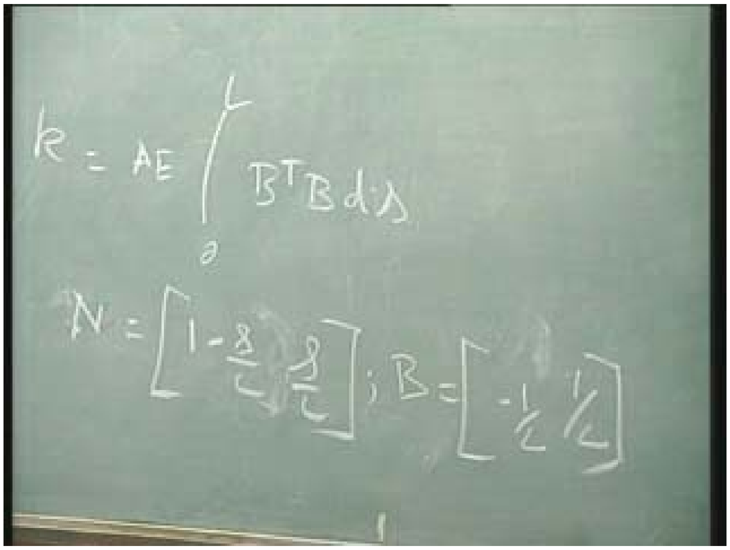

AE that is B transpose; AE is a constant, so, I can take it out, integral zero to L B transpose B ds. N, I have already developed, what is N? What is N? Yesterday we developed it. What is N? Sorry, sorry and s by L. from which from Now dx; please note what is dx? dx is the local coordinate. If you want, we can even, then again if there is a confusion, I can replace it by ds; your convenience, because we had written it in terms of s, so ds. What is B matrix? Dow of N. Dow here is, that dow operator is now, dow by what is that? dow s, correct. That is it. So it is B is equal to minus 1 by L 1 by L. period. B is equal to minus 1 by L 1 by L.

Substitute it, tell me whether you are able to get the terms. Completed? You should do along with me. so that when I say You try it out then only you learn these things. They are very logical. One of the reasons why I am asking you to do it is, because it is very logical to develop it. It is not a big problem to develop it. What is that you get? How many have completed it?

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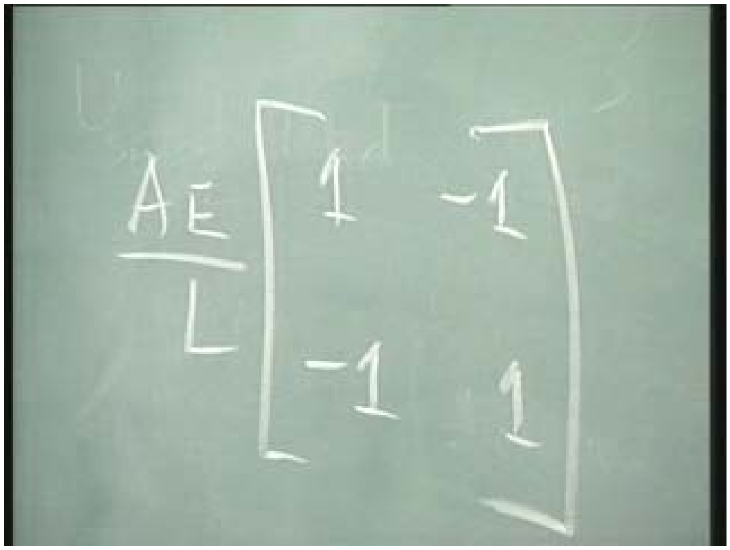

You know the answer anyway, I am writing it down. If you have a doubt let me know. That is simple but I am sure you do not know what r_e is. That is where you are going to have real, I should not say trouble, but real work. Let us get back to my old figure.

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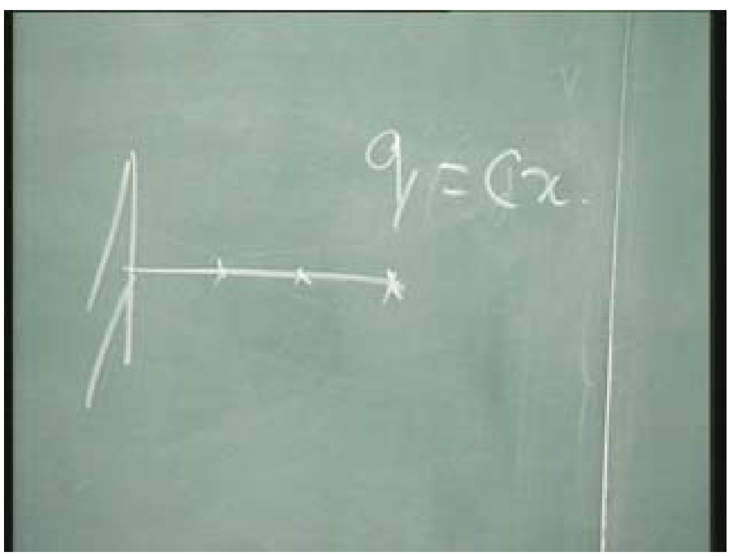

Let us say, I divide this into, how many elements will you divide? Say, 4 elements or 3 elements, so, 1, 2, 3, 4 nodes. What is r_e? In this case, what is that you have? You have q. q is equal to Cx, I have put. q is equal to Cx. But then, there is a small trouble.

That is, what I have used. What is the small trouble? What is r_e, by the way, for this problem? What is r_e?

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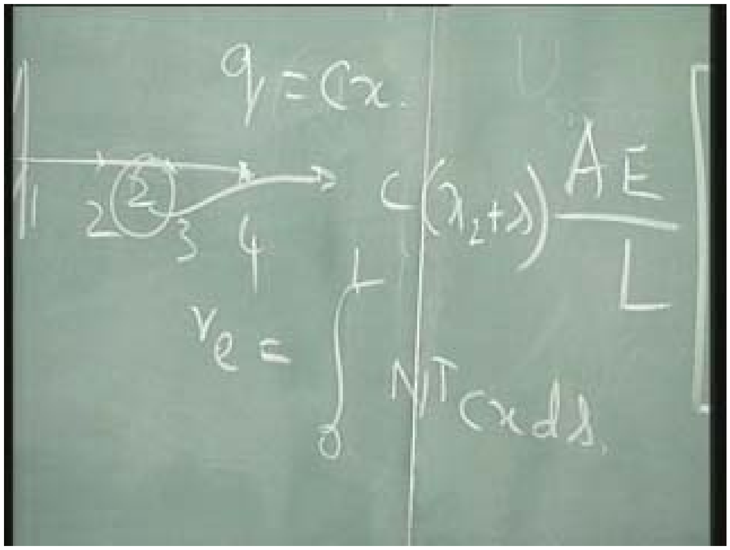

Yeah, integral, v is what in this case? Zero to L, depending upon how C is defined, so, Cx. We have done the similar thing last time in Rayleigh-Ritz procedure; C into x. There is a small difference between this x and s. What is the small difference? Take, say element 2. How will you write Cx? Let us say that we take element 2. How will you write Cx in terms of our terminology? For 2, how will you write Cx?

Yeah, look at that. C into, C into, say let me call this is 1 2 3 4; C, correct; that is it. So, C into x_2 plus s. We are using s, so, C into x_2 plus s. Let us not multiply by A because C takes care of that. Let us say that it is zero to L Cx ds; N, transpose cx ds let us write down this like that, where x, I should not write it as x.

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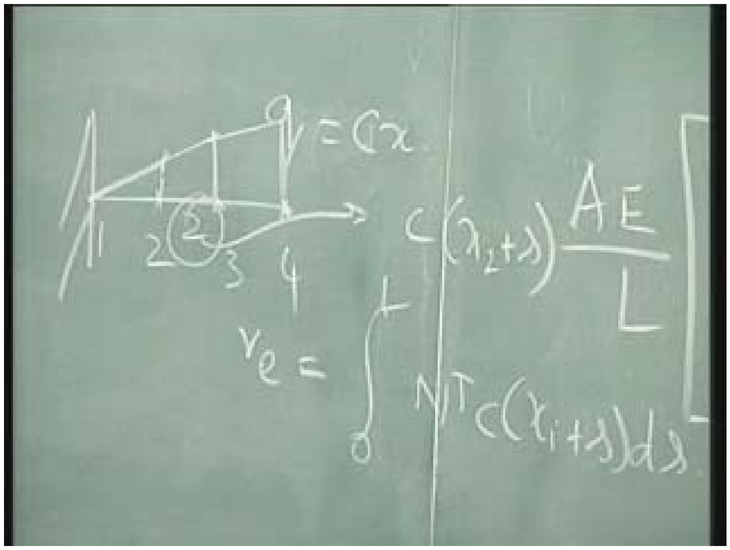

Now, that becomes C into, say let me call this as x_i, x_i plus s; C into x_i plus s. Yeah, fine that also you can do or you can write it x_i plus s. What you should remember is that this term starts from zero. If I plot it the other way, it starts from zero, go to this place, then goes like that, goes like this. When problem definition is given, I am giving the problem definition, so, I am not bothered whether you are using finite element or any other thing. I am only saying, for me, I have a, please note, yeah that is a good point that you brought out. There is the difference between the global coordinate system and the local coordinate system. Nicely you have brought it out, because x is now, as far as I am concerned, I am not bothered what you are going to use. As far as I am concerned, I am giving you a problem as an engineer for you to solve it. I will look at it only from a global coordinate system. In this case, I will say q is equal to Cx. That is all.

s is a local coordinate system, always finite element has a local coordinate system. The local coordinate system, I start s is equal to zero from left hand end and then go to s is equal to L on the right hand end. Hence there is going to be a transformation if at all from the global to local and local to global and so on. What I have essentially done is to say that it is C into x_i, the first one plus s. If it is two from second node from two and so on. For the first element it is actually Cx.

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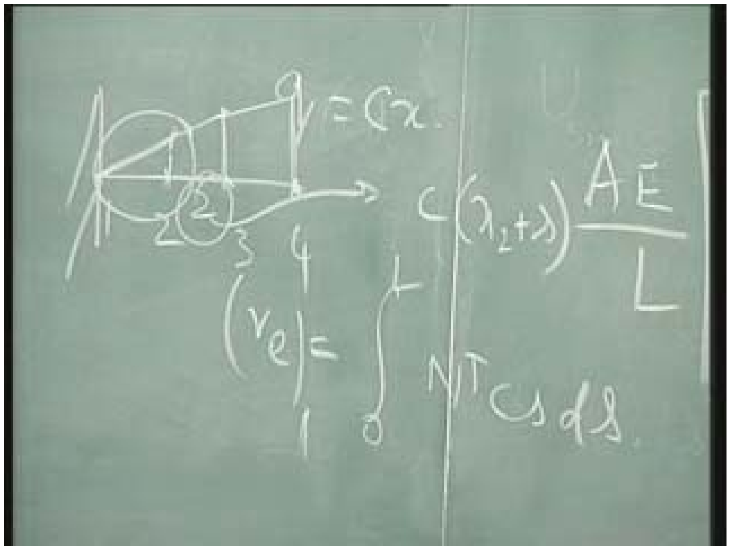

That is Cx for the first. Suppose I want to calculate r_e of 1, then it is that. Is it clear? Yeah, any questions, any confusions? I want you to follow closely every step so that you will not have any doubts. That is r_e 1. Zero to x_1 that is the length of the first one, this is the length. Let us say that the lengths are the same. x_1 is say L by 3, sorry, L T by 3, total length is say, LT. No, no, no; I am only talking for the second element. I am now looking at the first element. I am only looking at the first element. For first element it starts from zero. Look at this. This is for the first element. It is just Cs, zero to L, L is the length of an element; zero to L.

Let me say that the total length is say LT; I am dividing it into three parts, so zero to L. Is it clear? In other words, this I specifically wrote for the second element. For third element my x will become x_3. Now, I am only looking at the first element. Now, I am just substituting this and then I am getting this r_e. Is it clear? What you should also recognize at this stage is what small d is and how it is related to capital U, which is the vector. 1 2 is what I have written down; 1 2 3 4 as the node numbers, that I have written down. So, small d is just 1 2. Capital U which we used in our previous expressions is 1 2 3 4. So, r_e 1 will give me entries ultimately only for 1 and 2. This N is related to only 1 and 2. Is that clear? 3 and 4, corresponding 3 and 4 will go to zero when I expand it from small d to capital D.

Let us see, write it down. So, that will actually give rise to CL squared by 6 1 and 2; so, CL squared by 6 1 and 2.

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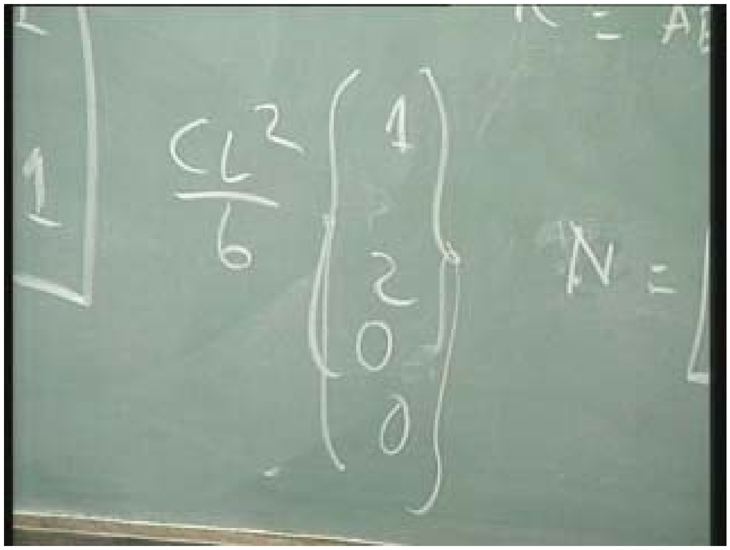

That will be the r_e. Please note, N transpose, so, it will become 2 by 1; 2 into 1 matrix and so it is a CL squared by 6 12. This was the confusion all of you had in the last class. What do you mean by expanding it to zero, sorry, expanding it to U? What do I do? I have just put zeroes. Now, what I have done? I have expanded. As far as the first element is concerned, I have expanded it to all the degrees of freedom. What does it mean? It means that I am now, what is that I am doing? I am building this element.

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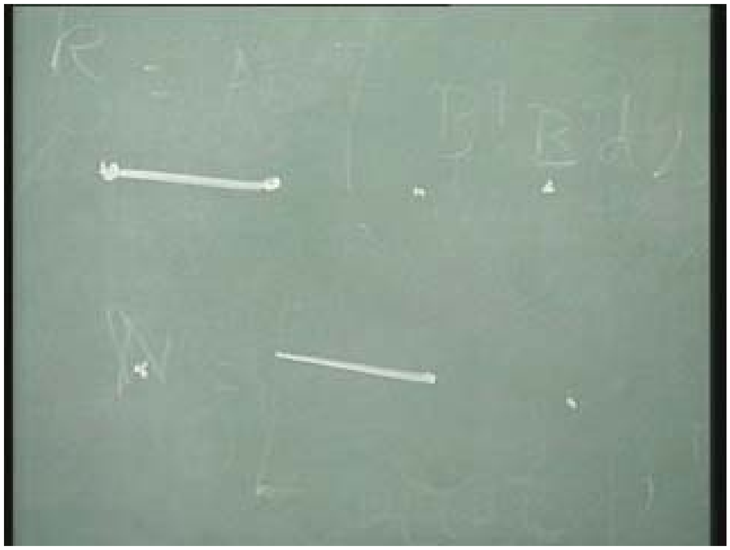

I am just putting this element here. I am just saying that there are nodes there. But this element does not contribute to forces in the other nodes. Other elements are there. That is not my job; that is not my jurisdiction. I have only one element and these two nodes that are present will be taken care of by other fellows. Then, I will put the second element here and I am saying the second elements jurisdiction is only say 2 and 3; the other fellows will be zero and so on. That is what I meant by expanding it completely to U. Does your doubt now become okay, is it clear? Just have a look, CL squared by 6 1 and 2.

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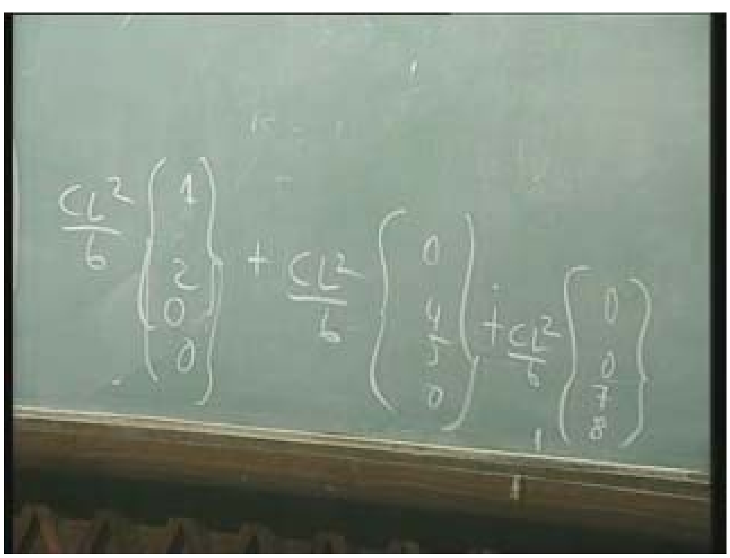

What is the use? Again I am repeating what is the use? So that, you can add now together r_e's, period; that is very simple. What is my second? For the second element, it so happens that it is CL squared by 6 4 and 5. How do I now write it completely? Plus CL squared by 6, just confirm it, 0 4 5 0. That is how I sigma I sum up r_e's. I think the last one is CL squared by 6 0 0 7 and 8. Is it clear? Now, the same way, No questions, clear now, this assembly? Now in the same way, you write down K also to assemble. How do I now write it down?

Let us get that to only this K. All these things we will remove, we will get back to only this K.47

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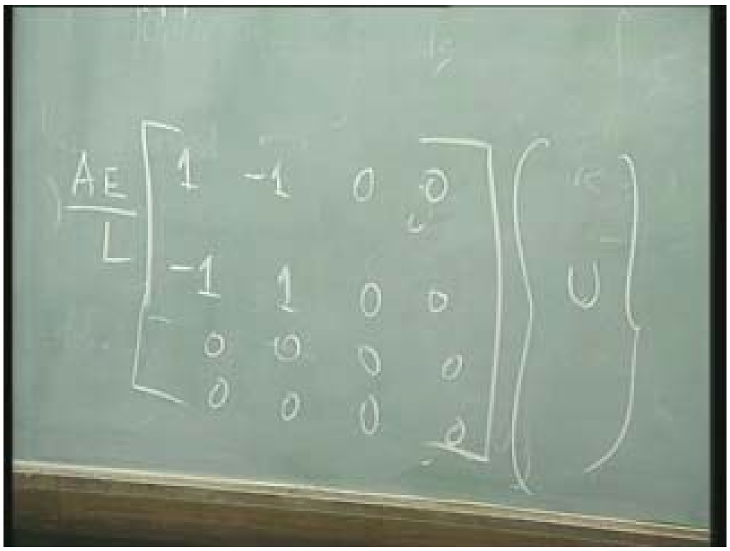

How do I now expand this completely? 0 0 sorry 0 0 and then 0 0 0 0 0 0 0 0, each of them corresponding to this capital U; U_1 U_2 U_3 U_4. The next one, the next fellow will have entries 1 minus 1 minus 1 1 and so on. Third fellow will have entry 1 minus 1 minus 1 1. Now, it is easy to add or else you cannot add, because each of them corresponds to different degrees of freedom; you cannot add it. You cannot argue that it is all 2 by 2, why not I add? Obviously you cannot add. Now, you can add it and get KU is equal to R. We will do that in the next class. Meanwhile, I would like you to I am going to write down only the final result in the next class. I would like you to work it out separately, so that there is no confusion.

Let us stop here. Before that if any question pertaining to this class you can ask me right now or we will discuss it in the next class also. We will meet in the next class.