Introduction to Finite Element Method
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Lecture - 15

In the last class, we saw the Rayleigh-Ritz procedure. As I told you Rayleigh-Ritz procedure is a forerunner tool in finite element analysis and I hope you understand. If there is any question, I can answer that. Yeah; one of the questions that was asked is how is that a_0 became zero? Because, if we remember we started with u is equal to a_0 plus a_1x plus a_2x squared and so on. Why did you make a_0 is equal to zero? In that procedure, it should be such that the displacement boundary condition that has been given, which is an essential boundary condition, should be satisfied by that particular polynomial. In other words, at x is equal to zero, u has to become zero, which means that a_0 becomes zero.

Now, we move over to the finite element analysis. What is the difference between finite element analysis and whatever we did? The difference is very simple. In the Rayleigh-Ritz procedure we considered the complete body. If you remember what we had put down, we had just put down this and then we said that we will consider it as a complete body.

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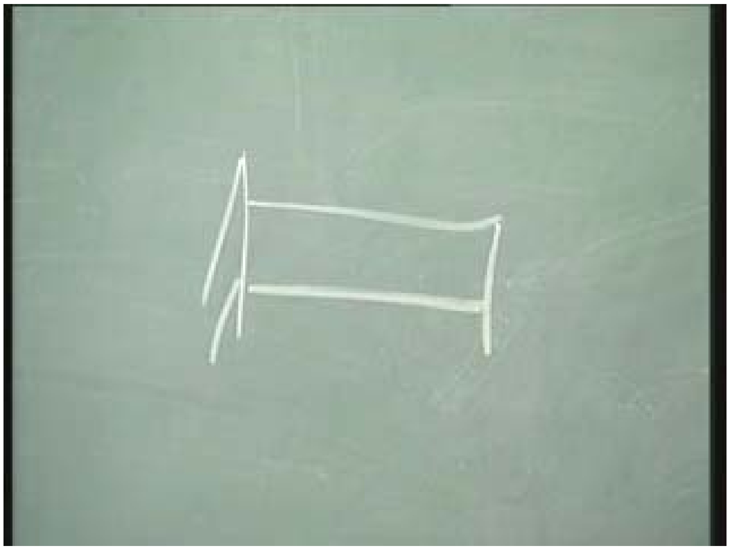

In Rayleigh-Ritz procedure, we had the body to be taken as if it is one element. But, what we are going to do in finite element analysis is to split this body into number of elements. In other words, we will proceed as if you are going to apply this procedure to more number of elements. Is it clear? That is what we are going to do, in this case. Before we proceed, let us look at how to write down the potential energy for a body. The potential energy term is written down completely like this. You will see lot of terms here. Have a look at this whole thing together and I will explain each of them very carefully; each of them a very carefully. Now, look at those terms.

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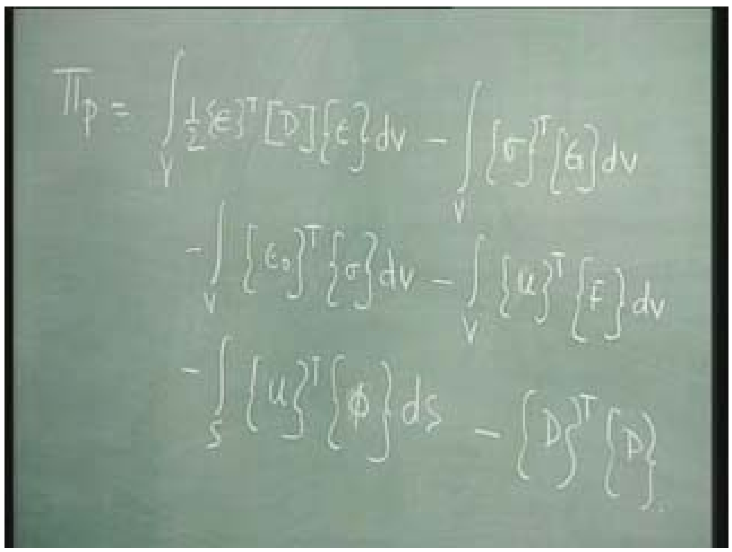

You have strain energy term. Please note you are used to writing half sigma epsilon. Now, what I have written? Essentially, for a multi axial case, I have written it in terms of a matrix notation, epsilon transpose D epsilon. Our old E now takes the value or takes the shape of D. Is it clear? So, the first term indicates in this particular expression, the popular strain energy term you all know. Look at the second term. You have now, a new epsilon zero, a new term called epsilon zero. In this case, what we have essentially done is to consider a body with an initial strain and an initial stress.

Let us see what happens to a body when there is an initial strain and an initial stress? Epsilon zero is an initial strain and sigma$_0$, which comes to the next term, is the initial stress. Is that clear or other words, what does it mean? It means that the body has some strain due to, say, assembly or fitting or whatever it is, some initial strain. It can be due to, say thermal stresses, this kind of thing can be present. So, sigma_0 and epsilon zero indicate the state of the body, when you start loading them. Or in other words, they are something external; due to something external that has happened to the body.

Now, look at this term sigma transpose epsilon zero. Look at the second term sigma transpose epsilon zero and look at this third term epsilon zero transpose sigma or these two terms are sorry, I think it should be sigma_0 transpose or epsilon transpose sigma_0; sorry.

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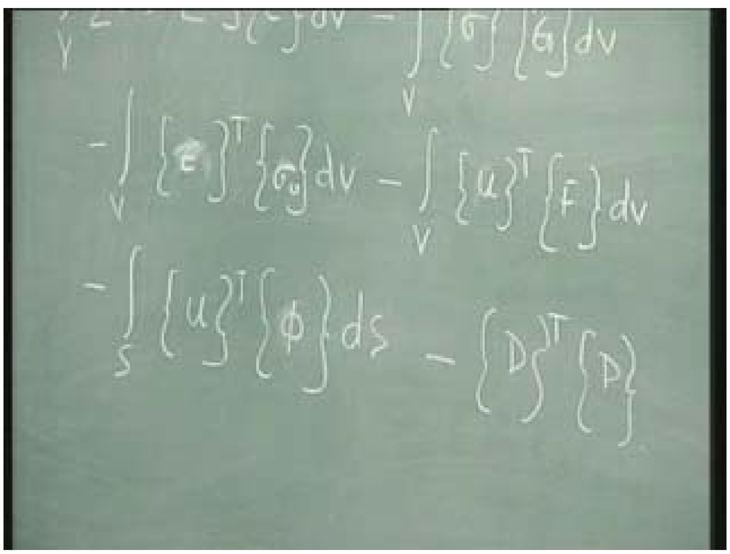

Yeah; because epsilon zero and sigma_0, I have just repeated that. Look at these two terms. What does it mean? It means that when I start loading the body, because of the pre existence of the strains and the stress, they start doing work and hence these two, there are minus signs. These two terms come from initial strain and the initial stress. Is that clear? What is the next term? What is the next term? Due to body forces? Please note these two are different from body forces. In the previous problem itself, you would have noticed that what we were specifying as, what did we specify, Cx? C is force per length squared and so on, was nothing but a sort of a body force that we were specifying. We are specifying throughout the volume or throughout the length of the body.

Now, what is this term, here? What is this term here? That term is due to surface forces. It is the work done due to the surface forces. So, you would see all those minuses there. What does it mean? It means that the potential for these external loads to do work is now getting lost. Is it clear and look at that last term. Last term, I have just added it now, specifically; may be you would not have seen it. I have just added it here, because we are going to also meet concentrated loads in a finite element situation. There are going to be nodes and you may put a concentrated load at a node. That is why, I have put that. The last term, I have added it. The last term depicts the work done by concentrated load.

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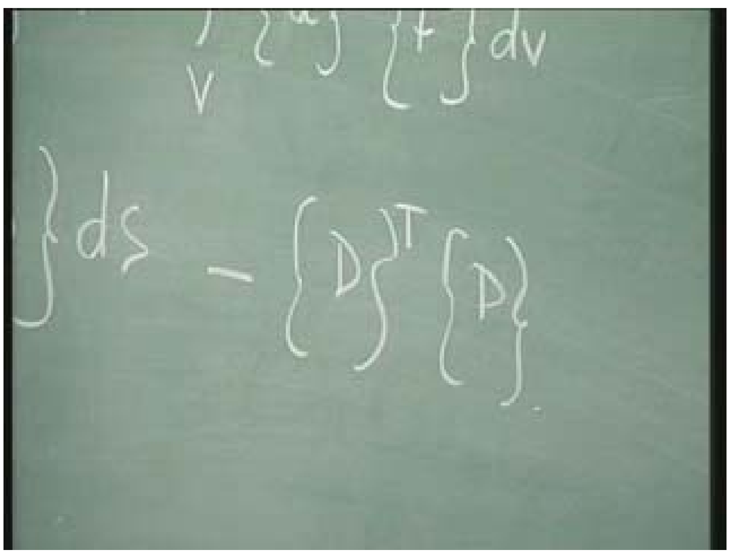

This completes the definition of potential energy for a body, when it has initial strain and initial stress, surface forces, body forces, concentrated loads and apart from an elastic type of constitutive equation. What is my goal? My goal is to now develop or reduce this into an algebraic set of equations. How am I going to do it? Let us see how we are going to do that? Please note that in my previous derivation on Rayleigh-Ritz procedure, what is that I did? I had put down a degree of freedom as a_1 and a_2 and so on. Now, what I am going to do right away, is to swap this a_1 and a_2 with u_1 and u_2 or u_i and u_j, to be more general. Is that clear? How am I going to do that?

The first thing that I am going to do is to take an element, say ij. Let us take that element ij.

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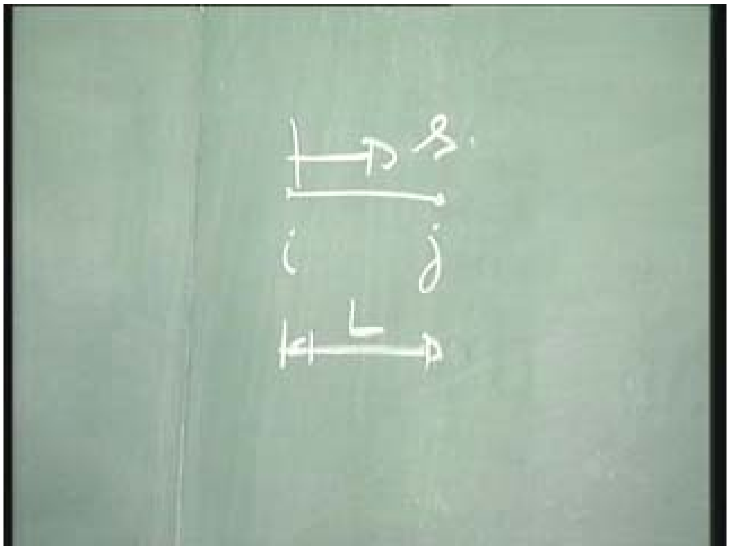

Let us say that the length of this element is equal to L. Now, I am going to put a local coordinate system to it. Let me put a local coordinate system, such that it is depicted by s. What does it mean? s is equal to zero at i and s is equal to L at j. Is that clear?

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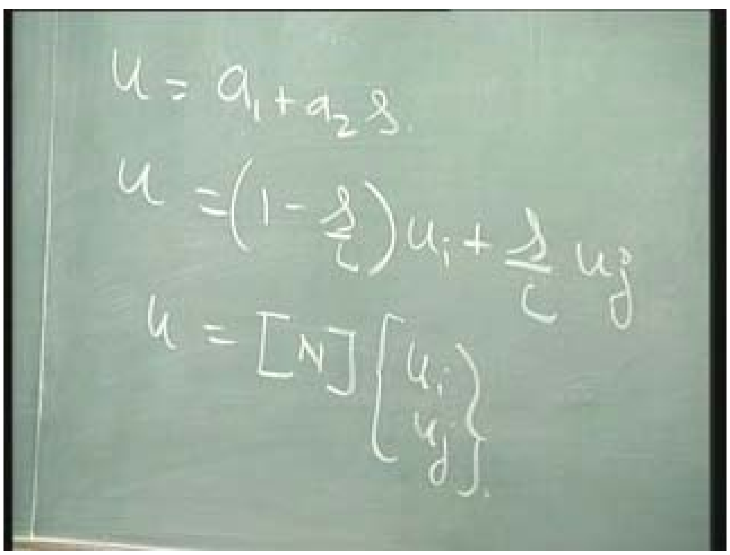

Let me now write down an expression for u. How do I write an expression for u? u is equal to a_1 plus a_2s. Now, what we are going to do is to replace a_1 and a_2 in terms of u_i and u_j; u_i and u_j, within quotes, known values or in other words, what we are going to solve? u_i and u_j are the ones which we are going to solve. You already know a procedure for solving them and so on. So, I am going to replace a_1 and a_2 by means of u_1 and u_2. Till now the procedure is the same. Please note here that I am not going to put a_1 is equal to zero, because there is no boundary condition here. We are treating a separate element, we are going to put the boundary condition later only. Is that clear? So, I am going to put first u is equal to a_1 plus a_2. Can you now substitute u_i and u_j and give me an expression for u? I will give you one minute, so, let us see whether you do it fast?

Let us see how you are going to express u in terms of u_i and u_j? What is that you are going to do? You are going to put s is equal to zero, u is equal to u_i. s is equal to L, s is equal to L, what happens? u is equal to u_j. So, you can express this as 1 minus s by L into u_i plus s by L into u_j. Is that clear? Let me write that as N u_i u_j. What is N? N is nothing but 1 minus s by L; s by L.

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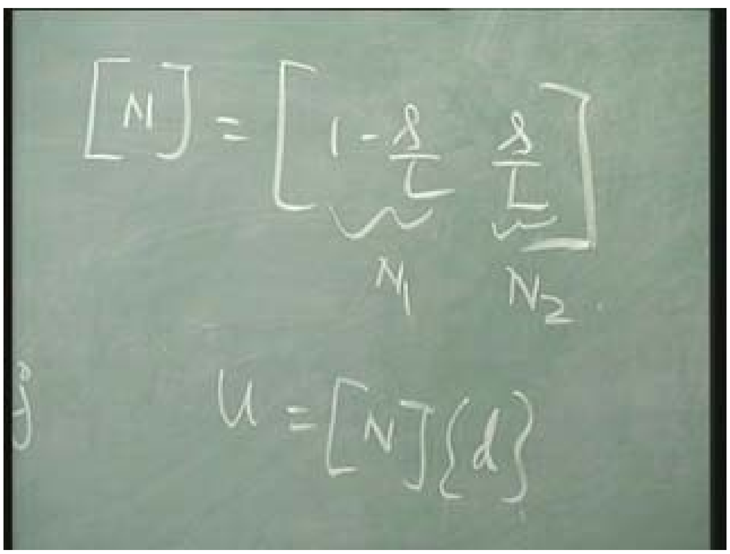

So, if you want, this N is what is called as the shape function. Note this carefully that N is called as the shape function. People call this also as interpolation function, basis function and so on. Essentially they are, either I mean, They are called by one of these names, but essentially they are the most important input for finite element analysis. Is it clear? Have a look at that and see whether you can straight away get one or two properties from it; 1 minus s L and s by L. Straight away, you can see one or two things. What are they?

They are nothing but an interpolation function, because it interpolates u by or getting the values of u_i and u_j. That is why they are called as interpolation functions. Is that clear and one more thing you can straight away notice there. What is it? That sigma of N is equal to 1. If I call this as N_1 and N_2, then sigma of N that is N_1 plus N_2 is equal to 1 and what is the other thing that you can see from this? What happens when s is equal to zero? N is equal to N_1, N_2 is equal to zero. So, 1 minus s by L and s by L is such that N_1 becomes, sorry, N_1 becomes 1 at s equal to zero and N_2 becomes, sorry, N_1 is equal to 1 at s is equal to zero and N_2 becomes zero at s is equal to zero and vice versa.

When s is equal to L, the first fellow goes off and second fellow remains. Is it clear, any questions on this? Now let me write down, u as Nd. I can write down the same procedure by means of a much more complicated matrix notation and so on. Because I am just introducing this topic, I have written down like that. This kind of moving from a basis to this N basis can be done for much more complex case as well. Is that clear? Have a look at that first expression, u is equal to a_1 plus a_2s; u is equal to a_1 plus a_2s.

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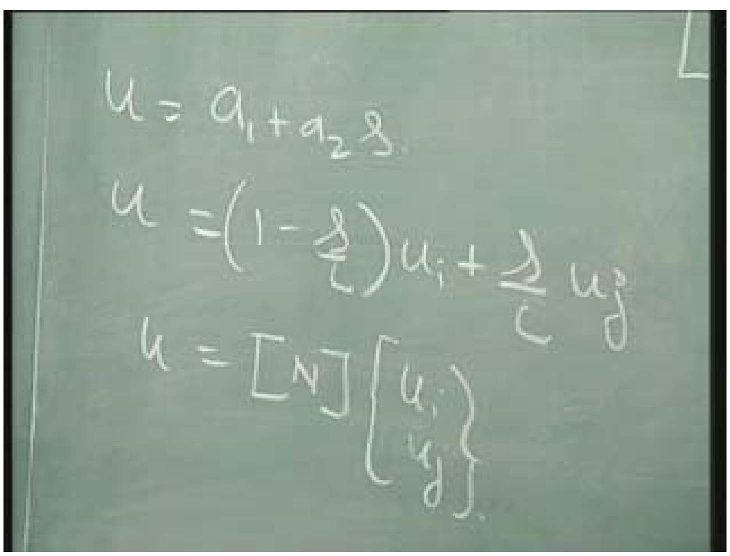

Yesterday, if you remember, that we had put a_2s plus say a_3s squared. Suppose, I put a_1 plus a_2s plus a_3s squared, what am I going to do now? Yes, pardon? Third node, in other words, I cannot swap now with two nodes. So, I have to put one more node.

Now I have, u is equal to u_i, u_j and u_k or u_1 u_2 and u_3, however you want to call it. You can now have three nodes and three coefficients, which can be swapped. What does it mean? It means that as you increase the degrees of the polynomial, I will also have to increase the number of nodes which go to define the element. Is it clear and these elements, where I have used higher order or higher order polynomials, are called as higher order elements. If I now have to do this, like replace a_1 a_2 and a_3 by means of u_1 u_2 and u_3, if I keep on doing this, this procedure becomes bit difficult, because I have to sort out or solve these equations like this. So, we are going to work out a better procedure to do it.

We are going to look at some well known interpolation functions later in the day or may be in the next class, we are going to look at what are called as Lagrange interpolation functions. Is that clear? Probably you would have already heard about Lagrange interpolation functions and Hermite interpolation functions and so on. We will look at that Lagrange interpolation, later in the class. Right now, let us say that we can write down u in terms of N and d. This is one dimensional problem.

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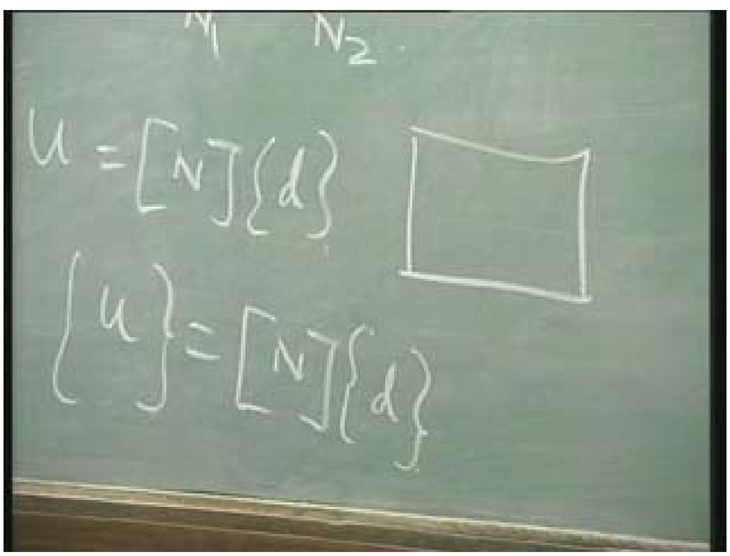

But if I happen to have a 2D problem with, say for example, an element like this, then how many degrees of freedom I am going to have? u and p, in which case I have to write to down u as like that which depicts u and v and what is going to happen now to d? How many entries did I have here? I had only entries, but when I now shift to a two dimensional case, how many d's I will have? Is it 4? Look at that in 1 2 3 4 and each of this nodes will have 2, correct. So, I will have 8 entries.

N will correspondingly change. So, in general I can write down u is equal to N d. Is it clear, any question? In general I can What is this N? Do not worry about it now. What is this N for this 2D case? Let us not worry about it. I can get you that very easily. It is not difficult, but right now, accept that u can be written down in terms of Nd or in other words accept that shape functions can be determined for different types of elements. When things become very complicated, we will move away from this very simple thing. We are going to move away, we are going to talk about what are called as iso parametric element and so on. Let us not worry about it right now. But right now, we will accept that u is equal to Nd can be written.

What is my next step? Just think for a minute and tell me, what is my next step? What do you think I should do as the next step? In order to do that, have a look at this expression here. Have a look at this big expression here and tell me what is it that I should get from u is equal to Nd?

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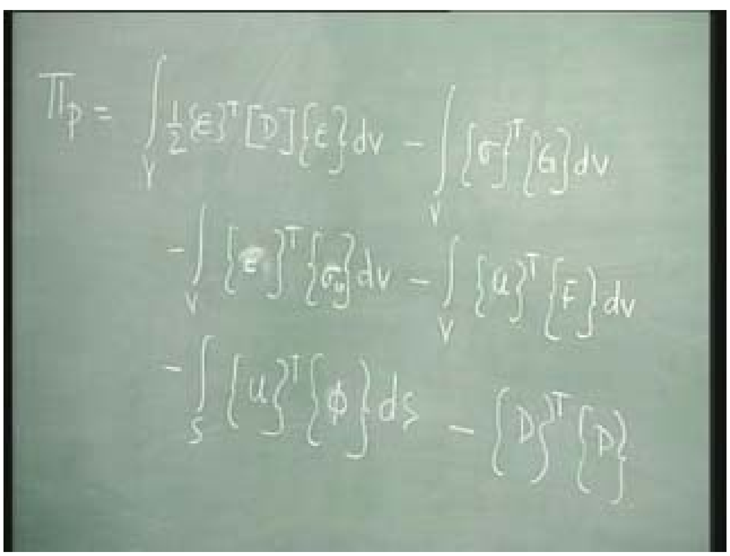

Epsilon, right, beautiful! So, you are very logically looking at things; epsilon. Fantastic, that is what I wanted; so, epsilon can be written in terms of my displacements. So, I am going to get strain displacement relationship. Remember, how did we write it? In terms of u, previously we had written it $epsilon_{11}$, $epsilon_{22}$, $epsilon_{12}$ and so on or $gamma_{12}$; we had all those things set up already. Just a second, we will write that term. Let us follow a notation and say that u, by the way you have to tell me which is that you are going to differentiate? Nd is there, because you have to apply differentiation.

Yeah, that is what I was waiting for. Is that d? What is d? What does d contain?

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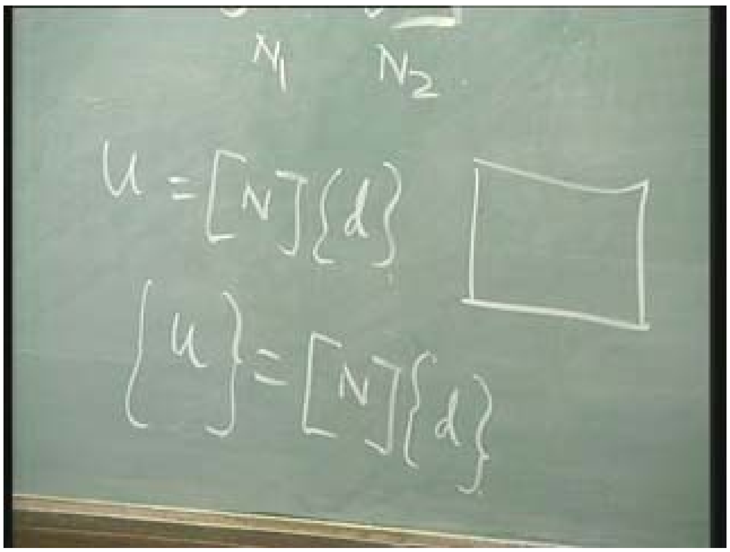

Yeah, they are constants, they are in terms of degrees of freedom. Say for example, I am going to differentiate this element with respect to s, which is the local coordinate system, which is something like x. Epsilon I have to differentiate with respect to x, so it will be in terms of N.

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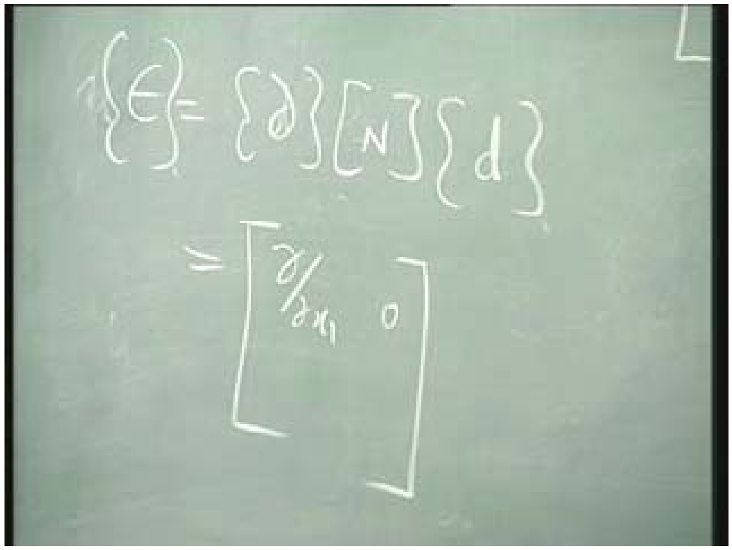

Let me call this as epsilon is equal to some dow term Nd. What is this dow matrix, can you think about it? What is this epsilon, I mean, sorry, I think I have to put like that. For 2D, for example, epsilon is epsilon$_{11}$ epsilon$_{22}$ gamma$_{12}$ and so on. You can keep expanding epsilon for 3D as well. What is this matrix, dow matrix? No; that is beautiful. So, that is it. So, that matrix is dow by dow x_1 zero zero and so on. Can you fill it up? The next line will be zero dow by dow x_2, then dow by dow x_2 and dow by dow x_1. That is now multiplied or this operator, you can say that, dow is an operator. Now, it operates on N; it operates on N. Is that clear? Now, I am going to ask you one more question. Just think about it. Even though you do not worry about what is inside N, you have to tell me what is the size of dow N and d? You have to tell me what is the size of dow N and d for a 2D case, let us say 2D case. Look at this.

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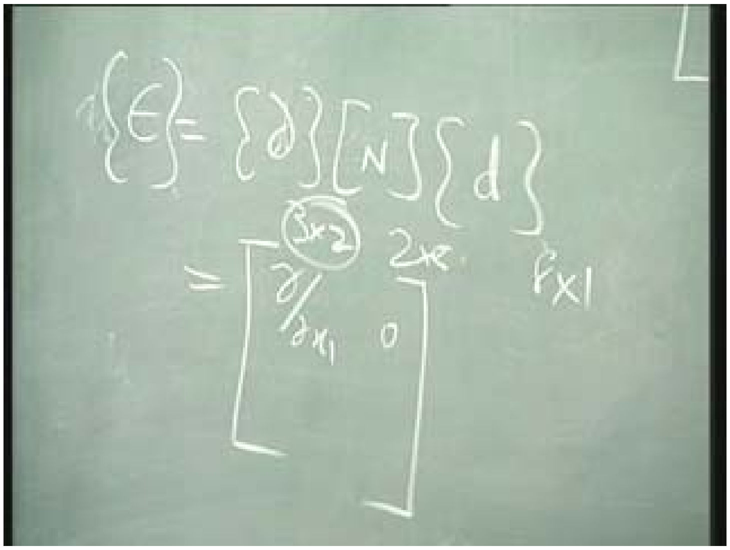

You have to tell me, what are the sizes of that? What is the size of dow N and d? What do you think? Dow-2 by 2; 2 by 2 or what? 3 by 2 and then N? N 2 by, common, 2 by 3, 2 by 4, 2 by 8; I get all types of answers. See, it is very simple, it is very simple. Look at d. d is what? What is d? 8 by 1; common, that is it. 8 by 1. So, what is N? 2 by 8, that is it; 2 by 8. What is that ultimately you will get? How do you get it? How do you see it? 3 by 1, so, that is what is epsilon?

Yeah, just have a look at that. You should not get confused. This is very, very important to understand what the sizes of this matrix are, because when we assemble, when we put them together, it is the sizes which are going to confuse you. That is why I am right now emphasizing what should be the sizes of these matrices? Having understood that, having understood that let me call dow N as a B matrix. Let me call this dow N as a B matrix, so that epsilon can be written as B into d. This is the most important part of the whole finite element, so, I want you to concentrate. If you have any doubts, just ask me. Yeah, so, I can write epsilon is equal to Bd.

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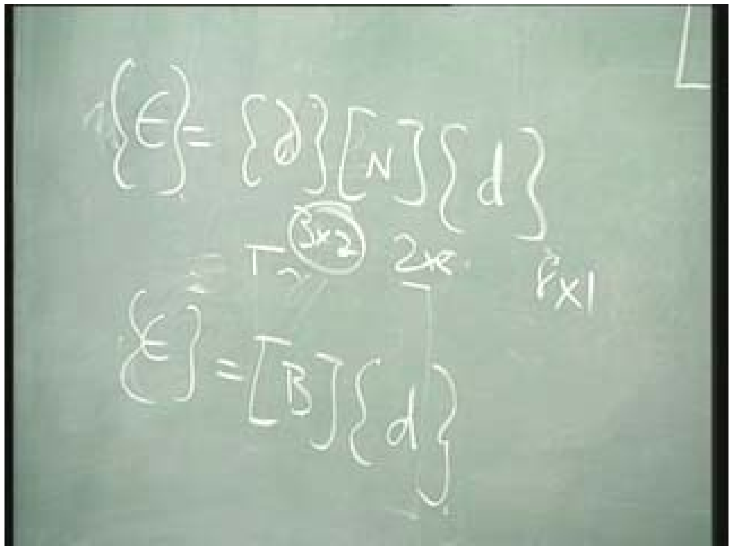

What is my next logical step? My friend, yeah, he said what is my next logical step? What is that I should get? What is the next logical step? What is that I should look for? Stress strain matrix, beautiful. We have already derived that. That is why I did all those things initially. We know everything now. I already know what this D matrix is. So, I can write down; that I have already written down, sigma is equal to, I am going to repeat that sigma is equal to, D epsilon. In our earlier classes we have spent lot of time looking at D. So, I will just write down sigma is equal to D epsilon.

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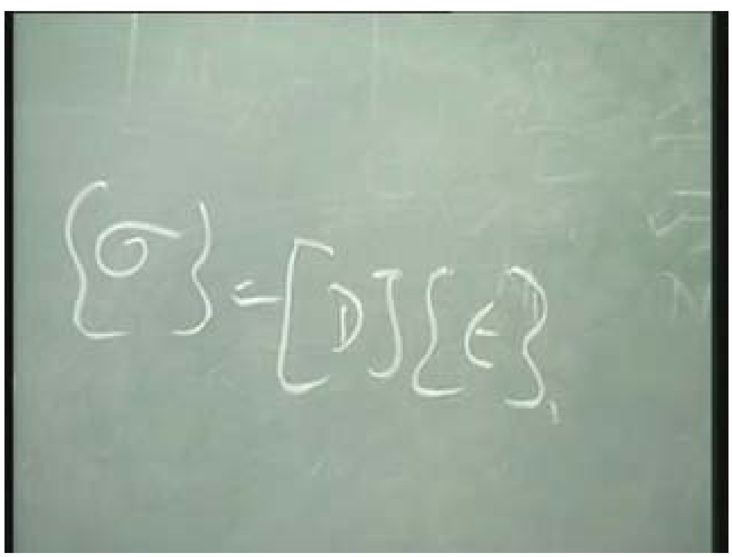

So, I can say that sigma is equal to DB into D and so on. Any question? Now, having got all these things, what am I going to do? I am going to substitute them into this expression. I am going to substitute all of them into this expression.

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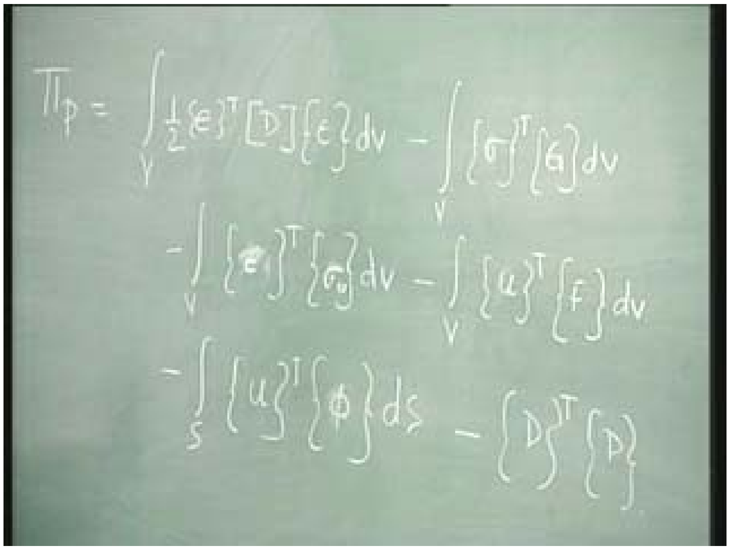

I would like you to do it, because it is straightforward and gives you an exercise. I will give you one minute to write this term, just attempt it. What is that you should know that a b transpose is equal to? What is ab transpose? b transpose a transpose, that is all you should know. Write it down, I will give you a minute. Have a look at this expression, this expression and then write it down and after a minute let me hear what you have done. This will give you a good exercise to remember each of these things, just put that down. Let us see, have a look at this.

Look at each of the terms and replace. All the while think about the size, for example, for a 2D problem. By the way notice that pi_P is a scalar quantity; pi_P is a scalar quantity.

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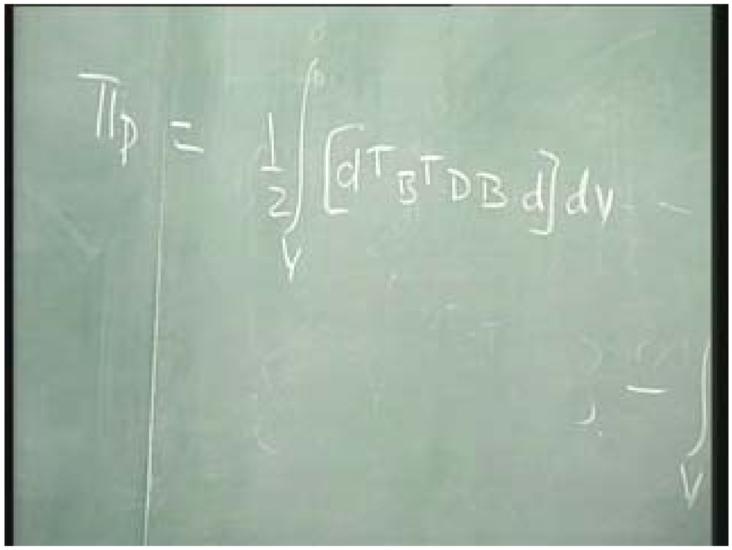

Do not look at what I am going to write now, because I would like you to write it. I am going to write the final result, but have a look at this and you keep writing it so that you will get a feel of what is happening. So, all of you have written the first term? Is that ok? Minus, epsilon transpose, what is epsilon, sigma transpose epsilon zero; okay what is sigma transpose? Yeah, epsilon transpose d transpose. So, what is epsilon? I want you to substitute till the end. Yeah, d transpose; see, all of them are matrices. I am not putting everything as a matrix, you know they are matrices. So, d transpose, B transpose, capital D transpose. Is it D transpose? B transpose D transpose yeah, so, what is the next term? Minus, where is that? Yeah, still you are writing? d transpose B transpose sigma$_0$ and then next term, minus v d transpose N transpose F dV that is the next term. That is for what? For body forces. Then for surface forces, d transpose N transpose phi sorry ds; yeah any questions?

Lastly, I have that D transpose P; minus D transpose, minus D transpose P. Just to make it simplified, I am not going to enter or I am not going to carry on those two terms that epsilon zero and sigma$_0$ terms. It is very easy to include them, because I have to write down big expressions. You can very easily take it down, so, I am going to just remove that for the time being. If you want, you can always add it to make it much simpler. Now, I am going to define a stiffness matrix as integral B transpose DB dV. Let me define K, stiffness matrix of an element to be integral B transpose

DB. Is that clear and this is for a general body. But, in the sense that is for an element as well or for the whole body; you can look at it either way.

But, since I have written down this last term here, it is for the whole body which has been already discretized.

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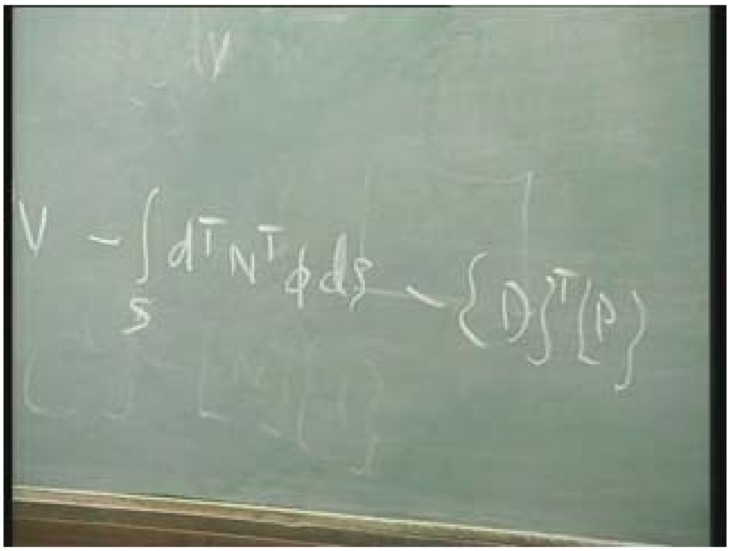

Is that clear? Now, I am going to do one more thing. This small d's here are actually, there is a small problem; the small d's here are for the element degree of freedom. I am going to expand them actually. That is why I am very careful in writing the last term. They do not go with this, so, there is always a trouble, as to understand that. What I am going to do is to expand that small d to the whole of the matrix. In other words, if I have an element which run say from 1 and 2 nodes, I am going to say I will write as if it is 1 2 and other things being zero up to say 20 elements, if the problem has 20 elements.

I am going to expand them. Why is that I am going to expand? If I expand like that, it is easy to add them, right now. But, computer implementations are not like that. They are quite different. So, let us not worry about that at this moment and so, let us see how we can write down this particular case or in other words what is that we are doing? We are expanding or in other words, we are substituting for small d by capital

D; substituting small d by capital D. Capital D indicates a vector which covers all the nodes in the body, which covers all the nodes in the body. Is that clear?

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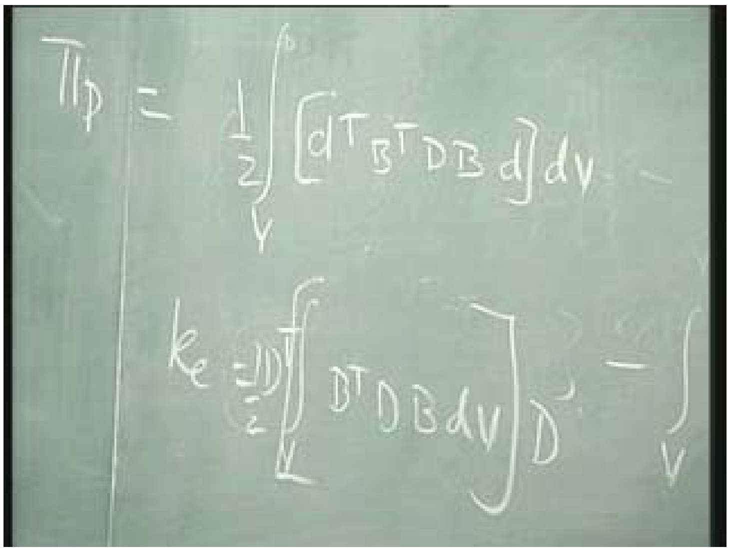

Write down the expression. Let us see, how you can write down that expression? Define k_e, k_e, as integral V B transpose DB dV and when I expand it to D, the total pi$_P$ of the whole system can be written in terms of capital D. So, I can write that D transpose D as a contribution of this element strain energy. Of course, half is there, half; as a contribution of strain energy of one element to the total system. Is that clear? No? Let me repeat that. Have a look at this again. What is this small d's? What is this small d's? They are for element. In other words, what it means is each element now contributes for pi$_P$. In strict terms, this pi$_P$, what I have written is valid only for an element. But, I have just added this term. This is not right to do it in the last term. Please note this is not the right way of doing it, but I just wanted to remind you that there is the last term there. There is one more contribution from concentrated loads.

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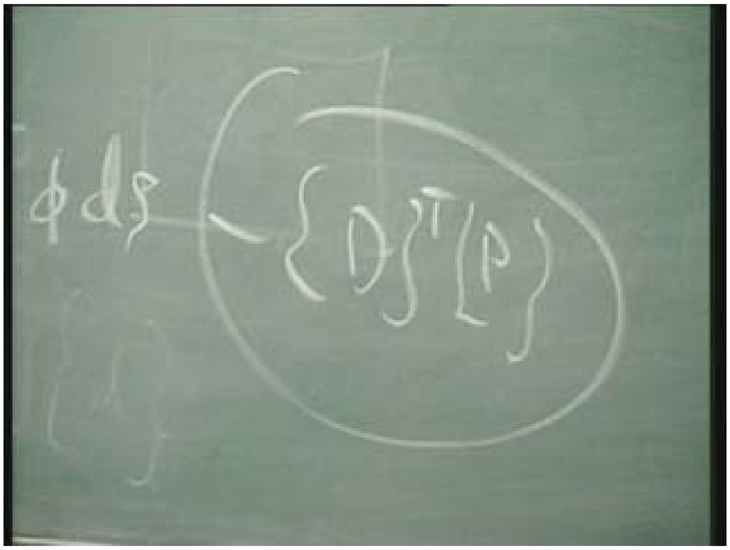

I would say that I have just added it more symbolically there. For the time being you forget it. If there is confusion, forget it for the time being. We will add it in the next step. I said that I have added it very symbolically. Each of the element contributes to strain energy and each of the element contributes to potential lost. What is that I should do, if I want to find out the complete pi$_P$ of the system? I have to add, add because each of the elements I have already divided the whole body into number of elements. I have to add the contribution of each of the elements as far the strain energy is concerned, as far the potential lost due to internal loads or in other words sorry, due to body loads or body forces and so on.

In order to do that, in order to add, I cannot add straight away because my d, small d is there, sitting there. They are different. The first element may have 1 and 2 nodes, the second element may have 2 and 3 nodes, third element 3 and 4 nodes and so on. So, how do I now add? Tell me a procedure how to add it? The best thing is to expand that small d to the whole of capital D, so that others will be zero. 3916I will just give you a minute to understand this. What I am going to do is to put down my old problem, you know the old problem and write it down in terms of the entire degree of freedom. What is that old problem?

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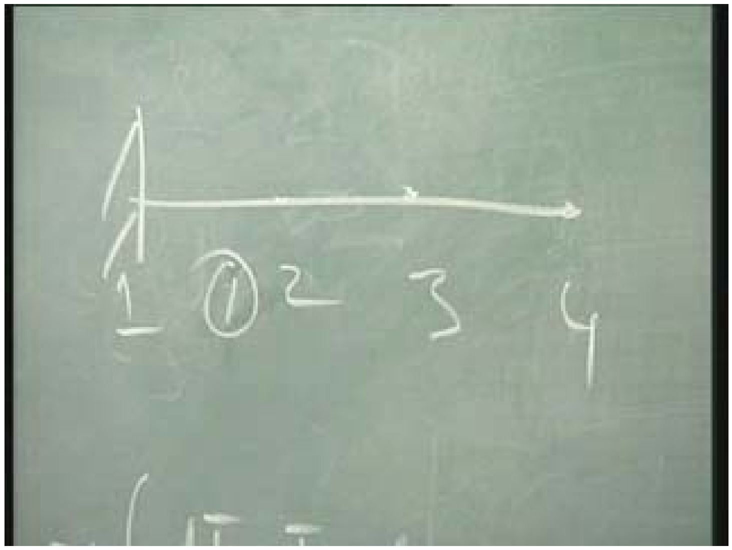

Say for example, let me say that I have 4 nodes. I have 4 nodes; 1 2 3 and 4 nodes. I have an element 1, which connects 1 and 2, element 2 which connects 2 and 3 and element 3 which connects 3 and 4. I have these three guys sitting there. Element 1 has k_1, say for example, k_1 minus k_1 minus k_1 and k_1 and the corresponding displacements are u_1 and u_2. I am saying u_1 u_2, make it u_1 u_2 u_3 u_4. Make it u_1 u_2; so, how will the k change now? That is what I want to ask you. Just a minute, write it down. Try it. It is very, very simple. I mean, very obvious question but I know that it is not very easy to follow. You please try it and write it down. I will write it down in a minute what exactly I mean by that?

Have a look at that and then have a look at this and then write it down in terms of, let us say that, this is the element 1. It is very straight forward question. What I am saying is that the k matrix will have now 1 2 3 4; u_1 u_2 u_3 and u_4.

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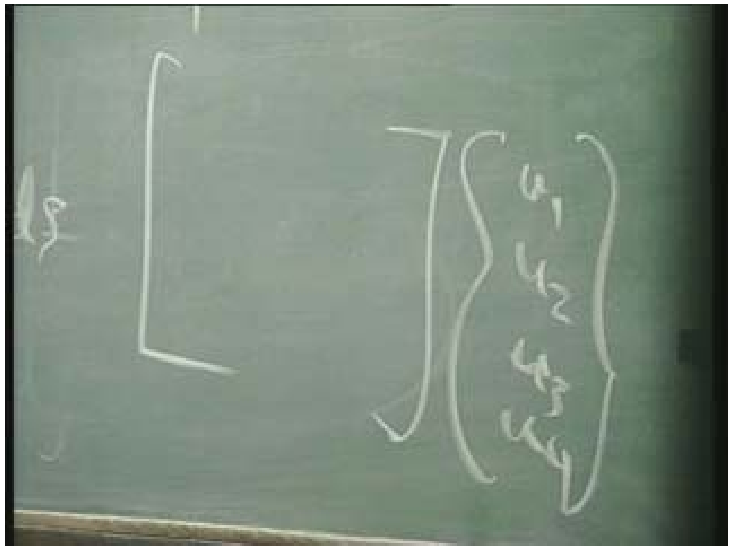

What will happen to the k matrix? I am expanding it. What will happen to the k matrix? Zero, that is it. You know that is what I have been looking for. It took you quite a while to find out. So k_1 minus k_1 zero zero, minus k_1 k_1 zero zero and so on. What is the advantage now? The advantage is that if I write the second element also in the same sense, how do I write it? Zero zero zero zero, first row zero zero zero zero. Then, second row becomes zero k_2 minus k_2 zero and so on. Then these are, straight away they can be added. They are matrix of the same size and not only that, this u's that are sitting there, they are the same u's. So, u can be taken out and I can add them. Is that clear? No, still not clear?

What I have done is to facilitate adding of k. What I have done is to facilitate the adding of k or else I cannot add k straight away, because this k_1 corresponds to this 1 and 2; k_1's or k's corresponds to 1 and 2, the k_2 corresponds to 2 and 3. So, I cannot add k_1 and k_2 straight away, unless and until I expand it like this. Though it is not an efficient way of doing things in a computer, let me make it very clear, this is not the way commercial code is written. Right now, as a student it is easy to understand what is happening. There are different ways of doing it. We are not looking at computer implementation straight away. We are not looking at computer implementation. We are only looking at how to do things, how things are done. Hence this is a nice way of looking at this.

Now, get back to this, get back to this question.

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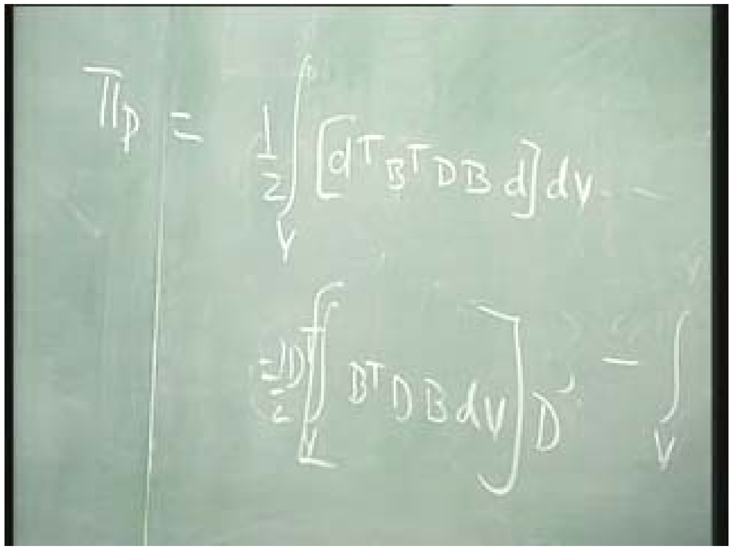

I have already defined k_e and u. This is not k_e, that is the u, the contribution of u of an element. Now, complete this thing. I am making you write because it is easier to follow when you start thinking and writing them. I will give you one minute, just write down or convert it and write down. I want something like this. I want ultimately K into d is equal to R. This is the expression I want, KD is equal to R. How are you going to write it down? That is the question. Just try it for a minute. I will give you all the clues that are necessary. One is, k_e can be summed up or k can be sum of k_e's, capital K. Let us call that as capital K. Capital K is equal to sum of k_e's. Then you look at also r_e's. What is r_e? What is r_e?

r_e is the other term, say for example, D; D transpose, take it out. So, r_e is equal to minus N transpose F dV and minus N transpose phi ds.

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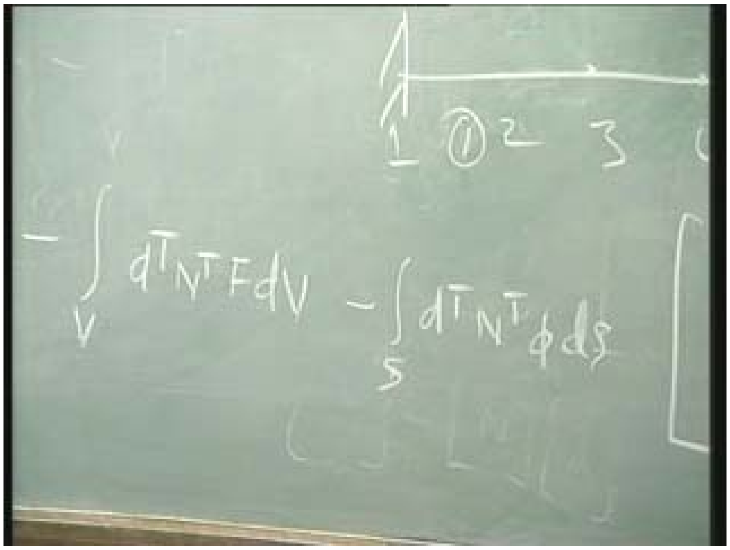

Is that clear? In other words, what is that you are going to get ultimately? Yeah, write it down, you have another 2 minutes, just write it down. I will do that. I will write it down, but just see how you are going to write down. I can write it down as half capital D transpose KD minus D transpose r_e. What is the next term I will have or R rather minus D transpose P will be my pi_P.

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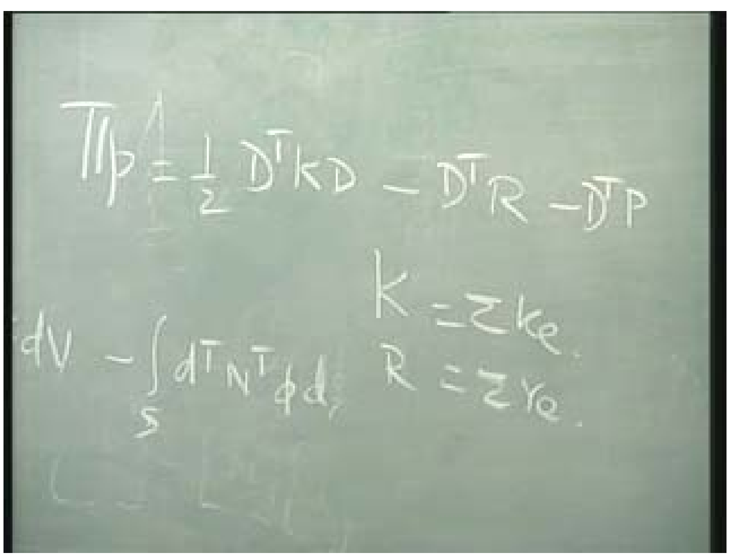

That is the total pi_P is equal to half D transpose KD. This K is the sum of sigma of k_e's minus D transpose R. This R is sigma of r_e minus D transpose P. That will be pi_P. K is equal to sigma of k_e. This is the matrix addition, because I have already expanded K in terms of complete D and R is equal to sigma of r_e's. Is that clear? Is that clear and the last term now I have brought in because, my pi_P now should include the contribution due to concentrated loads. Please remember what r_e is. r_e is nothing but N transpose FdV N transpose phi ds that is r_e. In other words, this will come out and that will come out.

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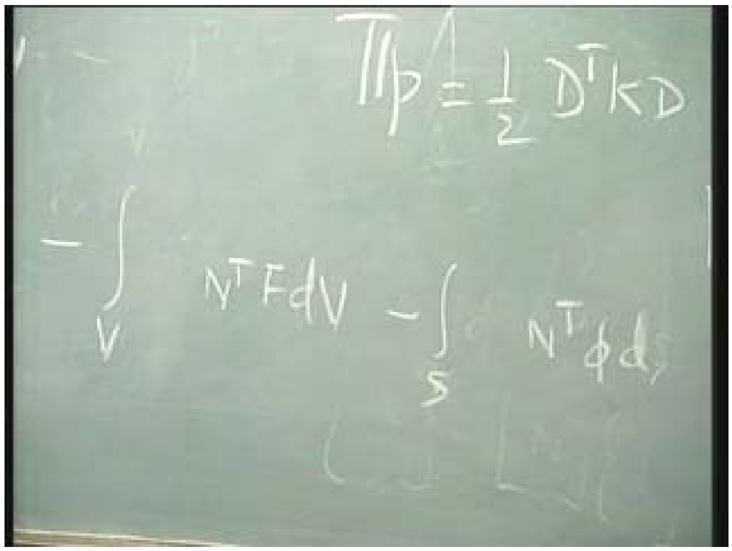

What is my next step, what is my next step? Yes, that is it. So, dpi_P by d D should be equal to zero, independently. In other words, dow pi_P dow D_{i2} or i, where i is equal to 1 2 and so on, should be equal to zero. What is it that I am going to get? A matrix, correct; but, what is that? Half D transpose KD, when you differentiate it with respect to D, becomes KD. You can work it by hand, KD.

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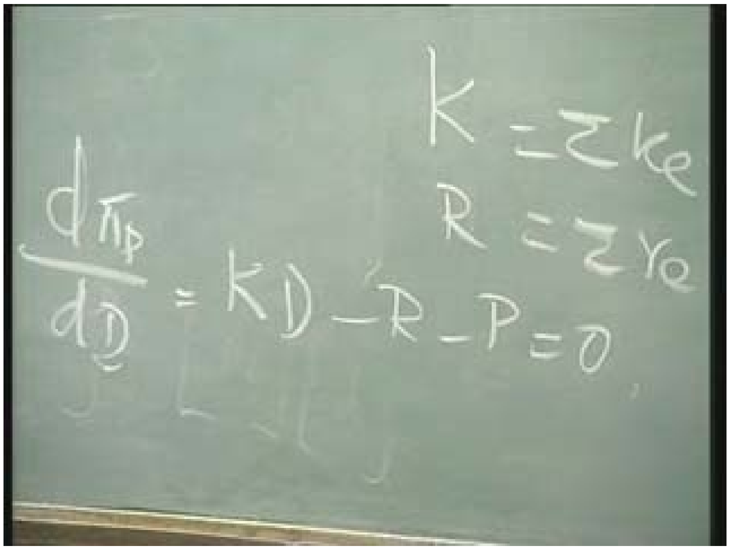

The first term now, after I differentiate this becomes KD minus R minus P and that is equal to zero. So, KD is equal to R plus P. That is my final expression. KD is equal to R plus P. K comes from small k_e, which is, what is it? Integral v transpose DB dV.

Yeah, 1. I think only small thing is that, see usually, I think there is a small error here. That sigma_0 is usually replaced by means of, I think that is the only mistake you did when you expanded, by D into epsilon zero. That is how you expand.

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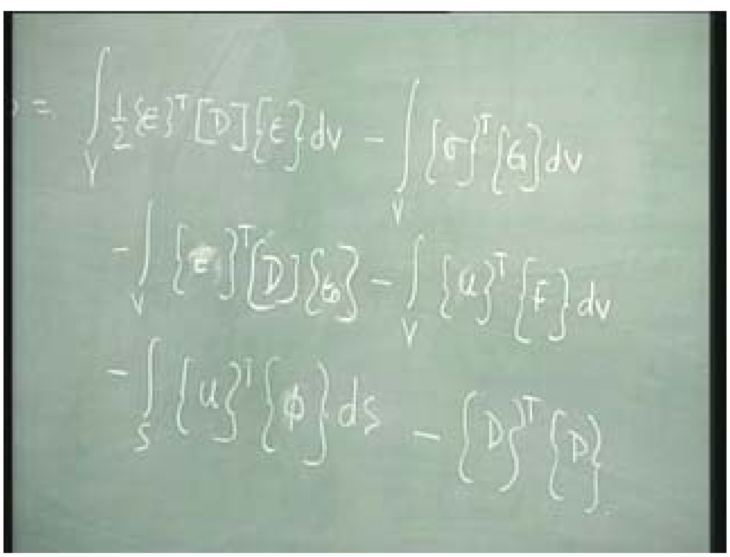

I think, you said d transpose. That is what I was surprised actually you write $sigma_0$ as D epsilon zero. Did you get it correctly? D epsilon zero and this is equal to sigma transpose epsilon zero. Anyway, I mean, this is very straight forward. If you understand sigma transpose epsilon zero and if you put it as $sigma_0$ D epsilon zero, I will explain this in the next class. What it really means is any strain, initial strain, you can talk in terms of initial stress or vice versa. We will discuss this in a later class. I hope you understand this. Any questions, any questions on it? Please go through it, if there are any questions, I will answer it in the next class.