Introduction to Finite Element Method

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Lecture - 14

In the last class, we saw the potential energy theorem. We actually derived it by using variation of principles. We also in that process saw the principal of virtual work and we said that both of them are the same, in the sense that, we can get solution of a problem either by going through this potential energy theorem or through the virtual work principal. I hope, you all remember. On one hand, we derived potential energy theorem from the virtual work principle. On the other hand, we also said that if I have to go through this route, what is the assumption that we made? We said that strain energy should be defined or in other words, as far as potential energy theorem is concerned, it is valid when its functional exists; in a more mathematical term, the functional exist.

In this case, what is the functional? Pi_p is the functional. In other words, the potential energy is called as the functional, because it is the function of the functions. This functional exists, only when strain energy density function exists or in other words for an elastic problem, we say that we can go through the route of the potential energy. On the other hand if you look at elastoplastic problems, then we cannot derive the constitutive equation through the strain energy density. Then we have to resort to the virtual work principal or in other words, our starting point will not be the potential energy theorem, but will be one step before it. That is the virtual work principle. We will see that later, because we are going to derive somethings now. So, we will do that first and then come back to this problem of defining or determining the stiffness matrix and in that process how do we go about either from potential energy principal or from the virtual work principle. We have to wait for, I mean that, that has to wait may be, till next class.

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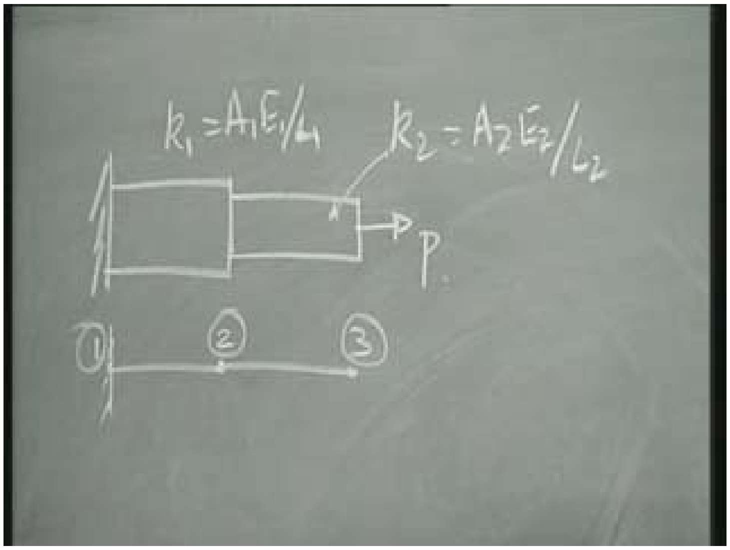

But, before we go ahead, let us look at the problem that we did or that we posed in the last class. We said that we have to derive or we have to determine what this problem is. I hope all of you remember this problem. This is one of our earliest problems that we did. What is that we determined? We determined u_2 and u_3; all of you remember this problem? Right. Now, my goal is to do the same problem by using potential energy principle.

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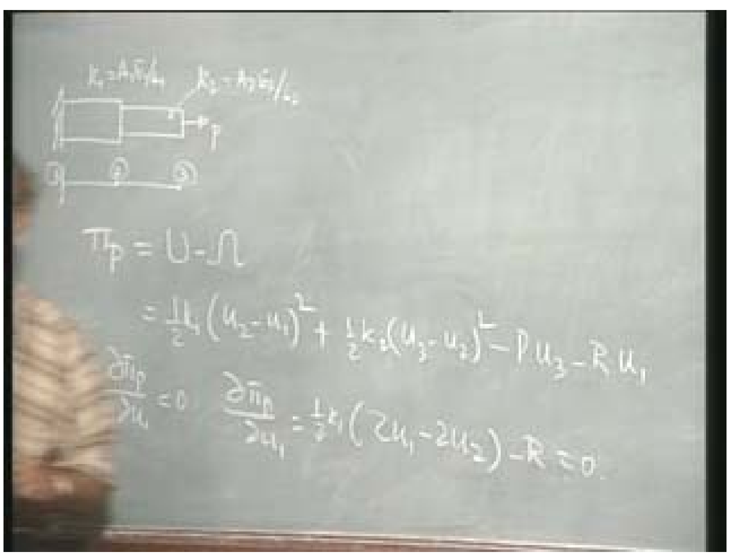

You know what potential energy is. It is the strain energy minus the potential lost due to external loads; due to the work done by the external loads. u minus say omega; let me write that as u minus omega. Omega is the potential or minus omega is the potential lost or in other words, in simple terms, it is just P into d. P is the load that is acting, d is the displacement for that load. Can you help me now to determine u? u is equal to, please note that I have already put for your sake 1, 2 and 3. There are two ways in which we proceed; just a word of caution, two ways in which we can proceed. One is to look at the boundary condition at this stage itself. We look at it like that and start writing down, but for a bigger problem this may cause a bit of confusion. There is nothing wrong, but it may cause a bit of confusion. Why, because actually the problem has it that there is a reaction that acts at 1; there is a reaction that acts at that point or in other words, the potential is also lost due to reaction. But, it so happens that the displacement at that place is equal to zero and hence P into d term would b equal to zero. Is that clear?

But, nevertheless in order to complete the picture, in order to complete the picture, it is a good practice to write it down completely along with the reaction. Help me in writing that. So, that is equal to half into K_1, one of you can say that; K_1 into say u_2 minus u_1 whole squared; u_2 minus u_1 whole squared plus half K_2 into u_3 minus u_2 whole squared minus P into u_3 minus R into u_1, where R is the reaction that happens to be present at 1. Is this clear? What is the next step? What are my degrees of freedom? u_1, u_2 and u_3; so, pi_p is the function of u_1 u_2 and u_3.

What is that we do as a next step? Differentiate this with respect to u_i's. So dow pi_p by dow u_i is equal to zero. Is that clear? So, what happens to dow pi_p by dow u_1? Yeah, work it out, because it is a good practice. Instead of just following what I write, it is a good practice; just you write it down. That will be half into K_1 into, so, u_1 squared plus u_2 squared minus $2u_1$ u_2. The first term here will be $2u_1$; $2u_1$ minus, minus $2u_1$ u_2, so minus $2u_2$. Is that clear? So, nothing will be present here and then we will have minus R; minus R and that is equal to zero. Is it clear? Straight forward.

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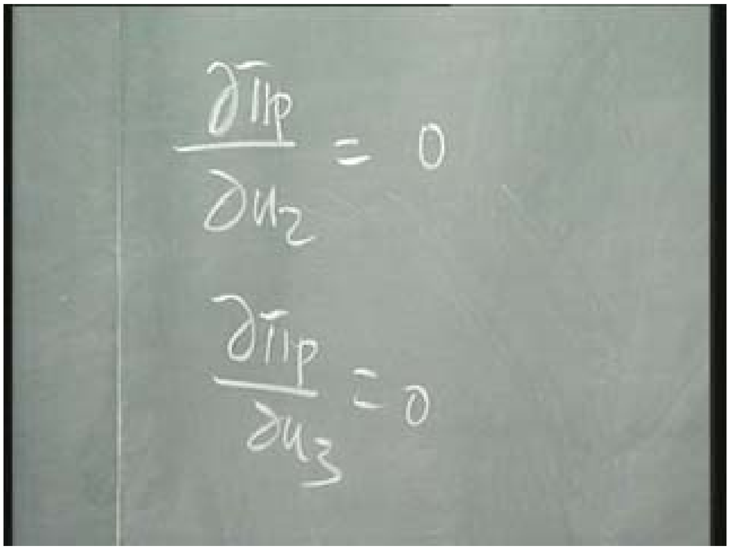

Now, write down what is dow pi_p by dow u_2? dow pi_p by dow u_2. What is it? Yeah, you do that. You look at this here; look at this thing here and then differentiate it with respect to u_2. May be, I think it is better to write it here, so that you can see it here. What will happen to the first term? Here also, u_2 is there. So, differentiate it and then, do the same thing with respect to u_3 as well. Both of them are equal to zero. I will give you 2 minutes. Just do that and look at the result and ultimately fill this up for me.

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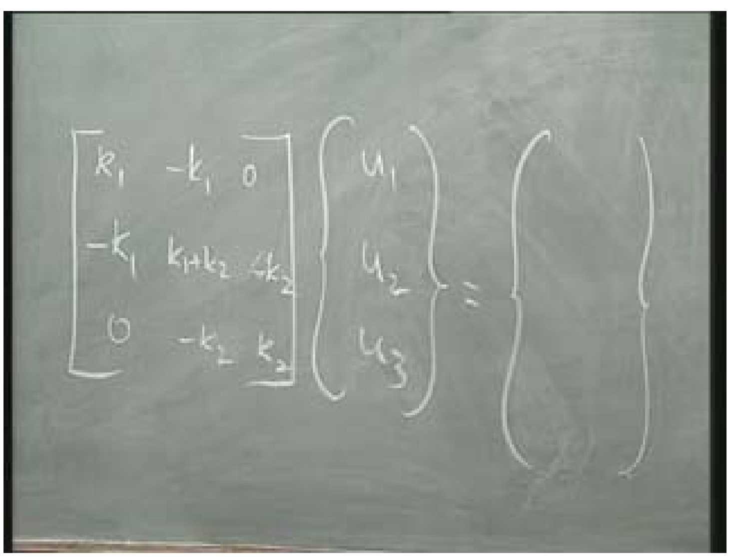

So, just do that you fill it up for me. So, what will you get? We will get K_1, there minus K_1, zero; here minus K_1 K_1 plus K_2, sorry minus K_2 minus K_2 and K_2. Look at that expression now. Is it clear? What is this? Go back and of course that is zero; go back and have a look at what we did in your earlier classes? Is it exactly the same? In other words, what is that you have done? You have got the same stiffness matrix, first from equilibrium equation; remember that we got that from equilibrium equation. Now, you have got the same stiffness matrix from potential energy theorem. What is here by the way?

R and minus P, because the directions are important; minus P, R and minus P, so on and so you can work it out. P, R and P, rather R zero P; R zero and P and so the direction of R automatically be determined from this. I am sure you can or in other words you can leave the first row, calculate the u_2 and u_3 from the second and third row and you can get this thing going. After this, I think I will leave this problem. I just wanted to show you that the same problem that you did earlier with the equilibrium equation can also be determined by means of this potential energy theorem. I would like to do another problem. The other problem is slightly more complicated and I am going to introduce Rayleigh-Ritz procedure today. Yeah, any question? Is that clear? Let us now look at another problem and we will introduce what is called as a Rayleigh-Ritz procedure, before we go into actually the finite element analysis.

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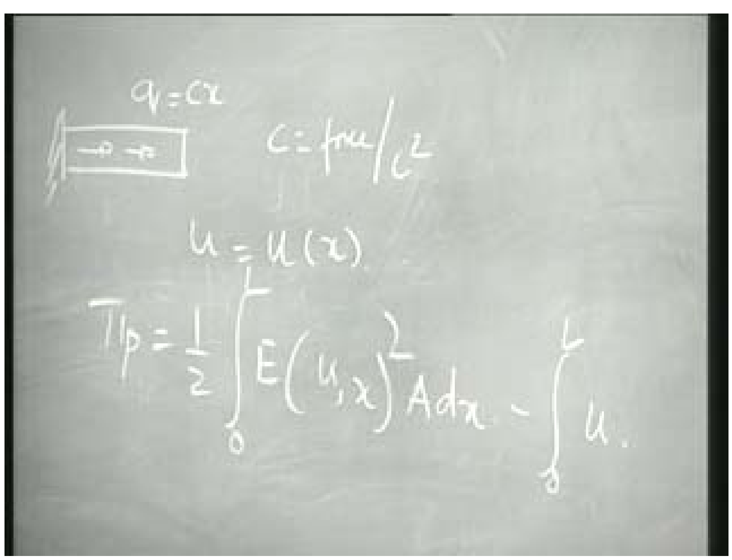

Look at the problem, q is equal to Cx. C is equal to force per L squared. Determine the displacement. Determine the displacement and the stress and the strain. That is the problem; q is equal to Cx and C is equal to some force per L squared. Determine the displacement. Now, you have to first write down pi_p. What is this displacement? Displacement is now function of x. So, u is some function of x. Is it clear? Now, write down pi_p first, as a first step write down pi_p. It consists of two terms; strain energy term as well as the potential that is lost due to external loads. What is the strain energy in this case? You know that, half sigma epsilon. What is sigma? E into epsilon.

What is epsilon? du by dx; now, this is integrated. This is per unit volume, so this is integrated throughout the bar. I can ultimately write the first term of pi_p as half zero to L E into u comma x whole squared into Adx. That is all; Adx. Is that clear, first term? What is the second term? Minus integral zero to L u into what is now the force?

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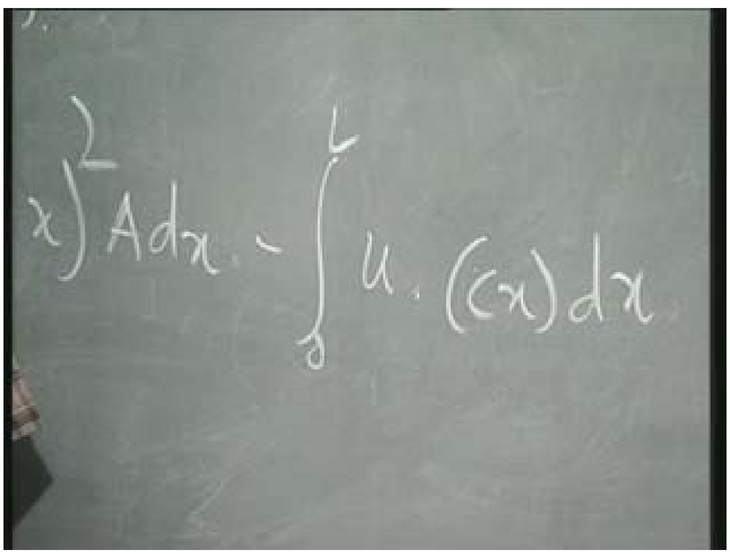

Cx into dx; Is that clear? So, force is u into Cx into dx. My whole job becomes slightly more complicated here, because I want to find out now u as a function of x. Now, we will follow what is called as Rayleigh-Ritz procedure, which actually is a precursor or a pre runner to finite element analysis. What Rayleigh-Ritz did was to convert an accurate problem into an approximate. In other words, he found out approximate solutions to this problem. What he did was that, he introduced an approximate function for u, introduced an approximate function for u.

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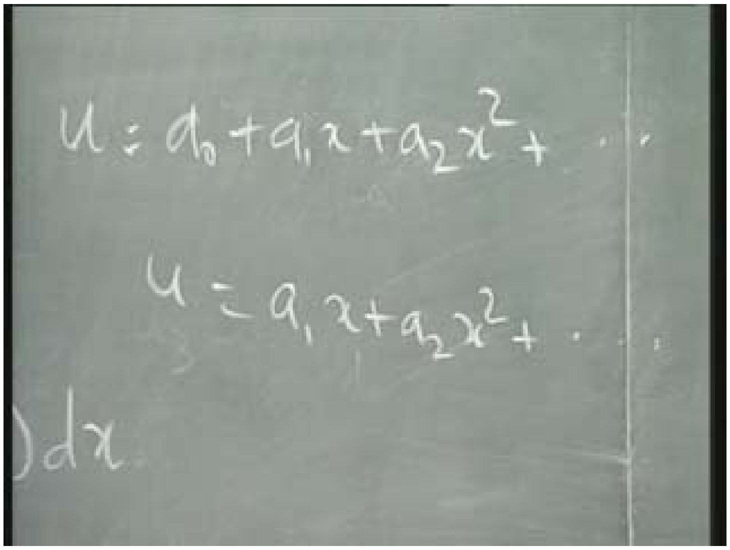

For example a polynomial can be introduced for u. u is equal to say a_0 plus a_1x and a_2x squared and so on. It was found that this polynomial should satisfy the boundary condition. In other words, u in this case happens to be a_1x plus a_2x squared plus so on. Is that clear? So, he said that I will approximate u in terms of some polynomial function which we say a_1x plus a_2x squared plus a_3x cube and so on. That is the first step. Now, what he did was to substitute this expression for u into the function pi_p. What happens when I substitute like that? This is an approximate solution; remember that it is an approximate solution. We will comment about the solution later in the class, but right now, let us say that what we want to do is to approximate u in terms of some a_1 a_2 and so on.

What happens when I substitute this into this expression, into pi_p? pi_p will now become a function of a_1's, sorry, a's; that is a_1 a_2 a_3 and so on. In other words, in the previous expression or previous problem, I had pi_P in terms of u_1 u_2 and so on. Now, I will have it in terms of a_1 a_2 and so on. In other words, a_1 a_2 a_3 can be looked at as a degree a freedom. That is correct. So, a_1 a_2 a_3 can be looked at as degrees of freedom. Is it clear? Yes, what is this degree of freedom? I know I can understand that you immediately do not catch. There is no physical meaning to this degree of freedom. On the other hand in my previous problem, you did not have any doubts, because the degree of freedom was very clearly written as u_2 u_3 and so on, for which physical meaning was very clear to you.

On the other hand, here I am writing it in terms of a_1 a_2 and a_3 and hence you may have a doubt as to what this means and that is one of the reasons why, finite element method modified this Rayleigh-Ritz procedure and introduced this concept of nodes and degree of freedom, corresponding to the displacements of the nodes. Is it clear? So, as a first step, let us see how to do the problem with, say, taking only one, u is equal to a_1x; only one term. We will do that with two terms as well, but before that let us look at the final result.

Just write it down at the corner of your note book, so that when you do this problem with single or a_1 alone or one term and then with two terms compare that with the final result which I am going to give.

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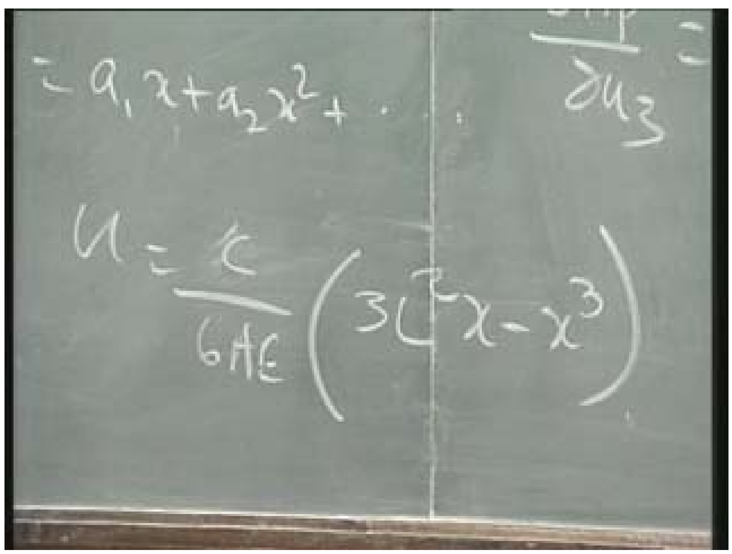

The final result can be written as u is equal to 6, sorry C divided by 6 AE into 3L squared x minus x cube; 3L squared x minus x cube. So, that is the final solution or accurate solution or analytical solution, by solving this problem analytically. Now, my goal is to get an approximate solution. In other words, whatever I am talking about, equilibrium is approximately satisfied; is not accurately satisfied at every point, but on an average it is satisfied. So, the first thing is take u is equal to a_1x, substitute it into this expression and see what you get.

Let us see how to do that. Yeah, any questions, any doubts? Yeah; half into sigma epsilon. What is sigma? E epsilon; so half E epsilon squared. What is epsilon? du by dx or u comma x; What is du by dx is what I write as u comma x. So, E epsilon, sorry, E epsilon squared becomes E u comma x squared. Is it clear?

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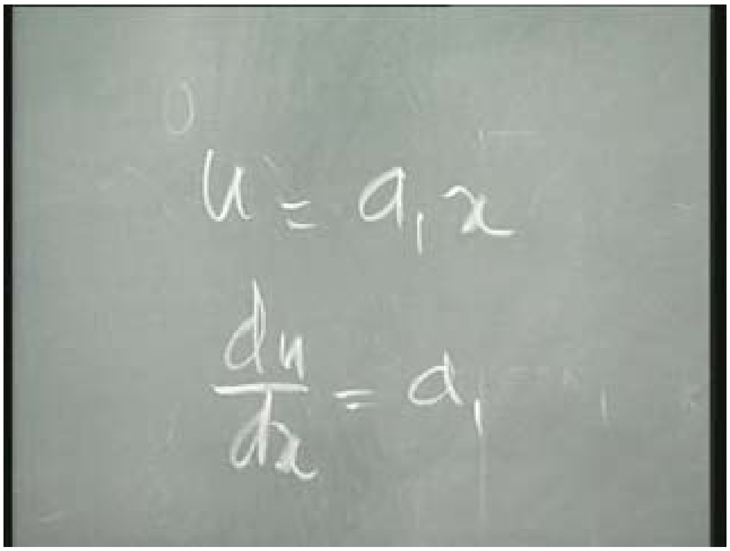

Yeah. Now, u is equal to a_1x. That is what I want you to work on, so that du by dx becomes a_1. Is that clear? Now, you have to help me. What is pi_p?

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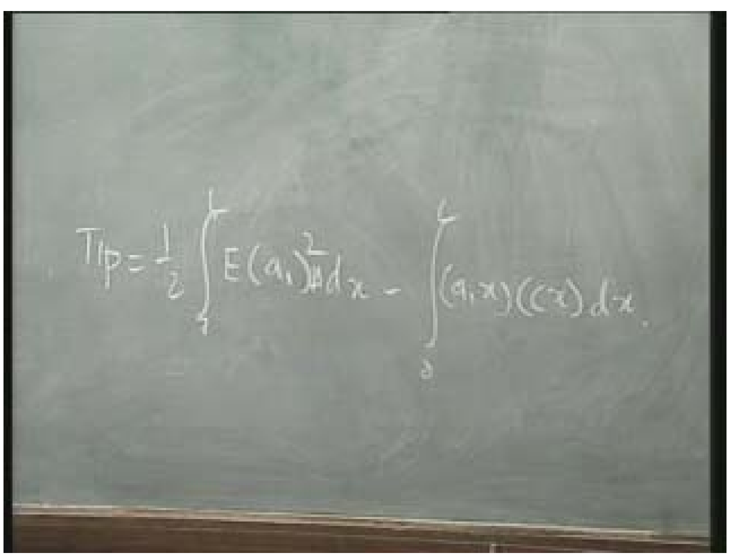

Half zero to L, E into a_1 squared into dx minus, Adx minus the other terms. Now, you fill it up. You know, I will just wait for a minute. I am not going to do this problem. I am going to give you some five, ten minutes, let us see. What I want you to do is, I want you to do the first problem with this a_1 alone and then I want you to do the second problem with a_1 plus a 2x. Let us get the results for these two problems and then, we will see, compare it with the final result. Before we proceed, just a minute; so, you will write it in terms of pi_p. The next solution is very simple. If you want, I can write that also. Zero to L, u is what? a_1x into Cx dx.

It is a_1x into Cx dx. So, pi_p is now function of a_1. Clear? Any question? No; a_1 is not a constant. It is a degree of freedom. What it means is that, u is equal to a function of a_1 plus a_2x and I am trying to find out what this distribution is? Please note this carefully. It is not a constant, because whatever be the loading, I will write it in terms of a_1 plus a_2x. This is the polynomial, irrespective of the loading condition. Is that clear? It is the polynomial irrespective of the loading condition; a_1 plus a_2x and so on. Now, I will find out that a_1 and that a_2 which will satisfy the potential energy theorem. That is dpi_p by da_i; dow pi_p the dow a_i.

What is that? That is what I am trying to find out. So a_1 a_2 is something like a variable. It is not that you can, it is a concept, whereas on the other hand, this form of the equation a_1 plus a_2x or a_0 plus a_1x plus a_2x squared plus a_3x cubed and so on, that form is the same. Usually polynomials are used; may be, sometimes sine and cos functions can also be used but 99% of the case, it is the polynomial that is used. So, a_1 a_2 a_3 and so on becomes the degree of freedom. They are like u's; u_1 u_2 u_3 and so on. Is that clear? Now what you do is, differentiate this. Then, calculate dow pi_p by dow a_i or dow a_1, in this case. There is only one a; so, a_1. dpi_p by da_1 and calculate a_1. That is the first step. Just a minute; you know, before you proceed, let us look at the other one.

The next problem or next step in this problem is to compare this result with two a's. What happens if there are two a's? That is a_1 plus a_2. u is equal to a_1x plus a_2x squared, so that du by dx is equal to a_1 plus 2 a_2x. Substitute it here. Find out what the results are and let us now compare the results with respect to one degree of freedom and two degrees of freedom or in other words, a linear term and the quadratic term.

We will stop for 10 minutes and see whether you are able to work out this solution and then later, one of you can come and work out this and then we will get some very interesting results to compare with actual solution or analytical solutions. So, you had some 10 minutes time to work out. I hope you would have done the problems, but let us look at the first problem.

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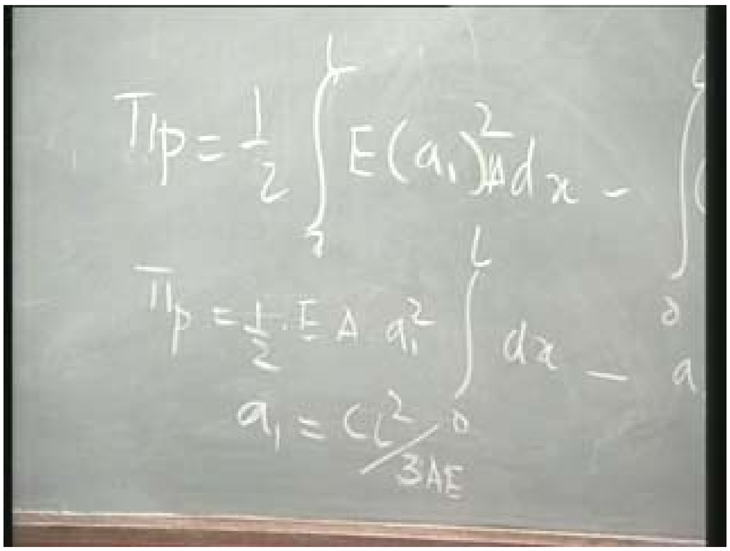

For your sake let me work out or let me look at the first problem or the first part of the problem. The first part we had said that u is given by means of a_1x. Substituting it here in this expression so that du by dx becomes a_1 and hence epsilon x whole squared becomes just a_1 squared, I can write down pi_p in this format. Then, integrating it, so, this term becomes, now this tern becomes just L and that term becomes, what is it? L cube by 3, so that, ultimately a_1 becomes CL squared by 3AE and note that dpi_p by da_1 is equal to zero and that is our initial condition for the potential energy theorem. Is that clear?

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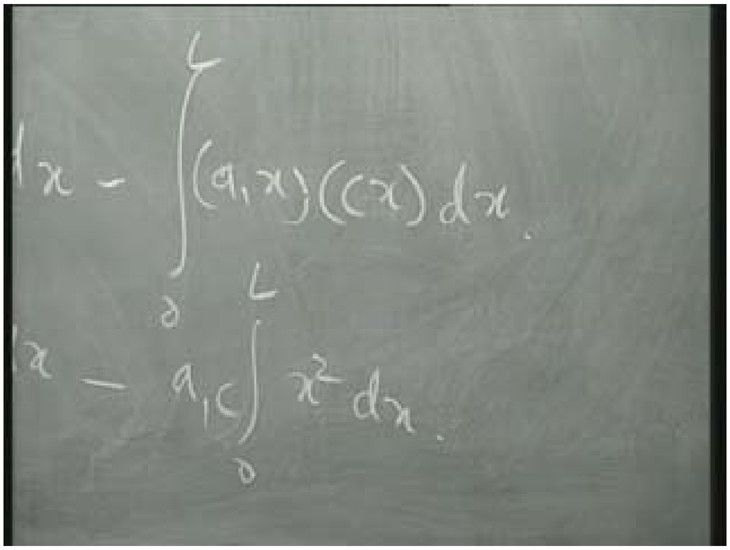

Now let us look at the second part of the problem. The second part of the problem has u to be defined in terms of a_1x plus a_2x squared or in other words what is that we have done? We have taken more terms in the polynomial and all of you know that as I take more and more terms in a polynomial I can fit, whatever be the order of u I can fit or whatever be the curve, I can fit it closer and closer. Any curve can be represented by means of polynomials. I go closer to the curve, as I take many more number of terms. That is the whole idea. You are right now clear that, when I take more terms to u then, I will go closer to the solution. We are going to see that in just a minute.

At this point of time compare u with, what you have got as the final solution or the accurate solution or rather solution that has been arrived at from analytical means and then compare this also, the stress and strain that you arrive at also, with respect to the other thing or in other words, immediately you can realize, how do you calculate stress, by the way?

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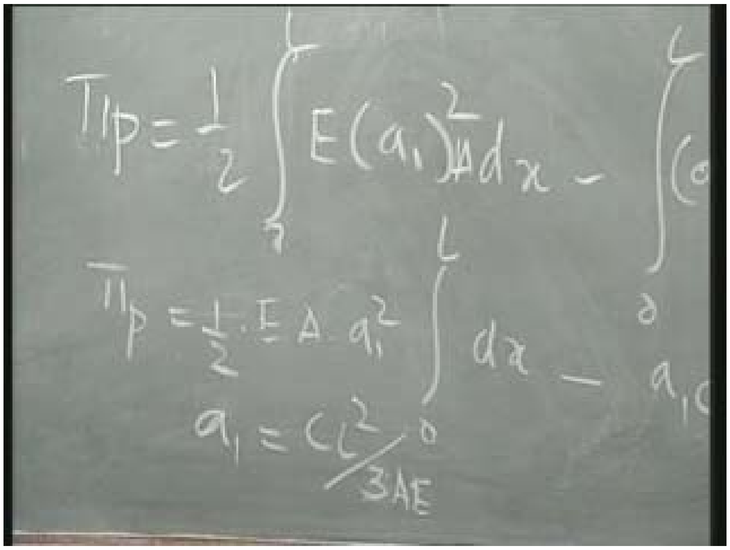

So, this is equal to, u is equal to a_1x. So, du by dx will give me epsilon x and that is equal to a_1 and that is equal to this expression. So, CL squared by 3AE gives me the, what is that it gives? Strain multiplied by E gives me the stress. So, what does it mean? It means that, now the stress, if I have taken, when I have taken the one term, is constant throughout the bar, which is definitely rather an approximation.

Let us look at two term solution. Let us see how better this two term solution is and let us see whether one of the students can come and workout a two term solution. Can you come? We can work out the two term solution. You can bring your paper, whatever you have worked out and look at that. Write boldly, yeah. You work it out; work it out as you have done; the second one. Yeah; so, you can point out if there is any mistake. Is it correct? a_1 squared; what happens to this term actually u comma x; u comma x is what? dow u by dow x. What is that here? a_1 plus a_2x whole squared. What? Is he writing correctly? So, 4; so, this is the second one. a_1 a_2x, so, the first term is right. What is it? 2, yeah, $2a_2x$; not x_2, yeah, that is it. Just x squared, yeah, I did not notice it, sorry. $2a_2x$ squared, so, second term is right. a_1x plus a_2x squared into C into xdx.

Now integrate this. First term becomes a_1 squared into x. When I substitute for L, that becomes a_1L. Yeah, yeah; you left out a_1 squared.

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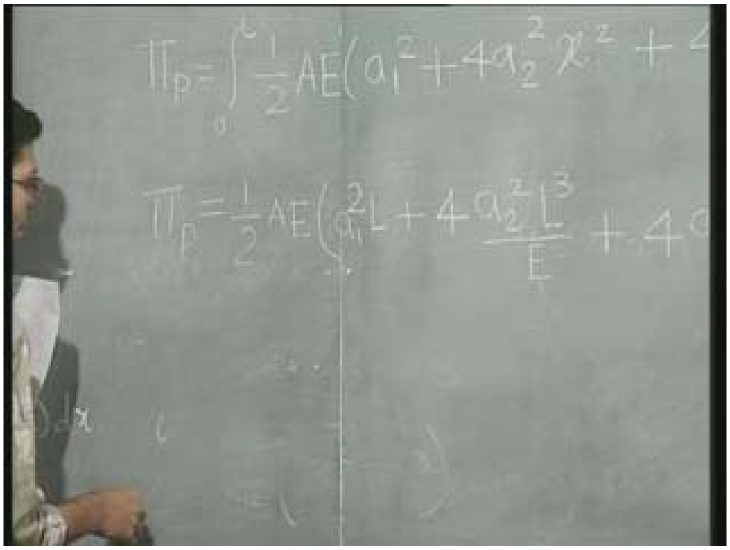

Yeah; is that the second step? What is the third step? Please note here that pi_p is now in terms of a_1 a_2 only, because we have already integrated. So, you can clearly see that pi_p, like in the previous problem or previous discussion, is a function of D_1 and D_2 and so on. Now, it is a function of a_1 a_2 and so on. So, obviously what we have to do is to differentiate pi_p with respect to a. That is a_1 and a_2; dow a_1.

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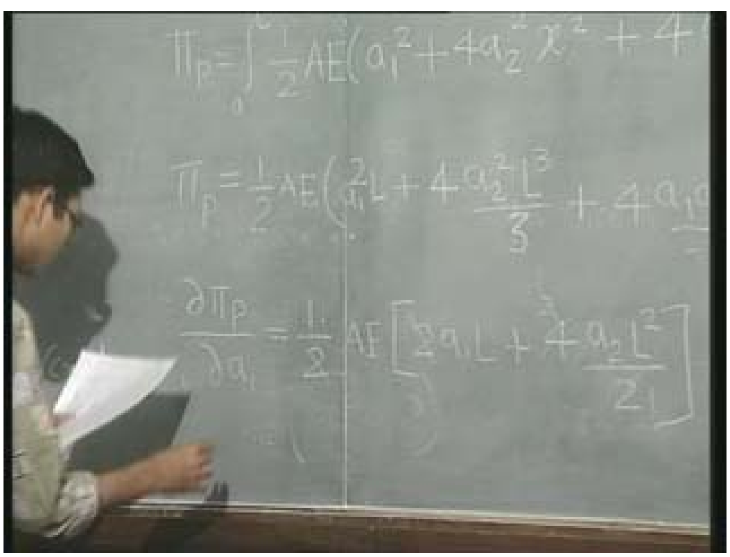

By 3; algebra can be quite complicated, you cannot leave out any term. Is that right? $2a_1$ L plus, this term comes from the $4a_1$ a_2 L squared divided by 2; from that term you get that. That is for a_1. Similarly, you write down dow pi_p by dow a_2; dow pi_p by dow a_2. Yeah, you can take this side dow pi_p by dow a_2. Is that okay? Any one has any objections to it? No algebra mistake; you can rub this off. So, we get now two equations in a_1 and a_2. Just I think you can write down the final value of a_1 and a_2, you can solve for it. Just get the final values of a_1 and a_2. You can write down the a_1 a_2 final values.

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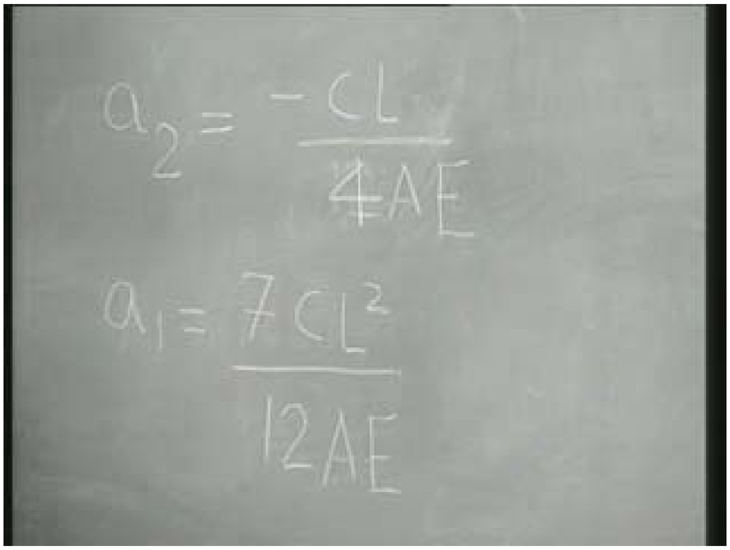

CL squared by 12AE; CL by 4AE, a_1 is CL by, yeah by 4AE; CL by four AE and a_1 is 7CL squared by 12AE.

How does this solution look like? Let us just draw approximately how this solution looks like and see where we stand with respect to the actual solution? Let us see what What is the actual solution? How many terms are there or what is the, I think, I had written down already, I think I have written in a piece of paper. What is the solution? It is cubic, in terms of x cube. First let us plot u versus x.

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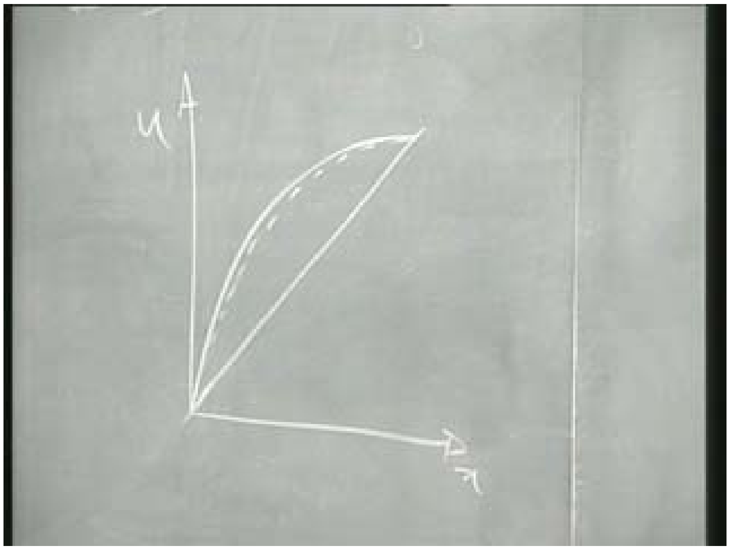

Let us say that that is the correct solution or in other words that is the solution. It has been obtained from analytical means. What is the solution when we took only one term? It is a linear solution and hence that is what if I plot it, that is what I would have got and the second term solution is actually very, very close to the actual solution and looks something like that. In other words, as I take more and more terms, as I take more and more terms, what is that I do? I approach the actual solution.

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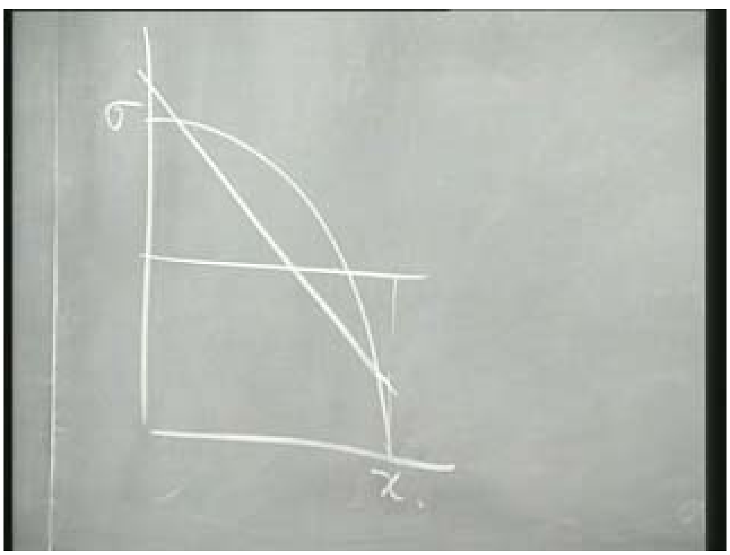

Now, let us see how the stress plot looks like? Now, what would be the stress plot for the first case that is for the actual solution? Will it be a straight line? What is the stress plot for the actual case? It is in terms of x cube. So, it will be a quadratic. The solution would look something like this, sigma versus x. Now, how would the first thing look like or the first solution or first part of the solution? It would be a constant. So, it is like this and the second one, what about the second one? It would be linear. That is something like this. That is the case. Is it clear? Can someone Just look at it for a minute. Can you comment on this graph? There are very interesting things there; there are very interesting things.

Number one is that, yes, there is a constant; there is a linear variation, that is very clear. But apart from that, there are other things as well. Can you look at that graph and tell me? Just take a minute to look at it and tell me. Beautiful; so in other words in other words That is a very good answer. In other words, what it means is that there are points where the solution is very accurate. There are points in this particular bar, where the solution happens to be very accurate. Later on you will see that in finite element analysis also, there are points where the solutions are very accurate and these points are called as super convergent points. That is a very interesting result. So, these two are very interesting, but the point you may ask is how many terms do I take? Do I take only one term, two terms or three terms and number two whether I can take only x cube terms, from x cube terms onwards? What will happen and so on? These are some of the questions, you may ask. So, there are some rules for writing down the displacement functions.

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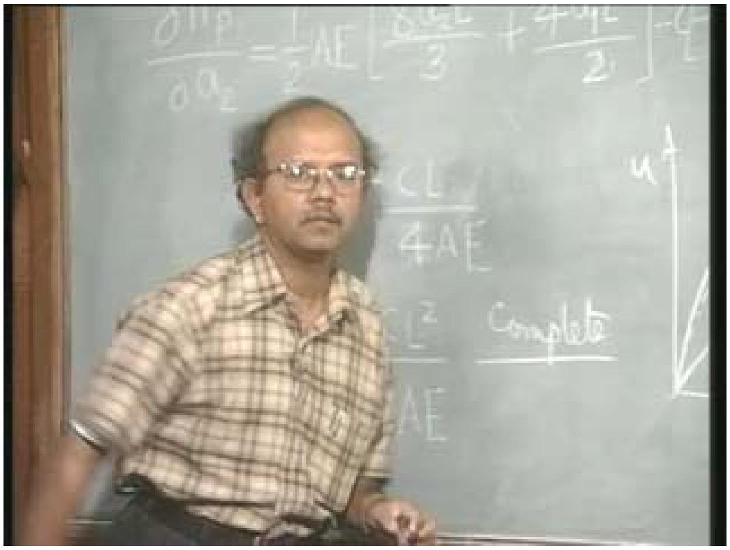

Number 1, the displacement function should be complete, should be complete. What does it mean? It means that if I take a higher order term, x squared term, then I should also take the linear term and so on. Why, because for example, if I had started this whole exercise of writing u in terms of x power 4, if I had taken this in terms of x power 4, what would have happened? Beautiful; so, we would not have got the solution at all. It is important that we have a complete series. If I take x power 4, I should also have x terminate, x squared terminate, x cube terminate and then only x power 4 terms; so, it should be complete in that sense.

Yes, I can leave out, after a_1 and a_2 I can leave out other terms, it does not matter. But I cannot go to the end or n? terms like, after cube or terms with higher order terms and then leave out lower order terms. Number 2, the u should be continuous. Whatever I choose, the u should be continuous and should be single valued, or in other words then only it will give you compatible results. It should be a single valued function; that is important. We are going to expand these conditions later, when we go to the second degree and so on. But right now, it is important to realize that this is the one these are the two important conditions.

Now, let us again look back this curve, look at this curve. Now, look at this curve.

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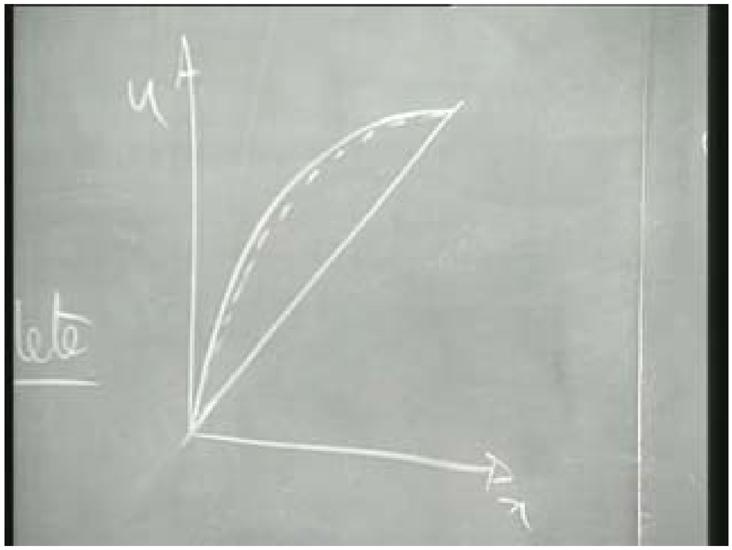

There is one more thing that you would notice from this curve. What is that you notice? That the displacement that we have arrived at is lower than the actual displacement; the displacement that we have arrived at is lower than the actual displacement. In other words, what does it mean, physically? No, what it means physically? Physically it means, yeah, that is it. Physically it means that the stiffness is more. The stiffness is more, when? When is that? When I restrict the degree of freedom; see, obviously in a one term solution, I had only that a_1 fellow. On the other hand, when I make it two terms, a_1 and a_2, I had two chaps to really tell me, how displacement takes place? So, lesser the degree of freedom, more stiff it becomes and hence the displacement terms are smaller or displacements are smaller.

Another thing you should notice very carefully between the two graphs. What is that, between this graph and this graph? What is that you notice between the two?

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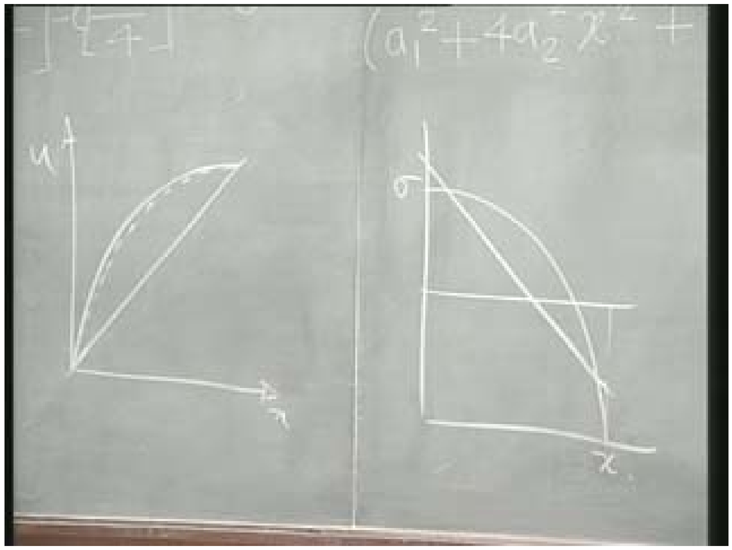

Please notice that displacements are much more accurate than the strains and the stresses. Notice that displacements are much more accurate than strain and stress. This is the same as stress or strain ….. the same; only thing is I have multiplied by E. But, in other words u is much more accurate than its derivative and notice one more thing carefully. These observations are very important because we are going to extend this same observations into finite element analysis, which is nothing but a small extension of the Rayleigh-Ritz principle.

One more thing can you notice from the stress graph. You see that at the end, stress is not zero, as is shown in other cases. In other words, stress boundary conditions will not be exactly satisfied but will also not be accurately satisfied, but also be approximately satisfied. You can look at Rayleigh-Ritz procedure, what we have done, in another sense; in the sense that we do Rayleigh-Ritz procedure by assuming as if the whole body is one element. Now, finite element can be looked at as a Rayleigh-Ritz procedure with the difference that we split up the body into a finite number of elements and start writing down this kind of equations, for every element.

There is quite a unity between finite element and this Rayleigh-Ritz procedure, because both of them are approximate, both of them try to write down the displacement conditions in terms of certain other factors like a_1 or u_1 and so on and both of them rely on, say, for example potential energy minimization or potential energy principle. Is that clear? We will start the derivation for finite element analysis from potential energy principle in the next class. But, meanwhile if there is any question, I can answer. Is this procedure clear? What we are going to do is to extend this procedure and apply it to a number of elements. I know that the criticism, as someone said that, a_1 a_2, what is a_1 a_2 a_3? That is not very clear. That is exactly what we are going to address, when we go to finite element or in other words what we are going to do is to replace the degrees of freedom a_1 a_2 a_3, which does not seem to have any physical meaning by means of degrees of freedom of the nodes, which seems to have much more or which looks like it has a much more physical sense rather than a's. How are we going to derive this? We will see that in the next class.