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Lecture - 11

We saw in the last class, the derivation for what we call as the elastic moduli or the matrix which relates this stress and strain. You remember that this is what we derived.

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We ended up that lecture saying that it is possible to simplify certain of these terms.

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For example we know that there are 6 terms or 9 terms, which can be brought into 6 in the stress matrix and similarly we have 9 terms which can be brought down to 6 in this strain matrix. Now, let us look at the position or let us look at what happens to a component which is very thin, say, you have a very thin gear or so on, a very thin component and the load is applied in plane. For simplicity, let us take just a sheet, a sheet whose thickness is very small when compared to the other dimensions and then apply load like this. What do we mean? We mean that that is very small. Is that clear? We assume that the load is uniform in this small area. In other words we can say that the thickness is so small that I can assume as if the load exists throughout that small thickness.

Let us study a problem or this problem when thickness tends to zero or very small. When thickness is very small there is anyway a strain in the thickness direction, there is a strain in the thickness direction, which for example can be written as epsilon₃, sorry 3 and hence epsilon₃₃. The strain in the third direction is epsilon₃₃. This strain, please note that this strain, is not allowed or not stopped by material to perform or in other words there is not enough material to stop this strain being present.

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Just to make this clear let us say that we have a very thick bar. Let us say that we have a very thick bar and now we are loading this uniformly throughout and let us now look at what happens, that is the third direction and what happens to this strain in the third direction? Let us say that we just take a simple prism, a small prism, imaginarily cut and see what happens to this strain. Especially let us look at what happens to Let us divide this prism into number of cubes. Especially now let us look at what happens to a small cube, say, cube number 1. It is a part of this prism.

This cube 1 is now bounded by two more cubes on either side. Say, let me call this as 2 and 3. Is that clear? It is bounded by two cubes. Assume that they are quite inside. Cube number 1 wants to come in; it wants to contract under this extension load, he wants to come in. What happens to cube number 2? They are all inside, so, cube number 2 also wants to come in. What happens to cube number 3? He also wants to come in. All these guys want to come in. What does it mean or what happens?

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Let me take the cube out. Say, I have 1 here, 2 here and 3 here. Let us say that this is face number 1, face number 2, face number 3 and face number 4; faces, they are the faces of the cube. That is face 1 is this face, in other words. Is that clear? Let us study this problem. 1 wants to come in. What does it mean? It means that the face 2 and 3 want to go inside. They want to move like this. They want to move inside, but the same faces are a part of two other cubes. Say for example, this face 2 is a part of cube 2. Let us now go and sit inside that cube 2 and see what I am going to experience. It is very clear that when I go and sit here as a cube, this face 2 would like to move in the opposite direction. That is in this direction and this guy here, 3 similarly because of cube 3 would like to go in the opposite direction. What are these effects?

These are Poisson's effect. These are Poisson's effect. Poisson's effect as far as cube 2 is concerned, would move the face 2 inside, cube 3 cube 1 is concerned would move the face 2 in the other direction and similarly face 3 is caught in a dilemma whether to go in or go out. Is the point clear? All these guys, who are sitting right inside this material, here this material, are guys who have very similar environment or in other words physically what does it mean? Similar environment; that means that this guy 1 will pull the face 2 with the same force or magnitude or whatever it is as the cube 2 which would like to pull it in the opposite direction.

So, what will happen to the Face 2? The fellow will not move. Two guys with equal strength, they are trying to pull each other; so, they will not move at all. So, face 2 will remain there. It will just be there only. It will not move. What will happen to face 3? Face 3? Same way that fellow also will not move and what will happen to face 4 and 1? They will also be the same way because there will be another cube. These are all small cubes. We are just trying to understand what happens? There are number of cubes, except cubes which are very close to the edges. In other words, more technically, these cubes will not have strain in the third direction. They will not have strain in the third direction on in other words epsilon₃₃ will be zero. Is that clear?

Intuitively you can see the change in length by original length; change in length is not there, does not exist and hence the face 2 and face 3 will not move and hence at that point if I take that infinitesimal cube and sit there I will not experience any strain. Is that clear? If I take all these cubes here, except the cubes which are at the either end, they will not have any epsilon₃₃.





Such a condition where, if I take any cross section, then their behavior is the same; the same is going to happen when I talk about $epsilon_{13}$ and $epsilon_{23}$. Again it cannot shear, because the next cube will stop it from shearing. How do I now represent the strain matrix? Instead of a 3 by 3 matrix, I can represent the strain matrix as a 2 by 2 matrix. What are the terms which go out of this? 13 23 33. So, row and column

which corresponds to 3 goes out. What I have here is only $epsilon_{11}$, $epsilon_{12}$, $epsilon_{21}$; $epsilon_{21}$ is equal to $epsilon_{12}$ and $epsilon_{22}$. That is all will be there in this matrix, so, it becomes a 2 by 2 matrix. When is this assumption valid? Assumption is valid when the thickness is very large. Now, how large? Strictly speaking, thickness is infinity, tending towards infinity. But I cannot have an infinite piece. There is no infinite piece.

What do I do? I compare this thickness with the rest of the body. When compared to the rest of the body, the thickness is large, quite large. I make an assumption, again an engineering judgment, saying that this is a plane strain behavior. 1148Plane that means in 2D; plane strain behavior. Is this point clear? Let us now look at a position where the body is thin, the body is thin like this body. I do not have, Let us say, that the thickness tends towards zero; very, very thin material. I do not have enough space to draw or to put down cubes, hardly. They are so thin that this kind of explanation would not be valid because I will immediately hit the surface. Under that case or In other words it is so small, imagine that I can put down only one cube, T tending towards it. On other hand, that one was T tending towards infinity, very thick; here T tending towards zero. It is very easy to understand when you start imagining that T tends towards zero.



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That means that I can put only one cube here. Let me draw that separately. So, I will have only one cube and that cube is such that it goes and sits and finishes off the whole of the thickness. This thickness here is equal to the thickness of this sheet. What happens under these circumstances? Pardon; compressed or in other words there are no cubes like the 2 and 3 guys, the neighbors who do not cooperate with 1, you know, here. In this case here, those two neighbors, they do not cooperate with 1. On the other hand this guy has no neighbors. He can do whatever he wants and hence these two faces on either side will go in.

What happens to this situation? How do you look at this situation? Here does $epsilon_{33}$ exist? Though the way the loads are there, $epsilon_{13}$ and $epsilon_{23}$ does not exist, $epsilon_{33}$ exists. But what about stress, what about stress? It does not exist. There is two dimensional stress condition, because unless this $epsilon_{33}$ is stopped, epsilon₃₃ is stopped, you will not get a stress matrix and hence that situation where this becomes 2 by 2 is called as the plane stress condition. Any question? Clear? Thickness, please note, tends towards zero.

Yes, because when do we have a stress? That is my next question. When do we get stress? See, unfortunately in all your earlier classes you would have studied stress is proportional to strain. In other words you would have studied that when strain exist, stress exists; stress exists, strain exists. That is a total misnomer; that is not right. In fact, you would have done problems contrary to it. I will explain that problem also. Before we go to that problem, look at this situation. Actually though I say strain does not exist, what has happened? Wait a minute, what has happened? There would have been a strain which would have been dictated by nu; nu into epsilon₁₁. There would have been a strain there.

In 1, in this cube 1, there would have been a strain. But, actually what is that the situation has done or the environment has done? It has not allowed that strain, the legitimate strain, which is due to this cube 1 to take place. That was not allowed by 2 and 3. Is this clear? It does not allow. So, what does it mean? It means that there has to be a stress to stop that strain not to be present. Is that clear? In this case there was a strain. In the thin sheet case, here there is a strain. There is a strain in the third direction; is it not? There is a strain. This strain in the third direction was not stopped

by this row cubes which are on either side. There is nothing like that there, because this thickness is such that I can put only one cube \dots and hence we get, what is the situation we get? We get a situation where there is strain, but there is no stress. Unless you go to stop material, you are not going to have stress.

Can you think of one problem which you have done in strength of material classes which is similar to this? You would have done, thermal problem; that is beautiful, thermal problem.



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Remember that you would have, The simplest of them would be a situation where there is a bar with the length, some L. We try to heat it, but it is rigidly fixed. What would be the stress? Actually what is that you have done there? You wanted the bar to go up, but did not allow the strain to take place. In other words you had stressed it to make it zero strain. You would have calculated the stress in a very simple fashion and maybe you would have had multiple bars with different alpha and so on and you would have done the problem. But what is that you did there? You stopped strain. Suppose you had used the formula, change in length by original length and said that the length does not change and hence strain is equal to zero multiplied by E. That is, stress is equal to zero. That would have been a nonsense result because you yourself know intuitively.

You should be very careful when you utter statements like when strain is zero, stress is zero or stress is zero strain is zero and so on. Is that clear? What is the lesson out of this? The lesson here is that, a three dimensional body can be approximated by a two dimensional situation by plane strain and plane stress. Is it clear? Let us keep this for a minute. Now, what I can do is, I can work out this C matrix. See, sometimes I will use C matrix or D matrix, because you should not get confused. C matrix is the stress strain relationship. Similarly, the C also in some books is specified in terms of d matrix; dijk instead of cijk, both of them are the same.

It is possible now to reduce this matrix for plane strain and plane stress. It is possible. How do we do this? Let us not get into that right now, because we want to do one more problem. But, can you tell me how you can reduce this?



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How do you reduce this? For example you want to look at plane stress. I want to reduce this for plane stress, how will you reduce this? Can I just take two rows two columns and finish it off? No, no. You have to take the complete matrix and say that look, sigma₃₃ is equal to zero. What is the expression for sigma₃₃? Because epsilon₃₃ exist, so, how do I express now epsilon₃₃ in terms of epsilon₁₁ and epsilon₂₂, go back and substitute it and then get the stiffness matrix. On the other hand, In other words though sigma₃₃ is equal to zero, in this column what is that I am going to multiply with the strain vector.

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What happens? My epsilon₃₃ still exists. So, I cannot make an assumption that I will just remove the third row, from third row third column onwards and put it up. On the other hand, what happens to the plane strain case? What happens to the plane strain case? Epsilon₃₃ is equal to zero. I have no problems there. So, I can cut it off and calculate sigma₃₃ with epsilon₃₃, with epsilon₁₁ and epsilon₂₂. Is that clear?

Now, let us look at a slide and see what the situation is, here.



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Let us say this is an LP motor, which we saw in our earlier classes. You remember that there was a defect. Forget for a moment that there is a defect in that component; it is a perfect component. Assume that you are only interested in the centrifugal forces. In fact, it is true that in most of these rotors like this, the stresses are more due to centrifugal force. So, let us assume that you are interested to calculate the stresses due to the centrifugal force in that rotor. Now, how does that rotor behave or in other words can I make a simplification to analyze this rotor?

Have a look at it again and tell me whether it is possible to make some assumptions regarding the rotor.



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Is there anything? Any clue you get? Fantastic; so, the first thing that immediately strikes you on looking at that rotor is that there is an axis of symmetry. Right at the center there is an axis of symmetry; right at the center, does not it? What does it mean? It means that if I take a section of it and then just rotate it about the axis, 360 degrees, I will get the complete geometry of the rotor. So that is very good. The geometry is axi-symmetric. What else do you get from that rotor? One is geometry is axi-symmetric. But look at it and just think for a minute. What else is there? I have talked about the loading. I have already given you a clue that, yeah, it is a centrifugal force. What does it mean? It means that, correct; in other words what it means is that if I take whatever be the section, all these sections would behave similarly; not

exactly plane strain. Just a minute, but it is a very similar behavior, it is a very similar behavior.

For example if I take a ring, now I can assume that this material is made up of rings like this, number of rings. So, whether I take a cube which is sitting here or the next cube which is sitting here and so on, these cubes behavior wherever I go and touch, touch it here or touch it here or touch it here, would be the same. From that point of view, it looks that it is a plane strain problem, but there is a small difference.



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In order to explain that let me put down a coordinate system. Let us say that What is the easiest coordinate system for this r, theta and z; so, r, theta and z coordinate system? Now, what is going to happen to this ring or let me go the other way. Since you say that it is plane strain problem, now my epsilons are slightly different; not to confuse with 33, 22 and so on, let me talk about this as $epsilon_{rr}$, $epsilon_{theta}$ theta, $epsilon_{rtheta}$ and $epsilon_{zz}$, so that you will understand what? Because we are in r theta coordinate system, so, that $epsilon_{zz}$ is in this direction, $epsilon_{theta}$ theta is in this direction; $epsilon_{theta}$ theta is in this direction and $epsilon_{rr}$ is in the other direction. Is that clear? So that would be $epsilon_{theta}$ theta. I have to be careful in answering my question. What is epsilon_{theta theta}? What is epsilon_{theta theta}? Many of you made a statement that it is like plane strain? Right, you said that it is like plane strain. That is what I wanted; a controversy that some people say it is zero and some people say it is not zero. Can one answer why it is zero? Absolutely; so, in other words you are giving an explanation which is very similar to what I gave for plane strain. You are trying to say that this fellow will stop this fellow. It is like the proverbial tortoise; one tortoise who wants to claim another fellow goes and pushes it down and so on. So, he wants to expand; this fellow will try to, okay, fine. I am not saying you are correct, but I understand what you are saying.

Let us hear the other part of the story. Someone said that it is not so; what is your explanation? Fantastic; so, that is the thing. You forgot about that; the ring will now expand. What happens when the ring expands? Because under centrifugal force all the guys who try to go out, so, the ring will try to expand. When the ring tries to expand what happens to the circumference? It will increase. If u is the displacement then, if u is the displacement in the r direction then, the new circumference will be equal to 2pi into r plus u. Get back to our original definition 2 pi into r plus u minus 2 pi r divided by 2 pi r. Then, you can see that it is u by r. What does it indicate?

It indicates that there is a strain in the theta direction, but it is independent of the angle and depends upon r. So, epsilon_{theta theta} is not equal to zero unlike the plane strain case. Is that clear?

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So, unlike the plane strain case, epsilon_{theta theta} is equal to u by r. Is that clear? Any question? This fellow will try to expand; this ring just to put the midpoint of the ring, that ring will try to go now to another position. The whole body will try to expand, will be subjected to strain in the r direction. Epsilon_{theta theta}, Please note that many students make the mistake that epsilon_{theta theta} is equal to zero. It is not so; it exists and is equal to u by r. Is that clear? Any question_{zz} is there, epsilon_{theta theta}, epsilon_{rr} and epsilon_{rz}; these are the things that exist.

I can still look at it as a 2D problem in the rz plane and then not forgetting that $epsilon_{theta theta}$ is not equal to zero. $Epsilon_{theta z}$ cannot exist because there this fighting between the two cubes will start. Then he has to deform in one side, so the other cube will not allow it to deform. So, all that will be stopped. Is it clear? Now, let us look at the next slide. What we have in that? Let us look at the next slide.

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What is that? That is the finite element model and what has been done is to model it using what is called as an axi-symmetric element. What we are trying to do, as a very first step in modeling after going through all these things, is to establish whether this problem can be looked at from mechanics point of view as an axi-symmetric problem, plane strain problem, plane stress problem or a complete solid problem, which has to be solved. In that process you will make some assumptions. Look at that for a minute and tell me whether I have made any assumption to make this problem an axisymmetric problem.

Is there any assumption? You look at those black lines there. What are those black lines? That is the first question you may ask. That is due to the centrifugal force of the blades. That is due to the force that is exerted by the blades onto the body, those black lines there. What are the assumptions I have made? I have made an assumption the blades as if it is continuously there, throughout theta. Though you may argue that they are not continuous, but for simplicity we have made an assumption which we feel will not affect to that extent; an assumption that the blade can be assumed to be continuously distributed throughout theta. The first thing that you come out to model finite element is to say what is the type of problem? This problem happens to be axisymmetric, you have to have that also in mind.

Let us look at the next problem. Let us see what we get over there.

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That is the solid model of a complete tyre. I am interested in two things in this tyre. What are the two things? The first thing I am interested in is the inflation - deformation due to inflation, stress due to internal pressure, inflation pressure; that is my first problem. The other problem I am interested in is inflation plus the contact between the body and the ground or the contact stresses. Just see that for all practical purposes you can assume that this is axi-symmetric, but can I do an axi-symmetric analysis for both the problems? Yes; No, that is very good? Let us see for what we can do and what we cannot do?

Let us look at the next slide.

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This is the model for, I hope it is clear, that is the model for inflation alone. That means that the tyre is on the ground, I have not put it on to the vehicle; I have put it on the ground and just inflated it. When you fix punctures or flat tyres, you take it out; you put it. That is the situation here. Under these circumstances, it is axi-symmetric, because the pressure is uniformly distributed. Go back to my ring here.



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The ring situation is that all these fellows, every chap, I do not have any preference, all the guys are affected by the same fashion. Hence the behavior is an axi-symmetric

behavior. We will later talk about how to derive element properties, but my first goal is to teach you, how to select an element? My behavior or the behavior of this material, of this geometry is axi-symmetric behavior and I have to choose an axi-symmetric element. Yes; I am coming to that, very good. One second, one second; let us look at the next slide to answer your question, what happens? Yes; look at that.



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What you asked is that what happens when this tyre is in contact with the ground? That is this problem. See that when the tyre is in contact with the ground, it is no more axi-symmetric because the guys who are right there sitting just below or just below the axle or just touching the ground, they are the guys who take the heavy beating by the normal forces. So, as I travel, radial ah sorry ah um along the along theta direction starting my travel from a cube which is right at the center of my contact region as I go out, I will decrease; now, I will be more relaxed. This equality is lost. You get an inequality situation and hence the problem is no more axi-symmetric. One cube here is much more stressed than the other cubes as I go around. Is it clear? Let us get back to that problem again.

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Let us look at the next problem. The next problem is connecting rod. Just before we go, only one lesson out of what you have learnt? What is the lesson? The lesson is that looking at the geometry you cannot say whether it is axi-symmetric or not axisymmetric. Is that clear? You have to consider both, loading as well as geometry and for the same geometry, a part of the loading may give you an axi-symmetric situation and another part of the loading may not give you an axi-symmetric situation. You have to be looking for both and always remember my cubes here and that will help you to realize whether the situation is axi-symmetric or not axi-symmetric. Is that clear?

Let us go and look at the slide now. That is the connecting rod of an engine. To be specific it is the connecting rod of a two, sorry, four stroke two wheeler. Now, look at that. I am sure all of you know the connecting rod there and how it is loaded and so on. What is the type of element that you think, you can choose? Yes, that is the thing. In other words, what you are saying there is a small confusion, you are not very clear. No, no, no; please note that you cannot mix up situations; you cannot say that one part is plane strain and another part is plane stress. In analysis you cannot split up like that. You cannot confuse issues there. It is As a component you are looking at whether plane strain, plane stress, axi-symmetric, solid. These are the four things we are considering right now.





What is the element that you think you can? Is it plane strain? Is it plane stress? Plane strain, plain stress, these are only two answers; plane that's what I am saying. A part of the class say it is plane stress, another part says plane strain. Is there any other answer apart from these two? Is there any other answer? I will give you clue. I already said that there are four things that I am looking for: plane stress, plane strain, axi-symmetric, complete 3D; complete 3D. Why is that you left it out? This is not just plane strain or plane stress, because if you look at this geometry actually, have a look at this geometry. What is the section? You know what is the section? It is an I section, correct; so, it is an I section.

In what way you can assume this as plane? You can clearly see that it is an I section. In what way can you say this is plane stress? Plane stress means when I cut, the section should be the same then only my cube or even in plane strain in the section should be the same. Then only my cubes will be in the same environment. If some portion is thick and some portion is thin, I cannot cut slices. I cannot consider as 2D. Ultimately, I am going to look at it as 2D, as you saw right one in the previous one. So, it is neither plane strain nor plane stress, nor definitely axi-symmetric and hence it is a complete 3D.





Though in that particular slide it is not very clearly seen, it is complete 3D. That is the solid model and the next slide clearly shows the results though the mesh is very complex. Why is that the mesh is not shown? We will see that later because the mesh becomes extremely complex to understand.

Let us now look at the next slide and see what you can get?

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Look at that. That is modified; now the design has been modified. Look at that. Let us forget about the green and red. Let us assume that it is only one complete element, one component. Can you now guess what would be the type of behavior? What would be the type of behavior? Yeah; I know you are getting confused, but you talk it out so that you will have a, plane strain; then, 3D. Of the four choices I think there are people who vouch for three at least, right; equally split almost. So, part of the class says plane stress, part of the class says plane strain, part of the class says 3D; very good.

That is actually the problem of modeling finite element analysis. It is not new, you know. You need not feel bad about it and that is exactly where many people fail in finite element analysis which has prompted even many people to say that I do not believe this result, you have made that assumption, this assumption and so on. It is not true. You have to be very careful in looking at assumptions, but at the same time you cannot keep saying that everything in this world is 3D. I will do all the time 3D problem; that is a pessimistic view because, in fact there are certain situations where bending is involved where 3D will not give good results. I have to resort to much more special cases like beam element where bending is depicted very correctly. So you cannot take that extreme saying that everything is 3D, but you have to be careful as to what you are analyzing.

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Look at this connecting rod. There are a number of forces that act on this connecting rod. Before answering you should have first checked up what is that you want. Look at that connecting rod and what are the forces that are acting. Yes; the first thing is how is the stresses developed or why is the stresses developed in the connecting rod? Because there is gas pressure; gas pressure will try to compress this. That is one thing. What are the other types of forces that may act in this connecting rod? Due to the reciprocating masses, the inertia of the reciprocating masses as well as the connecting rod itself starts moving and its own inertia also comes into picture.

What does this inertia give you or what is the type of force that happens to be present with its own inertia? It is a bending load; it bends, it bends. So, there is compression plus bending and hence it is called, actually this connecting rod is called, as beam column; beam, because it bends, column because it is compressed. So, these connecting rods are called as beam column. Is that clear? Now, of the three things, of the three forces that are going to cause you trouble, two important things are the compression, the gas compression as well as the bending.

The gas compression, the compression due to the maximum gas pressure is the one which is going to cause the maximum stress, but that is going to be compressive. In fact, if you really look at it, I have to do a whole series of analysis with this. I have to do a buckling analysis and I have to do other things. Let me not worry about all that. Let us just worry about static analysis only for illustration for this illustration. Maybe later in the course we will see a complete analysis, what all you can do with connecting rod; that we will see towards the end of the course. Right now, let us say that I am going to do only gas pressures or gas pressure plus bending.

When I take gas pressure plus bending, that is going to bend, say for example, like this. Then, what happens? Then, what is the type of analysis I have to do? Only for, say Let us take only for gas pressure so that you will not get confused. Then, what is the type of analysis you do? Plane stress, because it is so thin I can assume that this is very thin and the new connecting rod sections are the same as I translate the section; plane stress. If the connecting rod happens to be very, very thick, for example, look at a dam, a huge dam. Length is very large when compared to the depth, in which case what is the type of analysis you will do?

Plane strain; whereas, here it is quite thin. So, you can do a plane stress analysis. But, there is always a fear as to how best you are approximating it. If it happens to be crucial and if your thickness is quite large, you are not very satisfied, you can as well do a 3D analysis. So, even with bending, what happens? There is only an in-plane load. There is an in-plane load. Both the loads are in-plane; biaxial. So, I have no problem in doing this as plane stress, but you can do also a 3D. If it happens to be quite thick, then you can do a plane strain problem.

Let us just look at the next slide and see what is the component we have and what is the type of analysis we have to do?

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That is a gear, a gear rolling. Again, what is the type of analysis that you will do for this gear?

We will stop here and maybe we will continue this gear problem in the next class.