Introduction to Finite Element Method
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Lecture – 10

In the last class we started our discussion on strain energy density and strain energy density function. We said that for elastic materials, it is important that a function called strain energy density function exist. In other words, we postulate the existence of a function called strain energy density function.

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We also said that the relationship between stress and strain is given by means of an equation of the form sigma$_{ij}$ is equal to dow W by dow epsilon$_{ij}$. That is what we said. What does it mean? It means that if you are able to define a strain energy density function, how you define it looking at the behaviour of the material and so on. Then the stress strain relationship is obtained from this equation.

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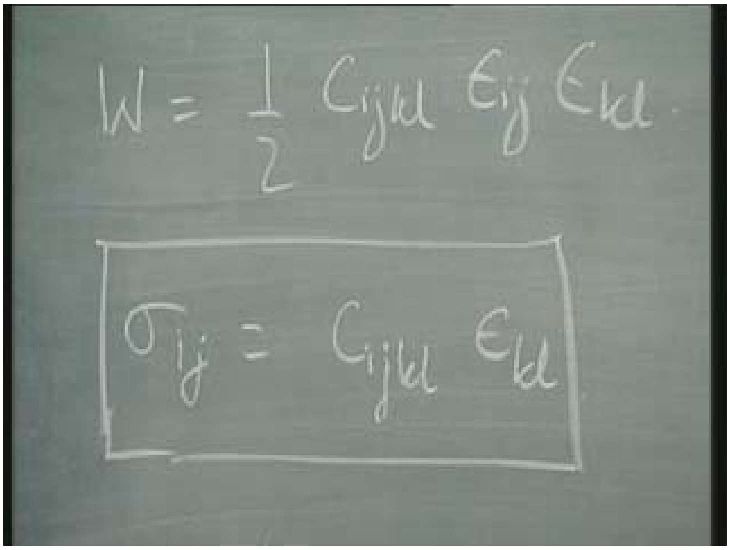

For example if you look at an elastic material, I can define a strain energy density function to be half of C_{ijkl} epsilon$_{ij}$ epsilon$_{kl}$. The definition of strain energy density function, say for example definition of this function, is quite involved. There are lots of things that you have to satisfy, but I am not going into the details of it; how I got this, because that is a separate course all together. But nevertheless satisfying some very fundamental criteria which are based on thermodynamics and so on, it is possible to write down, it is possible write down, for a class of material, functions. For example this is the function for a linear elastic material. Similar functions can be written down, for example, for an elastomer or rubber and so on and in fact you can look at a material to be distinguished by means of this kind of strain energy density function. These materials which are defined by means of this route are called as green elastic material. They are thermodynamically, very correct; mechanics wise or rather more broadly continue mechanics wise, they are well defined materials.

For example, if I define W is equal to C_{ijkl} epsilon$_{ij}$ epsilon$_{kl}$, then I can write down my stress strain relationship to be sigma$_{ij}$. That is dow W by dow epsilon$_{ij}$, which you can just expand it and write it. I will get C_{ijkl} epsilon$_{kl}$; C_{ijkl} epsilon$_{kl}$. In other words, what is this? It is a second order tensor; a second order tensor being transformed by a fourth order tensor. So, the fourth order tensor is a transformation function. I am not going to work out this step. I think by now you should be familiar with this. We will work it out, as well as, I mean very easily ……. That is actually what we call as elastic or linear elastic constitutive equation; material which obeys the Hooke's law or a broader, rather I should say that material which is isotropic hook in material is a sub class of this rather I should say that they are the elastic or linear elastic materials general equation.

The question that immediately comes to your mind is that look what have you put C_{ijkl} and you have asked us to vary ijk and l, you know, each three times 1 2 and 3. So, I seem to get how much?

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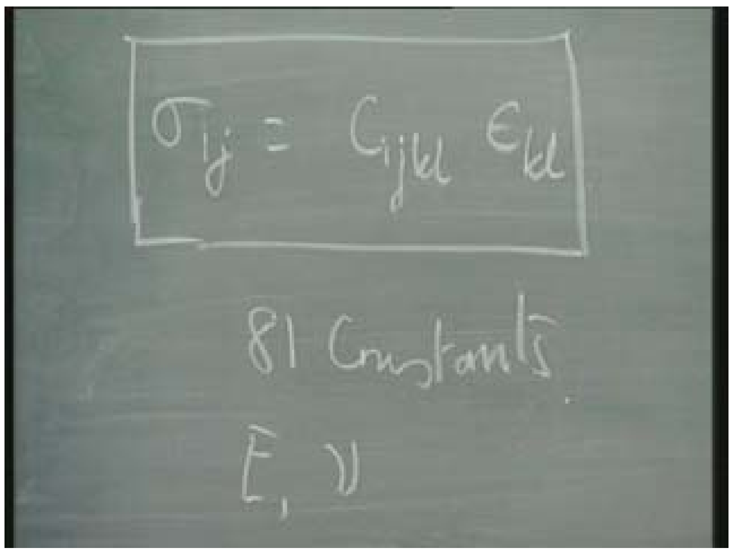

81 constants; so, I seem to get 81 constants by doing whatever you have said, but I know only of two constants. If you look at steel, I know only two constants. These two constants I am familiar with are Young's modulus and Poisson's ratio. What are you talking about or how do you connect now these 81 constants to these two constants? That is the question that is upper most in our minds. Is that clear? In order to get from here to here we have to see all the assumptions that have gone into defining a material, like for example steel; say, for example a connecting rod which we come across.

What are the assumptions that ….? This is a very general material; elastic material, no doubt, but very general. What are the assumptions that have gone into to get these two constants? Can someone guess? Isotropic.

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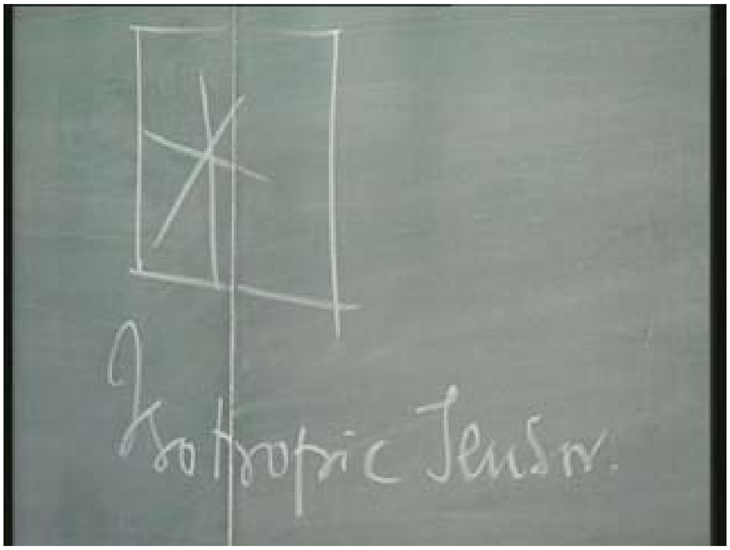

Yes; so, isotropic homogeneous material. What do you mean by isotropic material? Yes, properties are same in all directions; properties are same in all directions. However I rotate this, the properties are the same. More clearly, more in a mundane sense, if I have a sheet, if I take a tensile piece from the sheet with whatever be the orientation, I am going to get the same properties. Is that clear and that is a major assumption. What does this assumption give us?

We get into or we signified this or we put this mathematically; see how mathematics now plays into ……... We say that if you are talking about isotropic material, then I can put a subclass to this fourth order tensor. That subclass of this fourth order tensor is an isotropic tensor; it is an isotropic tensor. Now, what is an isotropic tensor? Fortunately for us there has been, there is lot of mathematical effort to define isotropic tensor. In fact a person called Jeffreys in 1973 found out that an isotropic tensor of the of the fourth order, say, C_{ijkl} of the fourth order can be written by means of or with the help of three constants only.

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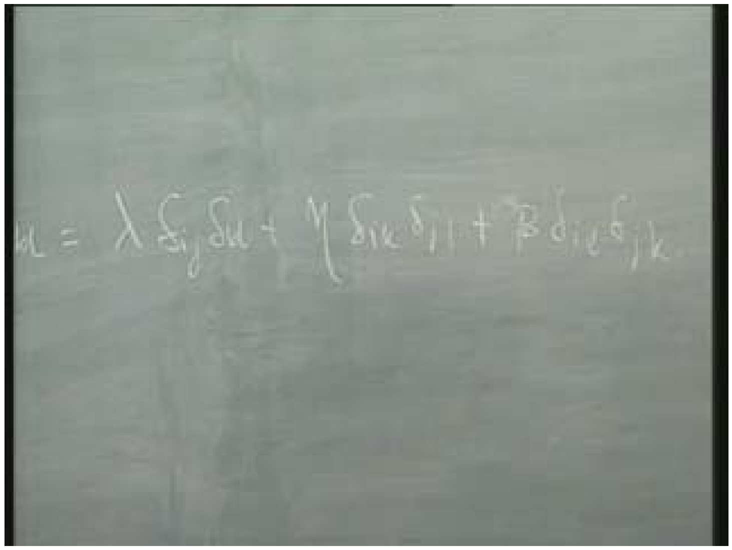

For example, I can say $delta_{ij}$ $delta_{kl}$ plus say eta $delta_{ik}$ $delta_{jl}$ plus say mu or beta $delta_{il}$ $delta_{jk}$. He said that if at all there is an isotropic tensor, fourth order tensor, he defined that this isotropic tensor can be written by this fashion. What does it mean? It means that whenever I do a transformation, we have not studied transformation, does not matter. Whenever I do transformation of the coordinate system, my tensor will not change. That is what is meant by isotropic property? This fourth order tensor will not change. So, he arrived at mathematically and said that, look if you are able to write it in this fashion, then this tensor happens to be an isotropic tensor. Is that clear?

For example he looked at second order tensor and said that $delta_{ij}$ is the only second order tensor, which is isotropic and so on. Let us not worry about his paper, but it is very important that we realize that this isotropy, the property called isotropy can be converted into a mathematical form and written like this. Is that clear? Let me substitute that equation into this equation C_{ijkl} $epsilon_{kl}$ and see what happens.

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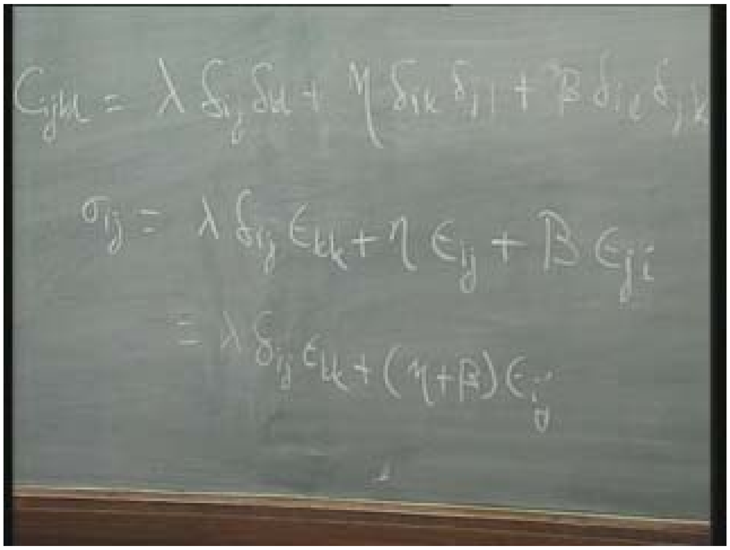

Now, this can be written as sigma$_{ij}$ equal to, note this carefully, so, you will get some practice in writing down the indicial notations. Sigma$_{ij}$ is equal delta into delta$_{ij}$ delta$_{kl}$ into epsilon$_{kl}$. What is that I will get? Delta, sorry lambda into delta$_{ij}$ delta$_{kl}$ into epsilon$_{kl}$ kk, yeah right; but, what happens to delta$_{ij}$? That remains. Correct, that is all. Delta$_{ij}$, k should become equal to l and so epsilon$_{kl}$ will become kk, beautiful; epsilon$_{kk}$ plus eta into delta$_{ik}$ delta$_{jl}$; eta into delta$_{ik}$ delta$_{jl}$ epsilon$_{kl}$. So, what will happen? k will become i, j will become l; so, this will become eta into epsilon$_{ij}$. Is that clear?

Yeah, because jl ik will become, i and k will become the same and j and l will become the same. So, you will see that this becomes eta into ik jl epsilon$_{kl}$ will become epsilon$_{ij}$. Is that clear? What will happen to the last term? Let us hear some answers from you. What will happen to the last term? Beta will remain the same, yeah. Epsilon$_{ji}$; that is correct, so, beta into say epsilon$_{ji}$. Can I, yes, you are ahead of me; that is very good. So, epsilon$_{ij}$ is equal to epsilon$_{ji}$; symmetric. So, this equation can be written as delta into lambda into delta$_{ij}$ epsilon$_{kk}$ plus say eta plus beta epsilon$_{ij}$. Eta and beta lambda and all are constants.

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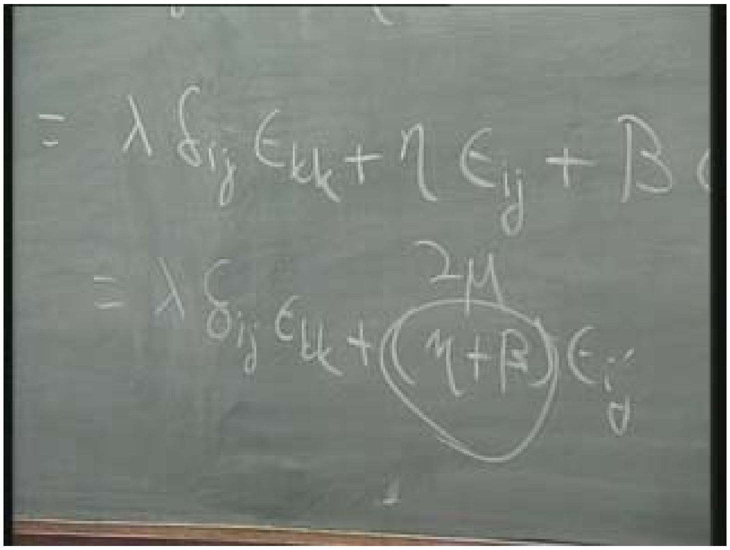

So, let me replace this eta plus beta as 2 mu. Let us write, because I just want to write this as 2 mu. There is some advantages of doing it. Let this be 2 mu, so that you can write that equation as lambda plus 2 mu into epsilon$_{ij}$.

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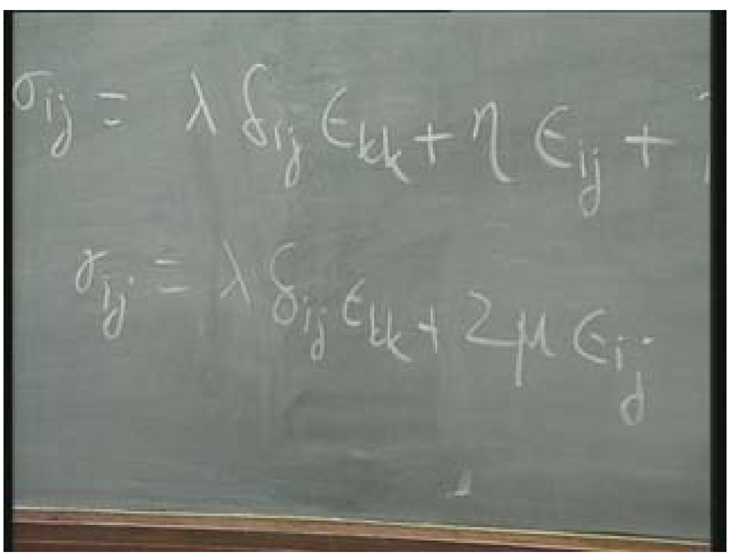

That is sigma$_{ij}$. Is that clear? Please remember we are trying to get to the stress strain relationship. Again just going back, we have strain displacement relationship. We had an equilibrium equation, now, we have gone ahead to determine the stress strain relationship. Lambda and mu are called as Lame's constant. They are more fundamental; they are fundamental parameters which come out because of the fact that the material is isotropic. See, how we have reduced now; very simple, by just using a mathematical property. When Jeffreys did it, he did not look at steel, but what he did was to look at the fourth order tensor. But then what did we do? We said, Oh God, this is isotropy; so, I can use an isotropic tensor. So, let me pick the property of an isotropic tensor and see what I do get out of it.

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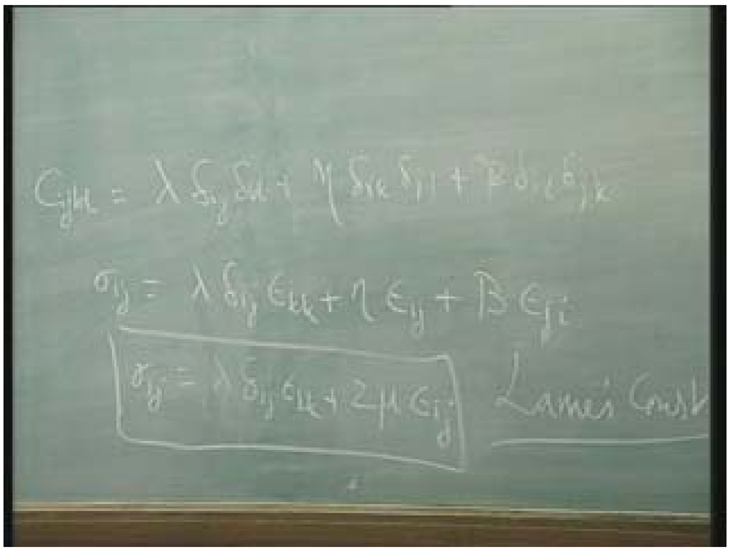

Apart from that, I also used one more that epsilon$_{ij}$ is equal to epsilon$_{ji}$ or in other words epsilon is symmetric and ultimately I arrive at an equation of the form where the equation defines this relationship between strain and stress by means of just two constants. Is that clear? Lambda and mu are called as Lame's constants. I know you are still not satisfied because you have been taught in your earlier classes very correctly so, that E and mu are the ones we should look for. Lambda and mu are not the ones which strike the bell. Okay sure; so, still E and mu. You will also see that or you would have also heard about shear modulus, bulk modulus and all that.

All those things still do not ring a bell. It is only E and mu. Let us see whether I can convert these two constants or can I relate or can I write first of all lambda and mu in terms of E and mu or can I exchange these two constants. For that let me go to my familiar trick of looking at a uniaxial case. Let us look at a uniaxial case, see whether I can derive from a uniaxial case, this relationship. How do I write for a uniaxial case?

First of all I am not very sure you have understood this equation. This looks quite complex, but may be you are still not so familiar with indicial notations. Do you want me to write down that one equation completely so that you will understand what it means? May be I have to, I think. Yeah, you could have asked me. I know that it is important that you understand.

Let us say that I will write it down $sigma_{11}$. I know lot of people want to avoid indicial notation. In fact, there is a higher or simpler way of writing it, but still that also is difficult. I can write this in a tensorial form; that is also difficult. The indicial notations are cumbersome to write, but they are sort of, they bring down the way you write.

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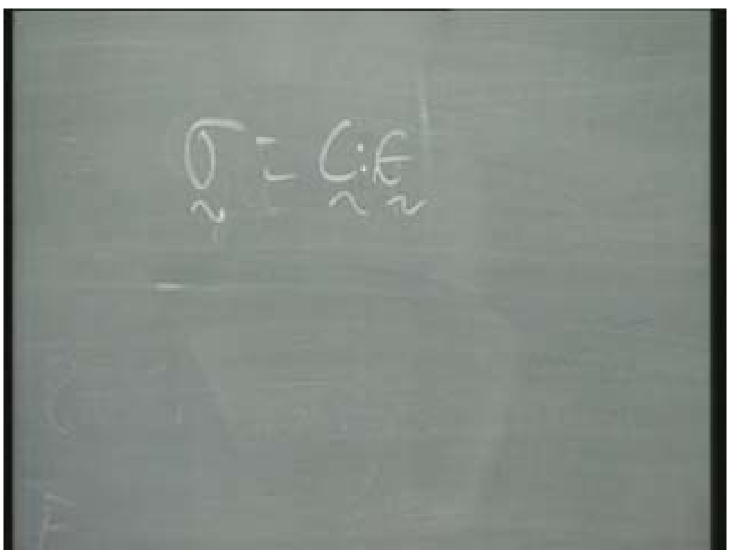

For example, our sigma is equal to C, I can write, C epsilon straight away. That is a tensorial form in which I can write. Some people write it as sigma double dot epsilon and so on. This is still contracting the whole thing. The more the lazy we become, the less we write, but it is not only for that, but also that we can do manipulations much more easily with this kind of forms. Is that clear? We are still like learning indicial notations. You know, it is important that we learn how to write $sigma_{ij}$. Let me write it for $sigma_{11}$. When I write $sigma_{11}$, what happens to $delta_{ij}$?

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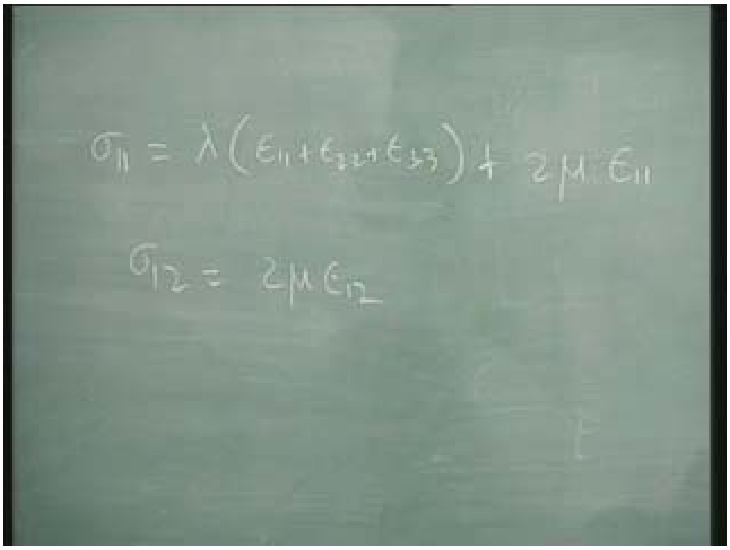

That becomes 1 and so I write this as lambda into epsilon$_{kk}$ and hence epsilon$_{kk}$ can be written as 11 plus 22; that is all. 11 22 plus 33 plus 2 mu into epsilon$_{11}$, that is it. What happens to sigma$_{12}$? 2 mu epsilon$_{12}$. What you recognize this as? 2 mu epsilon$_{12}$, what is that? What is it that relates sigma$_{12}$? I know that there is going to be confusion. The problem with us is that we are so familiar with E. What is shear modulus? What is the definition of shear modulus? What is the definition of shear modulus?

The problem with finite element and other thing is you have to be clear with definition. Yeah, what is it? Shear stress by shear strain, very good. What is this shear strain? 2 into epsilon$_{12}$; that means gamma, shear modulus relates gamma. Please note that it is, sigma is equal to G gamma. So, mu into 2 epsilon$_{12}$ that means 2 epsilon$_{12}$ is what? Gamma$_{12}$; remember, in last class we did that. So, sigma is equal to G gamma and hence sigma$_{12}$ is equal G gamma$_{12}$ and so, what is mu? Shear modulus. Is everyone with me? We are now comfortable writing this and hence let us write down an equation for a uniaxial case. Let us write down an equation for uni axial case. What happens in the uniaxial case?

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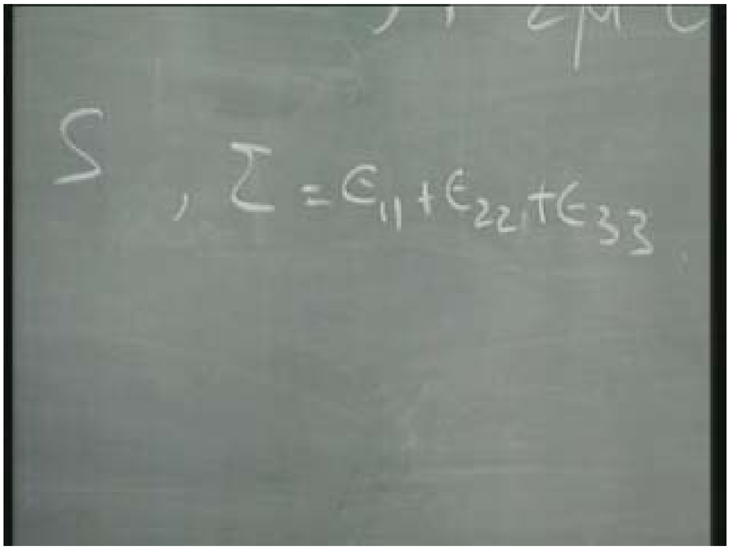

Let us say that the uniaxial stress is say S, some stress; I will just write S. Let us say that epsilon_{11} plus epsilon_{22} plus epsilon_{33} let me call this as sigma. Now, write down this equation.

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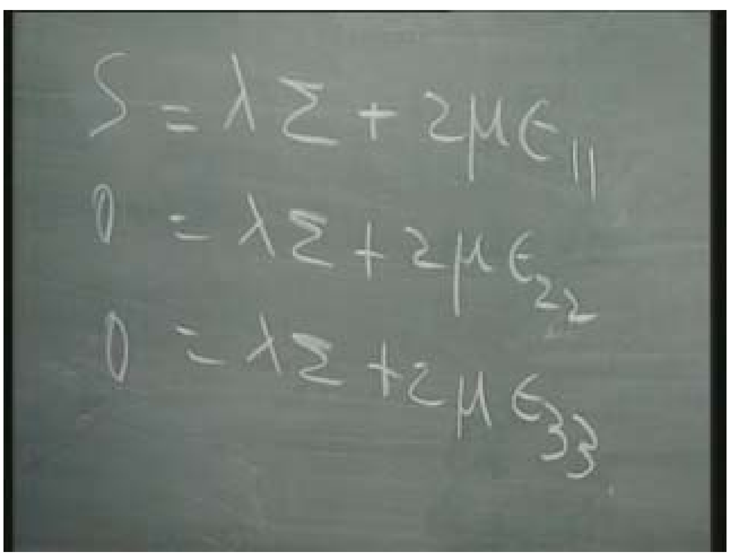

S is equal to, what is that I have? I have only sigma_{11} stress, I do not have sigma_{22} and sigma_{33}. Sigma_{11} that is S is equal to, how do I write? Lambda sigma plus 2 mu epsilon_{11}. What happens to the second equation? Say, 2 2, I want to write. Zero is equal to 2 mu epsilon not 12, we are writing 22, so, epsilon_{22} and zero is equal to lambda sigma plus 2 mu epsilon$_{33}$. Let us now sum them up, sum these three equations 1, 2 and 3. So, what is that we get?

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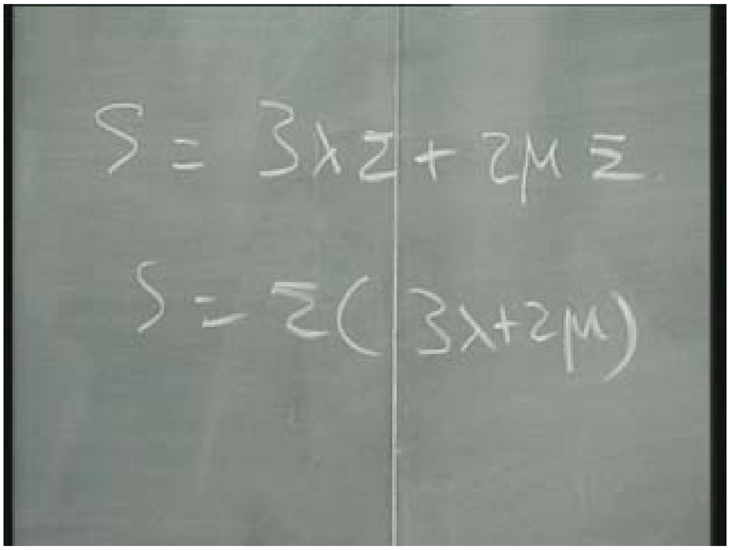

S is equal to 3 lambda sigma plus 2 mu sigma so that, that is equal to, S is equal to sigma into 3 lambda plus 2 mu. Is that clear? Let me now substitute that equation into this equation; from that equation let me write down or substitute it into that equation or other words for sigma let me substitute it in the first equation, so that I will get S is equal to, what do I get?

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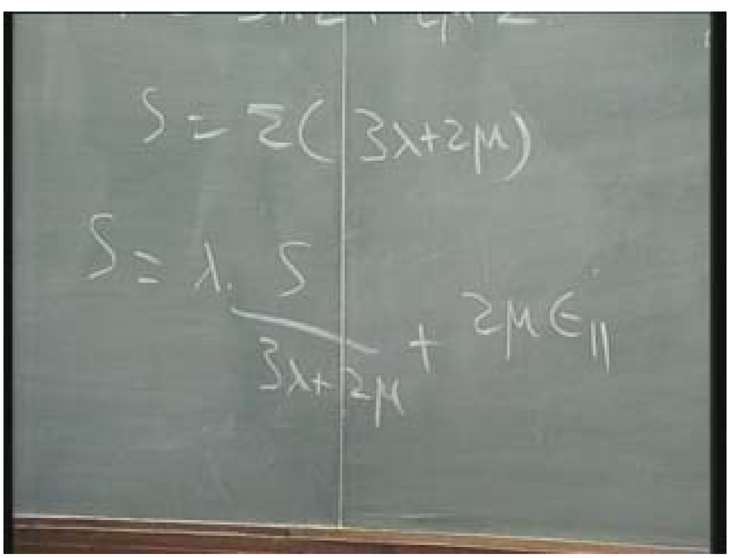

Lambda into S divided by 3 lambda plus 2 mu plus 2 mu ϵ_{11}. What is S, how do I write S? What is S? Stress; so, in terms of E what is that we are trying to do?

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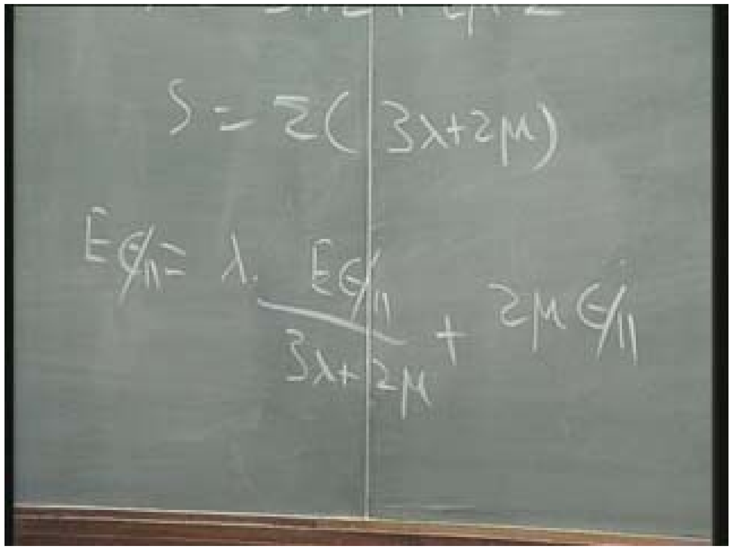

Please do not forget that we want to shift to E. So I can write this as E into ϵ_{11} and here also E into ϵ_{11}. Canceling ϵ_{11} throughout now, I can solve for E straight away. So, E into 3 lambda plus 2 mu. I will give you a minute to work it out and look at that equation, work it out and tell me what is the result for E? What is the result?

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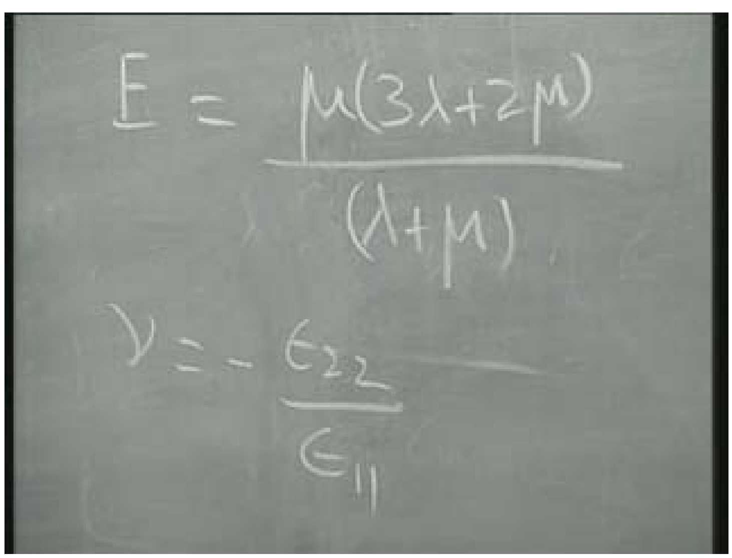

Just look at this equation. He said is it mu by lambda plus mu or yeah, mu into 3 lambda plus 2 mu divided by lambda plus mu, beautiful! We now see that there is a relationship between E and mu and lambda. Let us now bring into picture nu because we want to swap lambda and mu with E and nu, let us bring that into picture. Let us look at the definition of nu. What is nu, the Poisson's ratio? Lateral strain to linear strain. How do I define that? Say, it is defined as minus $epsilon_{22}$ by $epsilon_{11}$.

Now let me get $epsilon_{22}$, say, for example from this second equation, in this equation. Because this is equal to zero, so, $epsilon_{22}$ is equal to minus lambda sigma divided by 2 mu.

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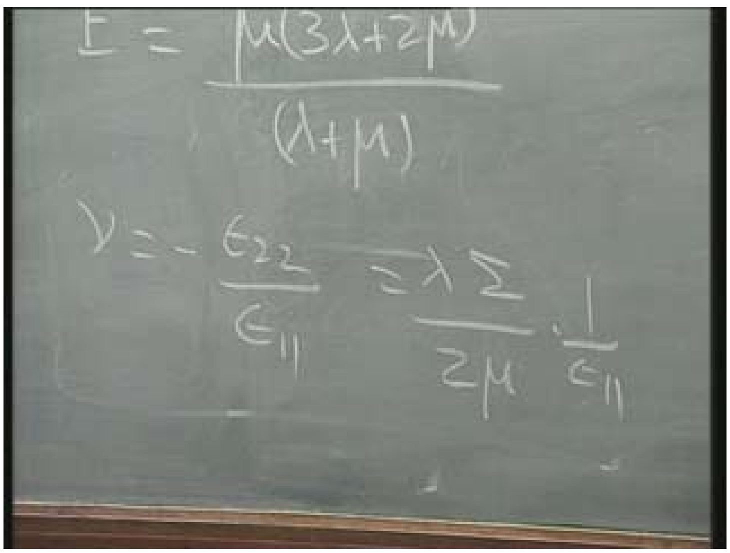

Let me write that down. That is equal to lambda sigma divided by 2 mu into 1 divided by $epsilon_{11}$. Now I know, what are the things I know? I know sigma is equal to S divided by 3 lambda plus 2 mu.

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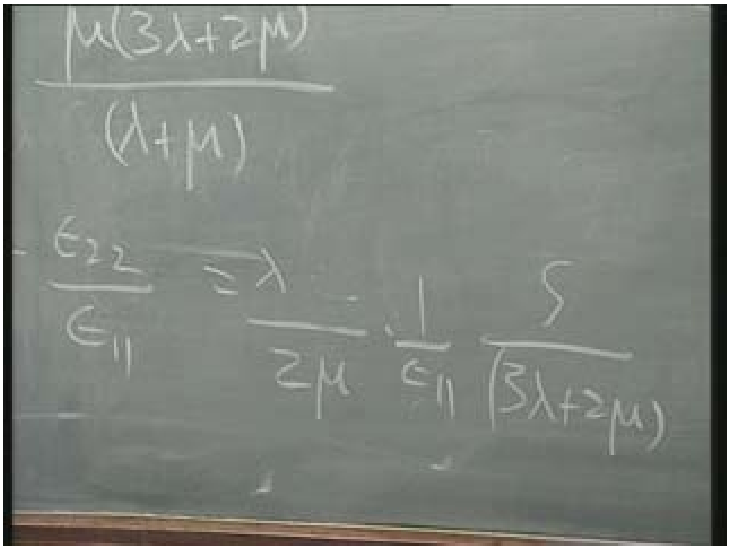

This I can write it down as, from here this is just small jugglery; algebraic jugglery, nothing I am doing. S divided by 3 lambda plus 2 mu. What is S?

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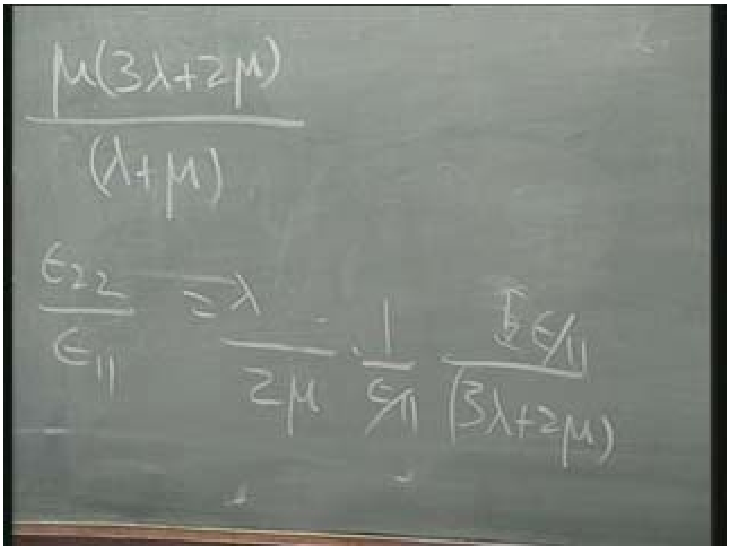

S is E epsilon_{11}; E into epsilon_{11}. This and this goes. What is E? E also I have it from there. What is E?

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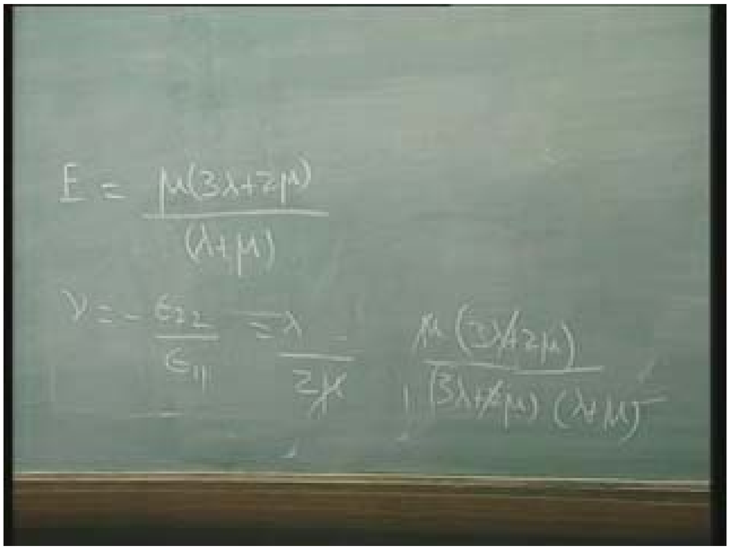

This goes off; mu divided by 3 lambda plus 2 mu, from here I am writing, divided by lambda plus mu, so that removing this, removing this, removing this, removing this, mu is equal to lambda divided by 2 into lambda plus mu. Is that clear? What is that we have done? We see that E and mu can very easily be represented in terms of lambda plus mu; lambda and mu, sorry. Any question? I can now invert the whole thing. I can now write lambda is equal to and mu is equal to in terms of E and nu.

Let us see what is the result? I am not going to do that algebra again; it is quite a simple algebra. Let us look at how or what is the expression when I now write it by inverting it. Any question, so far? Clear?

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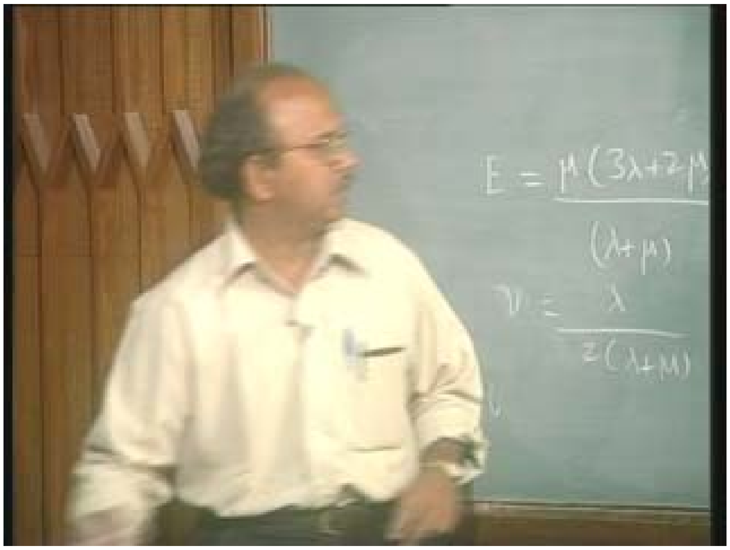

So, that E by inverting it, I get E is equal to mu into 3 lambda plus 2 mu divided by lambda plus mu and nu is equal to sorry that is what we derived now. I am just rewriting that or I think it is not necessary; let me write that down, lambda divided by 2 into lambda plus mu. That is what we derived.

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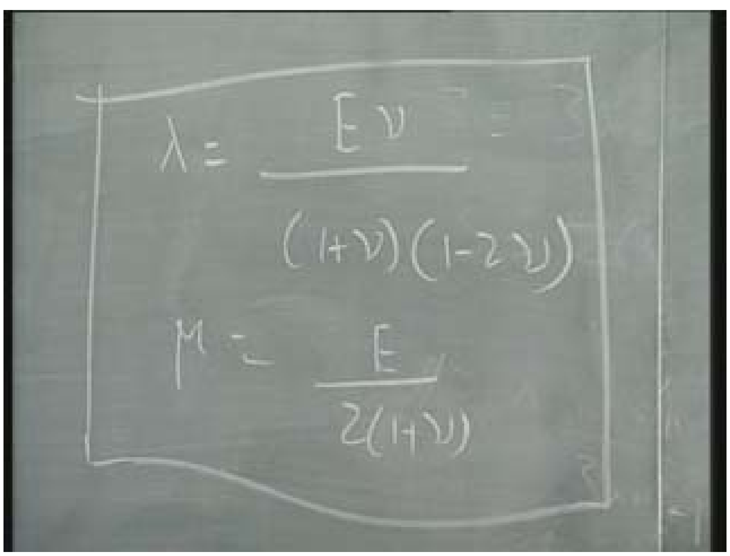

Hence I will write down lambda in terms of nu E and lambda is equal to E nu divided by 1 plus nu into 1 minus 2 nu; E nu divided by 1 plus nu into 1 minus 2 nu; this is very important, this expression and the next expression that I am going to write; mu is equal to E divided by 2 into 1 minus nu. Once I know these two expressions, I can now go back and write my sigma$_{ij}$'s in terms of E and nu. Look at this expression for a minute. What is that? Sorry; mu is equal to lambda plus 1 E is sorry E is equal to E divided by 2 into 1 plus nu. What is this? Shear modulus; may be you would have used it before and hence shear modulus E nu relationship comes out very clearly. Again you can identify that this expression is for shear modulus. Is that clear?

Now, let me substitute this simple algebra into my earlier expression for sigma$_{ij}$. Can you do that? Let us see what you get out of it. Let us write it for sigma$_{11}$ so that you do not get confused. Unfortunately I have removed that. Can someone write it for in terms of sigma$_{11}$? No, write it in terms of this and tell me. E nu divided by 1 minus nu or 1 plus nu into 1 minus 2 nu into epsilon$_{11}$ plus epsilon$_{22}$ plus epsilon$_{33}$ plus 2 nu plus E by 1 plus nu into epsilon$_{11}$.

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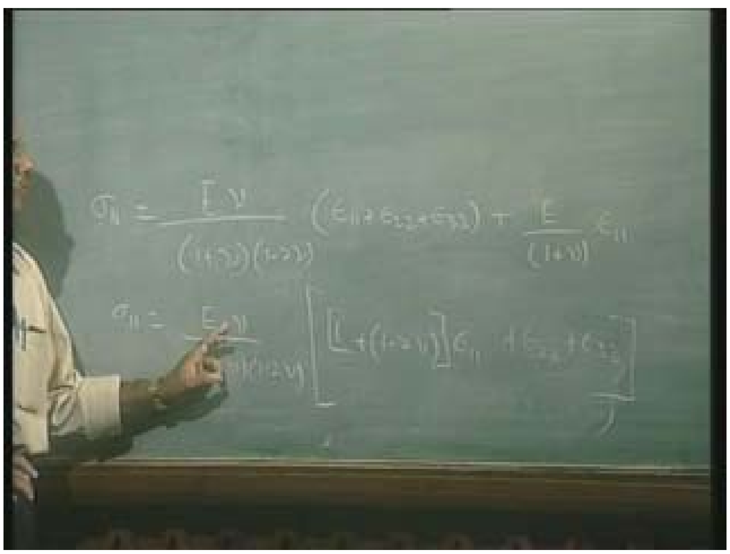

So, that can be written as sigma$_{11}$ is equal to, rearranging the terms E nu, let me take this out, E nu divided by 1 plus nu into 1 minus 2 nu; let me take that out. So, what happens to the epsilon$_{11}$ term? This will be 1 plus 1 plus 1 minus 2 mu epsilon$_{11}$ plus epsilon$_{22}$ plus epsilon$_{33}$.

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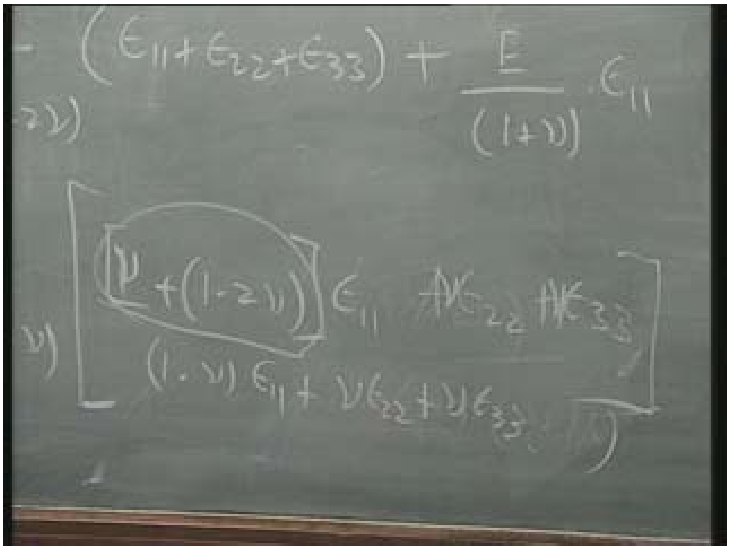

Wait, wait, wait, wait. E nu I have taken it out. Let me not take, so that it is easier. So that this will be nu; so, that will be easier to write. So, nu, nu. Is that okay? So that is easier to write so nu 1 minus 2 nu. This will now become, the whole thing becomes 1 minus nu into epsilon$_{11}$ plus nu epsilon$_{22}$ plus nu epsilon$_{33}$. Is that clear? I know the expressions are big, but I think it is very easy to derive. There is nothing much, it is simple manipulation. How will now sigma$_{22}$ appear? How will sigma$_{22}$ appear? No, no; tell me the full expression sigma$_{22}$ is equal E divided by same thing into, one person, one person; nu into 11 plus, correct, 1 minus nu into epsilon$_{22}$ plus nu epsilon$_{33}$. The same way we are going to get for sigma$_{33}$ as well. What happens to sigma$_{12}$? How does the sigma$_{12}$ term appear? Go back and refer to whatever we have done and tell me.

Yeah; so, let us now put them in a matrix form. Let us now put them in a matrix form to understand the relationship between sigma and epsilon.

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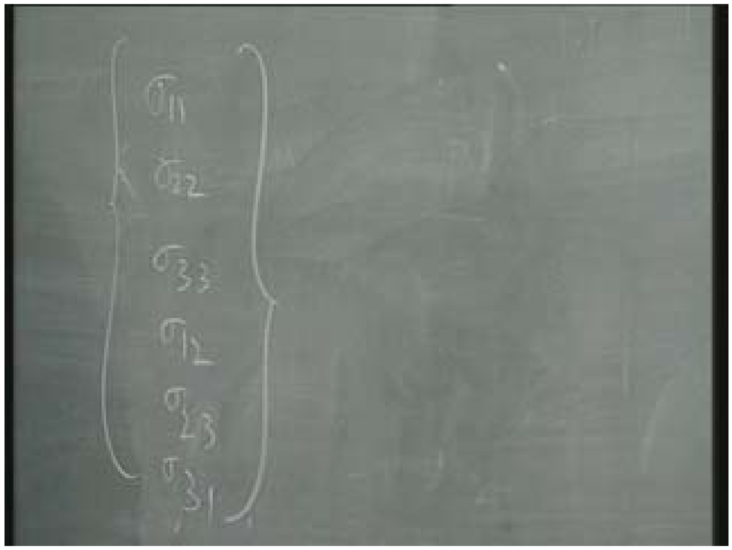

On one hand let me write this as sigma_{11} sigma_{22} sigma_{33} 12 13 and 32 or sometimes you write it as, so that just to maintain consistency let me write this 23 and 31; so, 12 23 31.

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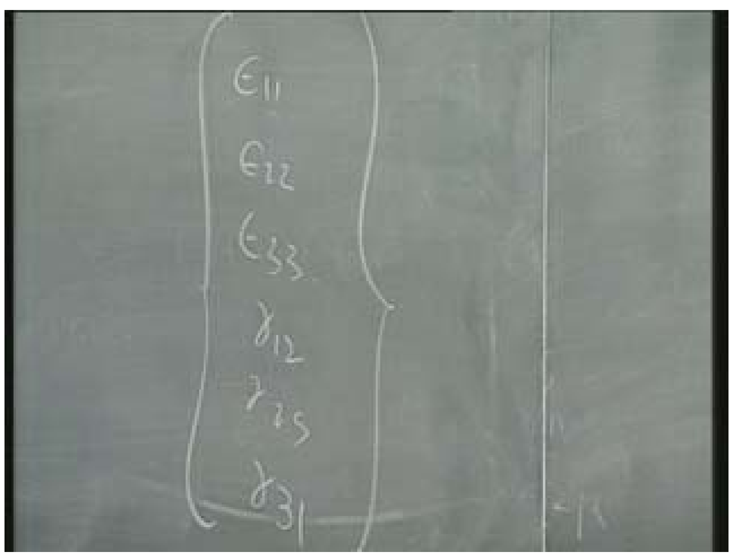

On the other hand, let me write this strain terms as epsilon_{11} epsilon_{22} epsilon_{33}. Note this carefully, the next thing what I am going to write. I am going to write gamma_{12}. Why is that I have written? Please note that traditionally many finite element books write, not epsilon_{12}, but gamma_{12} this is very important. Suppose you are going to go and implement a finite element scheme. You have to be careful as to what is this strain you are calculating. If you calculate epsilon$_{12}$, correspondingly you have to make adjustments in the strain displacement relationship. That 2, you should not forget that 2, that is very important because traditionally engineers have defined this strain to be gamma. The mathematicians came and said look this gamma is not very convenient, why not you look at it or why not you put it as 2 epsilon basically because they wanted to introduce tensorial properties. That is why they did it, whereas traditionally we have been using gamma. Hence, usually gamma$_{12}$ is put there. Gamma$_{12}$, gamma$_{23}$ and gamma$_{31}$ and what is in between the two is the stress strain matrix which is called sometimes as C matrix or sometimes as D matrix and so on. There are lots of things. Sometimes people call this as E matrix itself, sometimes people call it as C matrix, sometimes people call it as D matrix.

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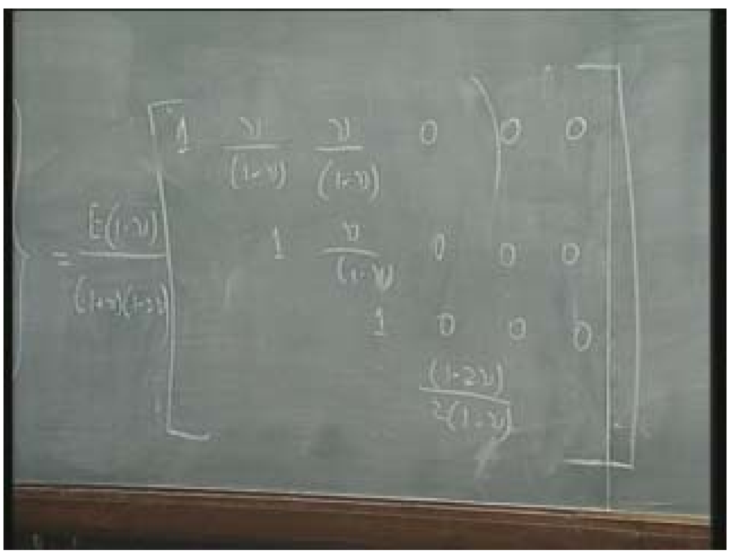

What is this matrix? E into, there are different ways of writing, so, I am writing it like this. 1 plus nu into 1 minus 2 nu; I am taking this out, sorry; 1 plus nu into 1 minus 2 nu, I am taking this out and then writing a 6 by 6 matrix. But what is important here? What is it that is important here to realize? Yes, so it is a symmetric matrix. Can you fill this up? Can you fill it up? What is the first term? What is the first term? 1. What is the second term? Nu. Why I am asking you this question is because it is so simple, so that you are not lost in these things. I want you to concentrate and just see that it is shear simple algebra and do not get carried away by these things; nu divided by 1 minus nu, then zero zero zero. Is that clear?

What does this indicate? This indicates that the shear strains have no effect on the tensile stresses or rather I should not say tensile stresses but normal stresses. They do not have any role to play. The shear strains, as far as they are concerned, they have no role to play. The second line starts with 1; that gets repeated and the third line starts with 1 000 and all these things are symmetric types. What is the fourth line? Rather I think my lines are not very clear. Let me write this separately. It is a big matrix so that we know what it is?

Let us say that this is 1, nu divided by 1 minus nu, nu divided by 1 minus nu, 000. Let me put that like this and then 1, nu divided by 1 minus nu, 000; 1 000. What is the fourth term? No, fourth term cannot be zero all the way. I have to fill it up here; yes all these things are zero because first these three, they are symmetric; so, three are zero, so, there should be an entry in the diagonal term. What is that entry? 1 minus 2 nu. 1 minus 2 nu divided by, correct, 2 into 1 minus nu. Why because, we have used gamma there. Is that clear? You have to be very careful. This is nothing. This is very simple but you have to be careful. If you use epsilon, correspondingly it will change there. Now, multiply this and that; E divided by 2 into 1 plus nu which is G. You are correct, so you will fill this up for the next two problems.

Have a close look at this equation, have a close look at this equation, this part of the equation. can you can this equation Can this part of the equation or can it spell some danger? Look at this equation. What is that? Nu not gamma. Where is gamma here? Nu is equal to half. What is happening? What is happening? It becomes infinity. Yes, so, the material, do you thing nu will be 0.5. No; possible. Can it be minus? can can can poison See, these are some of the questions which you have to answer. Can nu be minus? Yes; why? Tell me one material for which nu is minus? What does minus mean? Please not this carefully. When nu is equal to minus, see, look at the way I defined it.

Absolutely; there are certain foams which have been manipulated such that when you take this foam and apply stress, instead of other side coming in, the other side would start going up, going out; instead of coming in, it will start going out. When I pull a bar, when I pull a bar what will happen? The other side would start coming in. Then I defined nu to be positive. On other hand when I pull the bar, these sides start going out. What does it mean? Yes, absolutely you are correct that there are certain foams which have been discovered, where the fellow will, instead of coming in start going out. But, my question is can nu be 0.5?

Yes. For example most rubber materials; you know, you would have studied this, nu is equal to 0.5; there is one case where probably you would have studied. Where? Plasticity; for example, some of you would have studied. Yes, correct; so nu is equal to 0.3 in the elastic region and when the material goes to plastic you will say nu is equal to approximately 0.5 and in fact many of you might have studied that rubber materials nu is equal to 0.5. Actually what does it indicate? That indicates that the material is incompressible. Nu is equal 0.5 and it can be shown that the material is incompressible and it becomes very difficult to solve the problem using this simple multiplication. It is not possible because it will become infinity.

Apart from that, there are a few things that we can look at in this equation. What is this? This is a general three dimensional equation. The point is that should we use this kind of three dimensional equation all the time or can I make a reasonable, meaningful engineering assumption to reduce this 3D into 2D. In other words can I solve a problem as a two dimensional problem? Is that possible? In fact the whole finite element modeling rests on concepts like that. In fact, no doubt they are outside the view of finite elements per say but in order to do a model, you should understand you have to come to this extent to understand what this is.

If you wrongly write nu is equal 0.5, then definitely you know that you are getting into trouble. That is why you should know this kind of equations and there are special tricks of the trade that you will do in order to satisfy this constraint called incompressibility. But if time permits, we will come to that kind of tricks later in the course and in fact one more question I asked. Before we go to other simplification, where can nu be above 0.5?

Yes; unfortunately yes, many course even do not accept it, many, say, reinforcements in tire; some of the nylon reinforcements they are believed to be above even 0.5. So, many course do not even accept it. There are problems from that point of view also. Let us not worry about that; they are material aspects. Let us come back and look at whether we can simplify this 3D situation into 2D situation.

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Basically there are three simplifications that are possible. They go under the name plane strain, plane stress and axi-symmetric; plane strain; plane stress and axi-symmetric.

We will talk about this simplification in the next class.