Fundamentals of Operations Research

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Lecture No. # 08

Introduction to Duality

In the last lecture we saw termination conditions for linear programming problems and we ended the lecture by considering the problem that had unrestricted variable.

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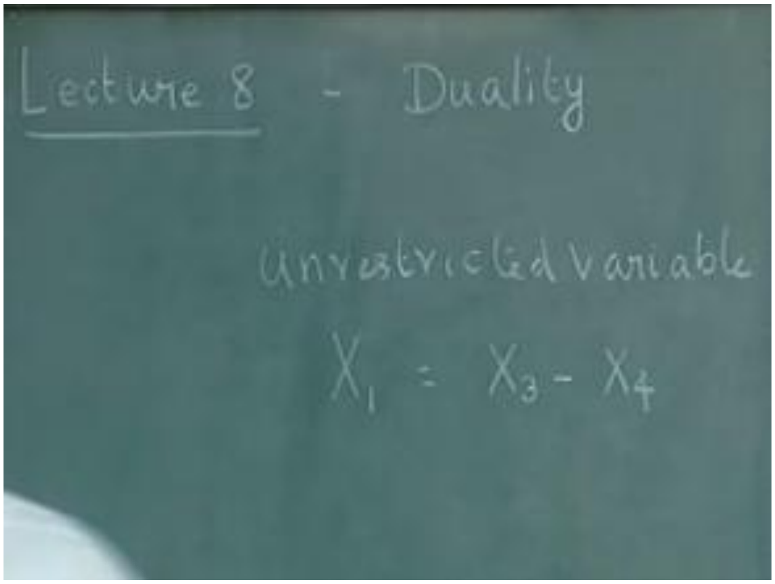

In the example that we chose, X_1 as the unrestricted variable and X_1 was written as $X_3 – X_4$ and we solve the problem optimally. The simplex table is shown below.

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		4	-4	5	0	0		
		X_3	X_4	X_2	X_5	X_6	RHS	θ
0	X_5	2	-2	3	1	0	8	8/3
0	X_6	1	-1	4	0	1	10	5/2
	$C_j - Z_j$	4	-4	5	0	0	0	
0	X_5	5/4	-5/4	0	1	-3/4	1/2	2/5
5	X_2	1/4	-1/4	1	0	1/4	5/2	10
	$C_j - Z_j$	11/4	-11/4	0	0	-5/4	25/2	
4	X_3	1	-1	0	4/5	-3/5	2/5	–
5	X_2	0	0	1	-1/5	2/5	12/5	6
	$C_j - Z_j$	0	0	0	-11/5	2/5	68/5	
4	X_3	1	-1	3/2	1/2	0	4	
0	X_6	0	0	5/2	-1/2	1	6	
		0	0	-1	-2	0	16	

In our discussion we said at the end, that the optimal solution is $X_3 = 4$; $X_6 = 6$ with Z = 16 but we also observe that X_4 which is a non basic variable has a $0(C_j - Z_j)$ at the optimal. X_4 in turn tries to enter the basis but when X_4 enters the basis we realize that we do not have a living variable. We get a feeling that we see unboundedness in this example. We may also feel that because the non basic variable has $C_j - Z_j = 0$ at the optimum it could indicate alternates optimum. What is happening is it is indicating neither. The reason being, then we had substituted $X_1 = X_3 - X_4$ and in this example $X_3 = 4$.

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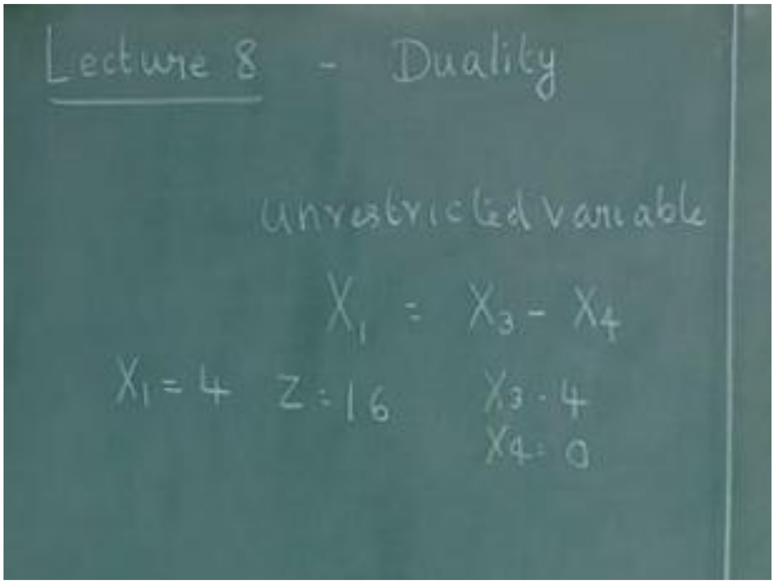

This will be taken back to the original problem with variable X_1 taking a value 4 and Z = 16. The original problem has the solution $X_1 = 4$ and Z = 16 but that is reflected as $X_3 = 4$; X_4 which is non basic is = 0 From X_1 it has to take a value 4 in the optimum solution, then one way of getting this 4 is (4, 0). For example we could have a solution $X_3 = 5$; $X_4 = 1$ and X_1 could still be 4. So that is being reflected by the presence of this 0 under X_4 where X_4 is trying to enter the basis.

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		4	-4	5	0	0		
		X_3	X_4	X_2	X_5	X_6	RHS	θ
0	X_5	2	-2	3	1	0	8	8/3
0	X_6	1	-1	4	0	1	10	5/2
	$C_j - Z_j$	4	-4	5	0	0	0	
0	X_5	5/4	-5/4	0	1	-3/4	1/2	2/5
5	X_2	1/4	-1/4	1	0	1/4	5/2	10
	$C_j - Z_j$	11/4	-11/4	0	0	-5/4	25/2	
4	X_3	1	-1	0	4/5	-3/5	2/5	–
5	X_2	0	0	1	-1/5	2/5	12/5	6
	$C_j - Z_j$	0	0	0	-11/5	2/5	68/5	
4	X_3	1	-1	3/2	1/2	0	4	
0	X_6	0	0	5/2	-1/2	1	6	
		0	0	-1	-2	0	16	

It is suggesting that we could have alternate combinations of X_3 and X_4 but effective value for X_1 will remain at 4.

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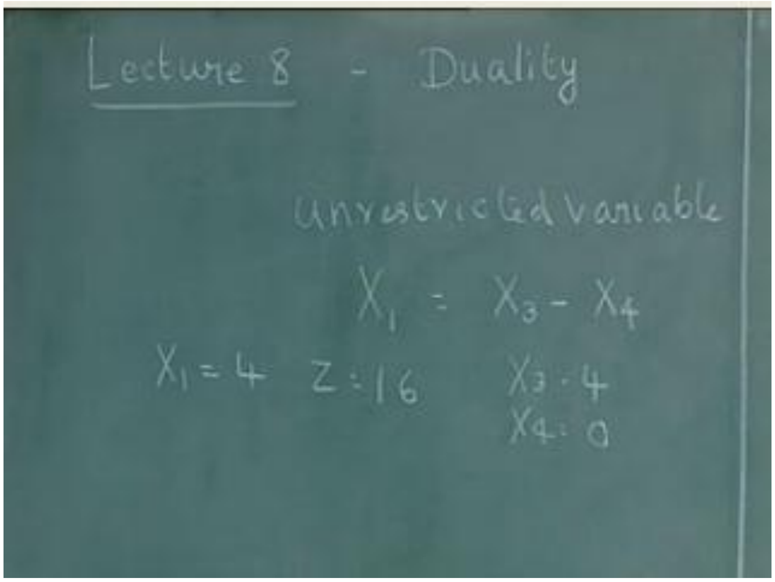

Also because there are infinite ways of getting this 4, it could be for example (4 0) or (5, 1) or (6, 2) or anything. One gets a feeling that X_4 can actually take any positive value and X_3 can be correspondingly adjusted. Therefore we get a feeling that, that we are going have unbounded solution.

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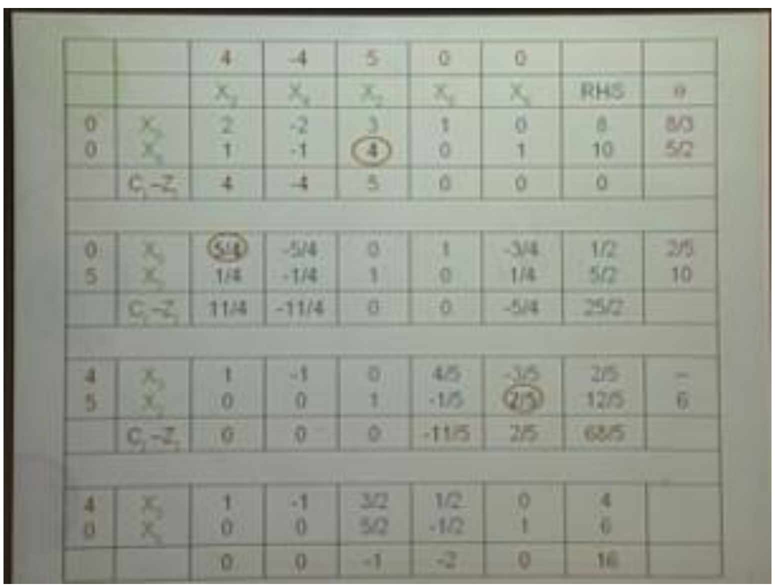

		4	-4	5	0	0		
		X	X	X	X	X	RHS	θ
0	X	2	-2	3	1	0	8	8/3
0	X	1	-1	4	0	1	10	5/2
	$C_j - Z_j$	4	-4	5	0	0	0	
0	X	5/4	-5/4	0	1	-3/4	1/2	2/5
5	X	1/4	-1/4	1	0	1/4	5/2	10
	$C_j - Z_j$	11/4	-11/4	0	0	-5/4	25/2	
4	X	1	-1	0	4/5	-3/5	2/5	–
5	X	0	0	1	-1/5	2/5	12/5	6
	$C_j - Z_j$	0	0	0	-11/5	2/5	68/5	
4	X	1	-1	3/2	1/2	0	4	
0	X	0	0	5/2	-1/2	1	6	
		0	0	-1	-2	0	16	

So this phenomenon will occur when ever we are working with the problem that has unrestricted variable. If that variable has to be in the solution, then the variable is being represented as a difference of two variables.

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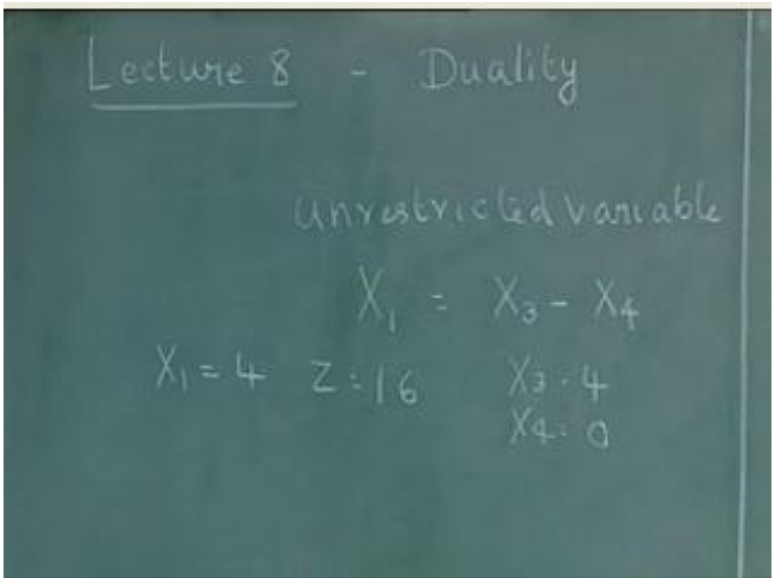

One of them will be in the solution. In this case, this unrestricted variable takes a positive value. So this component will be in the solution with 4.

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		4	-4	5	0	0		
		X_3	X_4	X_2	X_5	X_6	RHS	θ
0 0	X_5 X_6	2 1	-2 -1	3 4	1 0	0 1	8 10	8/3 5/2
	$C_j - Z_j$	4	-4	5	0	0	0	
0 5	X_5 X_2	5/4 1/4	-5/4 -1/4	0 1	1 0	-3/4 1/4	1/2 5/2	2/5 10
	$C_j - Z_j$	11/4	-11/4	0	0	-5/4	25/2	
4 5	X_3 X_2	1 0	-1 0	0 1	4/5 -1/5	-3/5 2/5	2/5 12/5	– 6
	$C_j - Z_j$	0	0	0	-11/5	2/5	68/5	
4 0	X_3 X_6	1 0	-1 0	3/2 5/2	1/2 -1/2	0 1	4 6	
		0	0	-1	-2	0	16	

The other one will be non-basic at 0 but will try to enter the basis by having a 0 value for $C_j – Z_j$ and will indicate something very similar to unboundedness. This is something which we will have to have to understand whenever we are solving the linear programming problem where the original problem has unrestricted variables. Once again in the last lecture we looked at the various termination conditions and let us have a quick recap of these before we move to the next topic.

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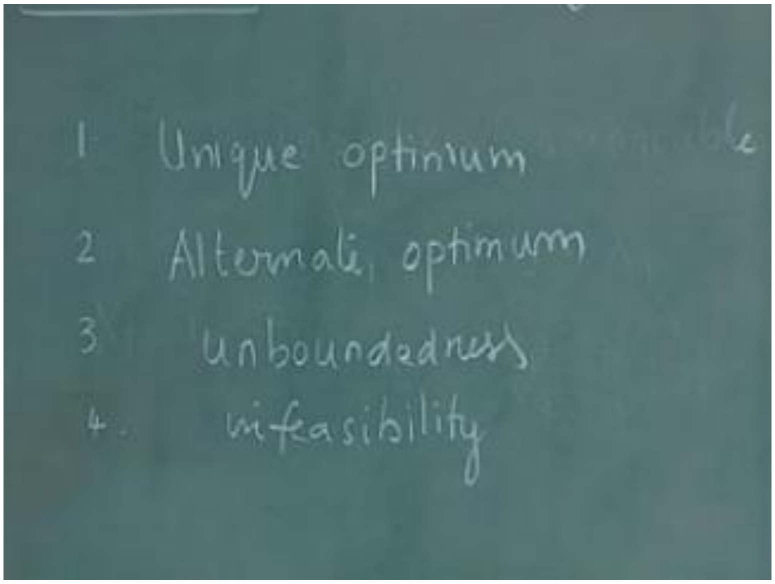

The various termination conditions for unique optimum, alternate optimum, unboundedness and infeasibility, were dealt with in the last lecture. Now the alternative optimum is always represented by non-basic variable having a $C_j - Z_j = 0$ at the optimum and trying to enter. Unboundedness will be indicated by having an entering variable but unable to get the corresponding living variable. Infeasibility is indicated by the optimality condition being satisfied but an artificial variable will be in the basis with a strictly positive value. They also said if the artificial variable is in the basis with value 0 then it indicates that we try to solve the linearly dependent system of the equations. From all these termination conditions we generalize and we state what is called the Fundamental theorem of a linear programming. The fundamental theorem of linear programming is as follows.

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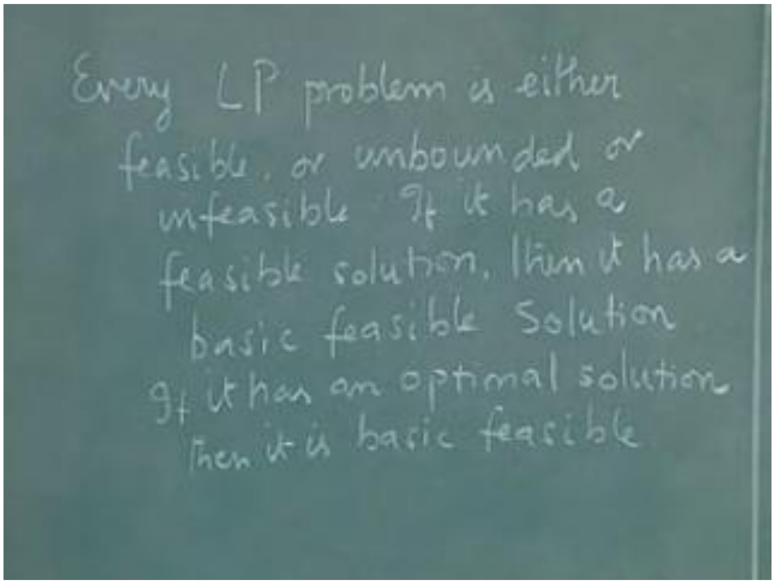

Every linear programming problem is either feasible or unbounded or infeasible. If it has a feasible solution then it has a basic feasible solution. If it has an optimal solution then it is basic feasible. Now this theorem is important for many reasons. All the termination conditions get reflected in the first sentence. So every linear programming problem has to satisfy one of these three. It will have an optimum which could be a unique optimum or an alternative optimum. It could be unbounded or it could be infeasible. So these are exactly the things that we looked at in the termination conditions. If it has a feasible solution then it has a basic feasible solution it means that there is at least one corner point solution for this.

This comes from the fact that the feasible region in every linear programming problem is a convex region. So if the problem is not infeasible then it is feasible. It means there is at least one corner point solution and if it has an optimum solution then the optimum solution has to be a corner point solution and it has to be a basic feasible solution. So this theorem is important. We will come back to the theorem. Refer to this theorem a little later when we look at some aspects of duality. In summary of the linear programming problem, we have so far we looked at solving linear programming problems for maximization objective and minimization objective. We have looked at different types of constraints. We introduced variables, negative slack or artificial variables. We also understood the artificial variables are needed to get a starting basic feasible solution.

We also looked at the various aspects of the simplex algorithm in terms of initialization, interaction and termination and then we defined the fundamental theorem of linear program.

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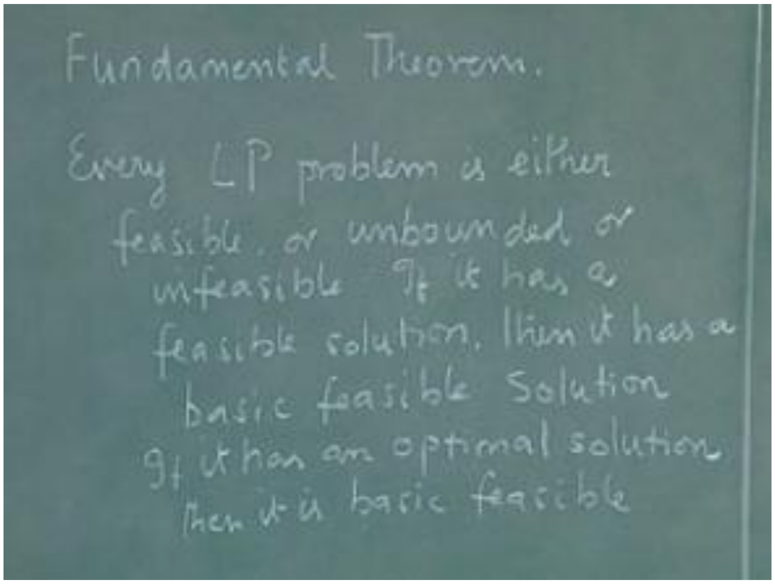

Then we move to a very important aspect of linear programming called duality.

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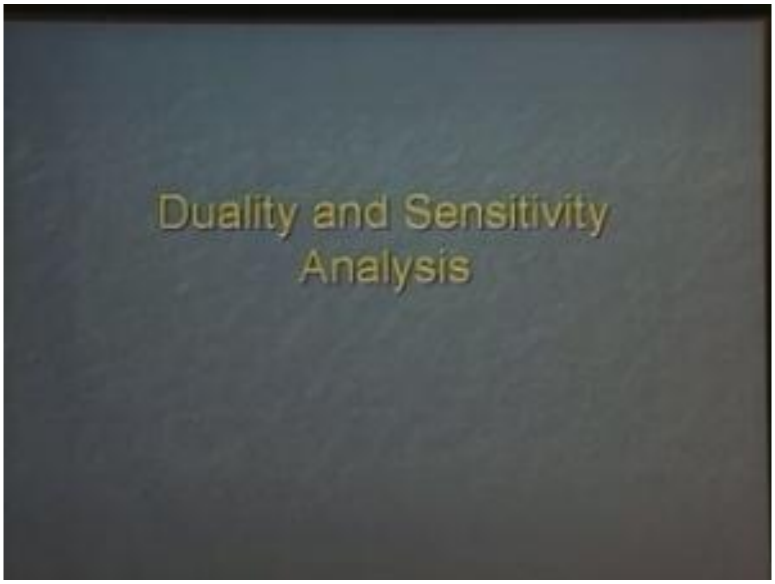

Now let us explain the duality by taking the same example that we looked at the beginning of this course.

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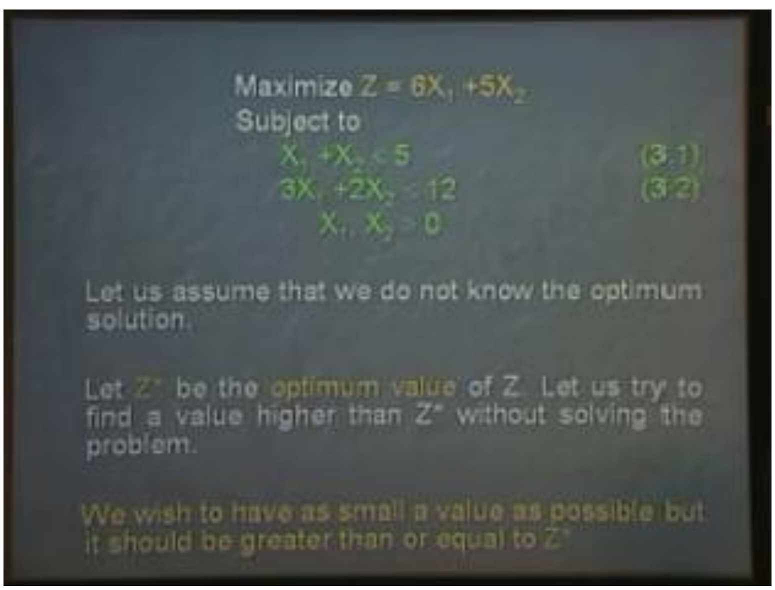

The problem is to maximize.

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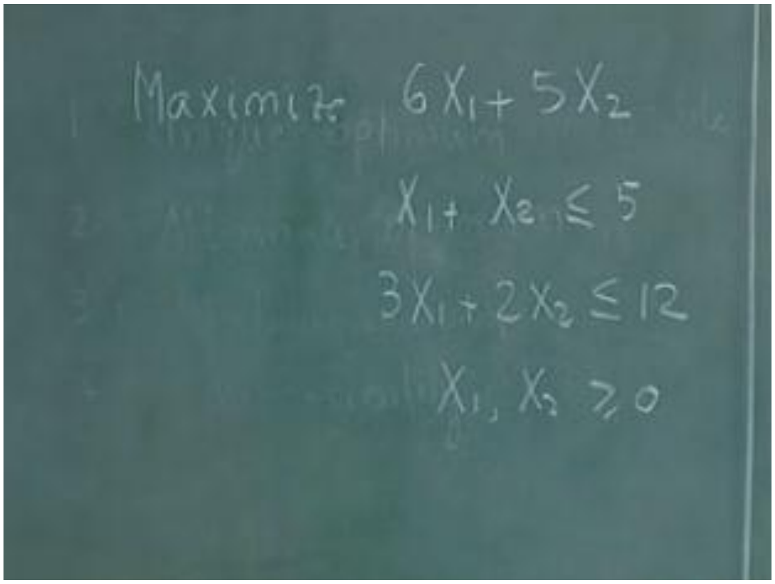

The problem is to maximize $6X_1 + 5X_2$

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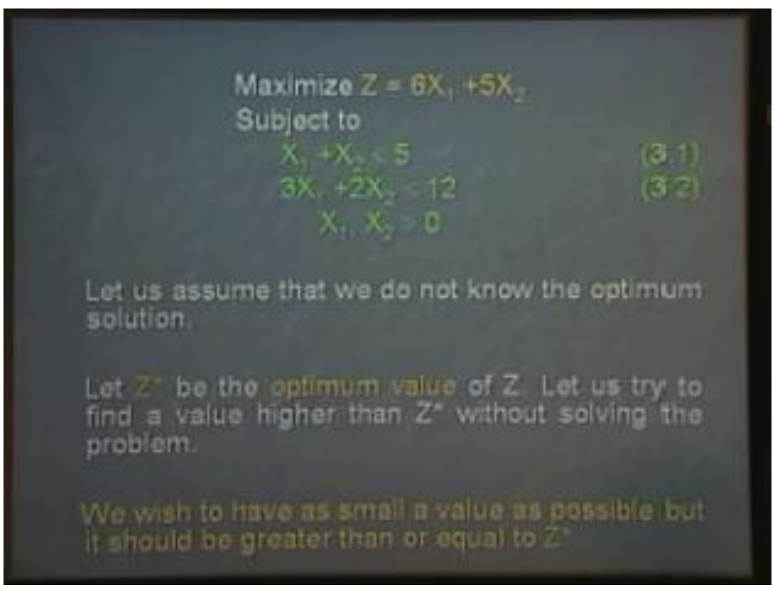

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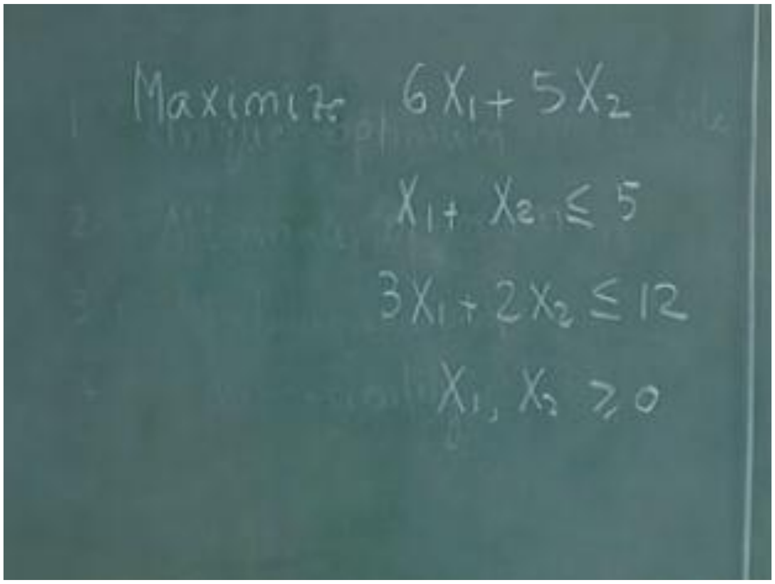

$X_1 + X_2$ less than or equal to 5; $3X_1 + 2X_2$ less than or equal to 12; X_1, X_2 greater than or equal to 0. Now without solving this problem, let us try to get some estimates of the objective function. For example if we did not have either of these constraints or if the problem did not have any constrain at all, then an obvious answer would be X_1 = infinity; X_2 = infinity; Z will be infinity.

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And since we have these constraints, the Z value is at the optimum.

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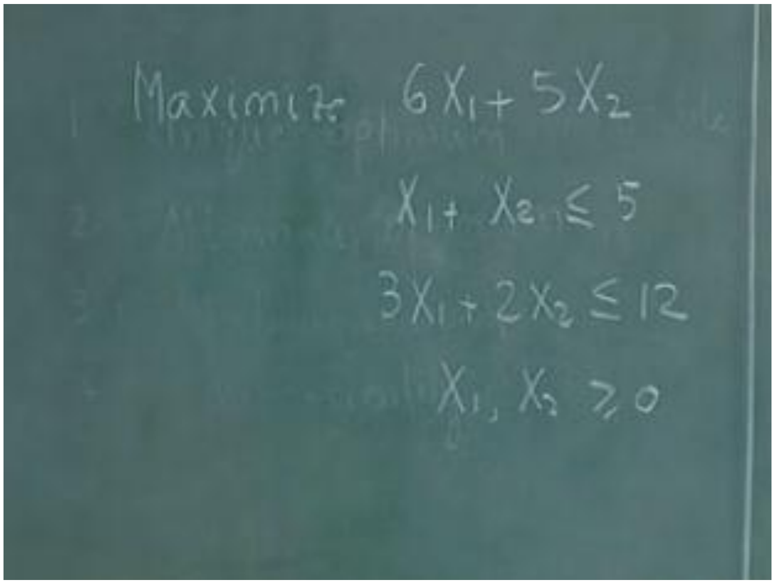

It will have to be less than that of infinity.

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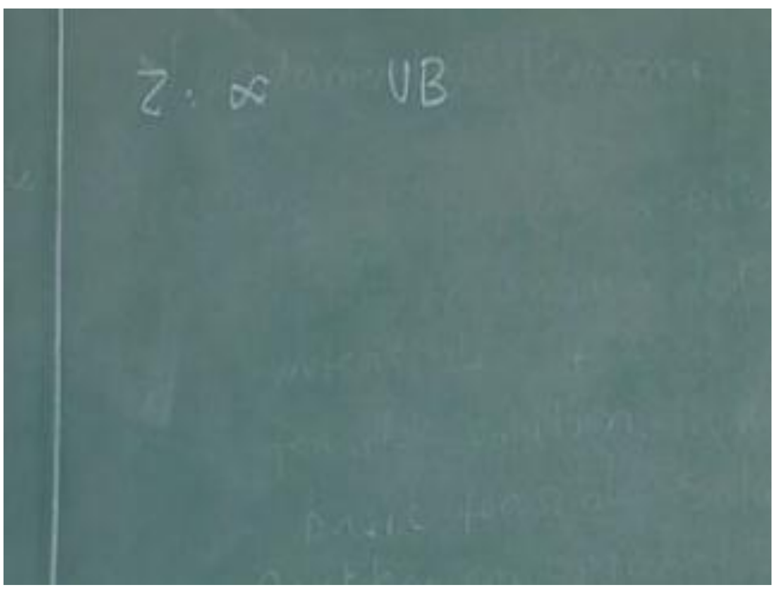

Putting it differently, this infinity is very easy, simple, upper bound to the value of the optimal solution. Now let us look at other ways of getting better and better upper bounds.

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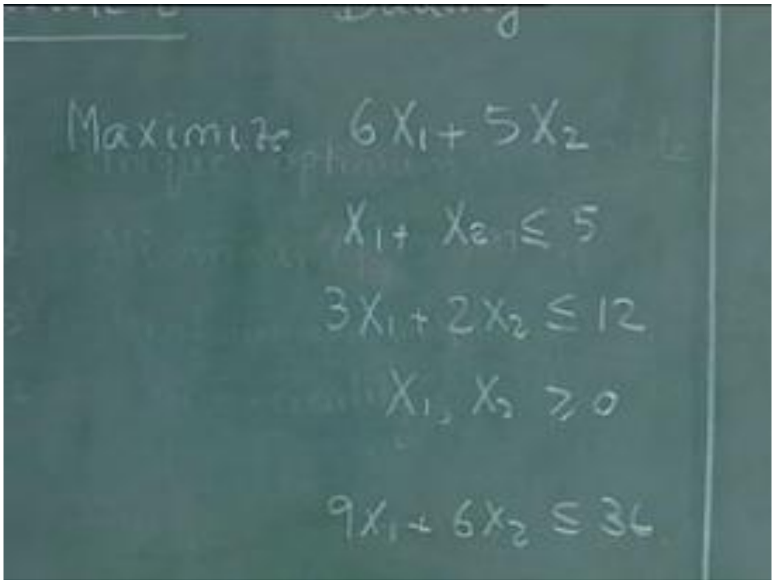

For example if we can look at this problem, we confidently say that the Z value at the optimum cannot be more than 1000, then 1000 becomes an upper bound to the objective function at the optimum. Now one of the simplest things to do is to multiply the second constraint by 3. If you multiply the second constraint by 3 we get $9X_1 + 6X_2$ is less than or equal to 36. We know that the optimum solution has to be feasible therefore X_1 and X_2 are greater than or equal to 0. The optimum solution has satisfied all the constraints. The optimum solution has to satisfy this constraint $9X_1 + 6X_2$ is less than or equal to 36. $9X_1 + 6X_2$ is less than or equal to 36; for X_1 X_2 greater than or equal to 0; $9X_1 + 6X_2$ is greater than $6X_1 + 5X_2$. Therefore $6X_1 + 5X_2$ should be less than or equal to 36. Therefore the optimum solution should have an objective function value less than 36. 36 is upper bound to the objective function value at the optimum.

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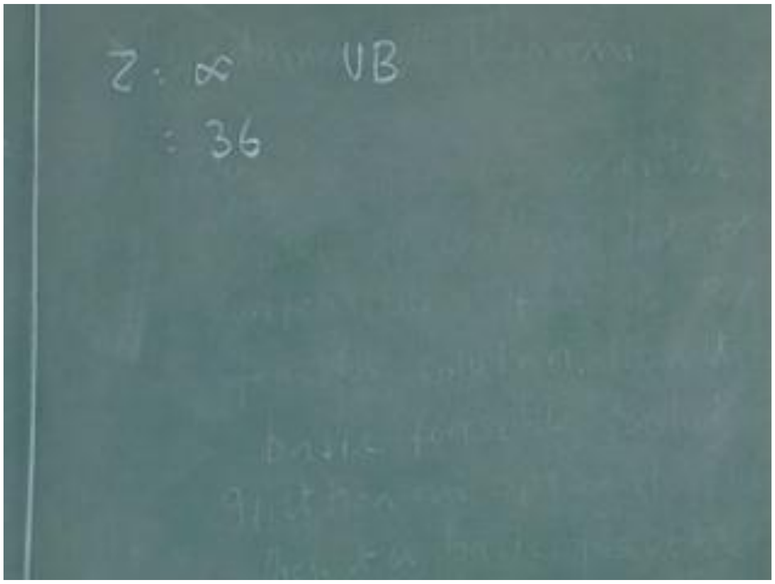

So we can get Z = 36; Let us do another thing.

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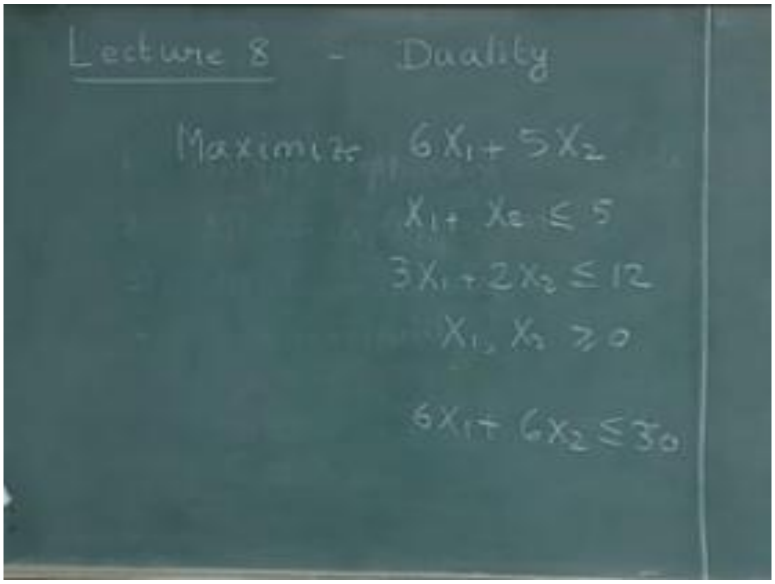

Let us multiply the first constraint by 6. We get $6X_1 + 6X_2$ less than or equal to 30
By the same logic X_1 and X_2 are greater than or equal to 0; $6X_1 + 6X_2$ is less than or equal to 30
$6X_1 + 5X_2$ is less than $6X_1 + 6X_2$ which in turn is less than 30. Therefore 30 is upper bound to the value of the objective function at the optimum.
We can go back and say this problem cannot have an objective function of more than 30 at the optimum.

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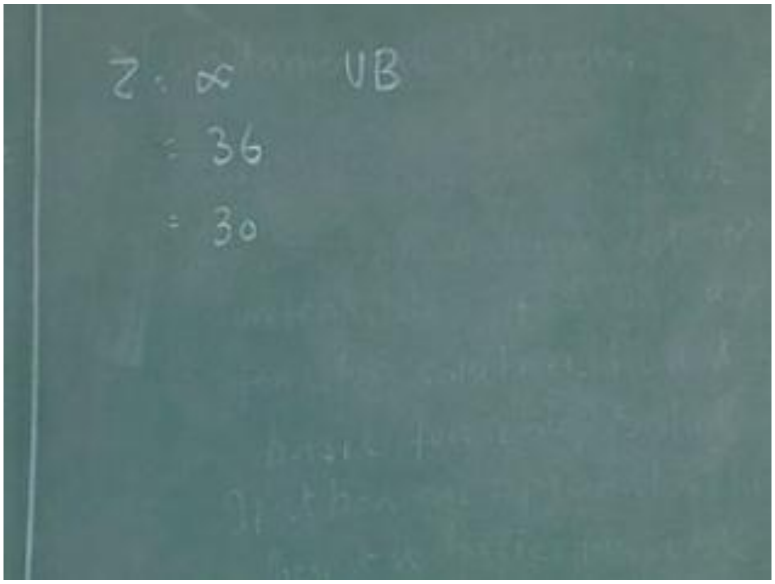

We now look at 30. Now let us look at the third now. We realize this (Refer Slide Time: 12:17) 30 adds more value to us than this 36 because we are now zeroing on and we now realize that this problem cannot have an optimum objective function value of more than 30. It can be 30 or less. Let us look at the third thing.

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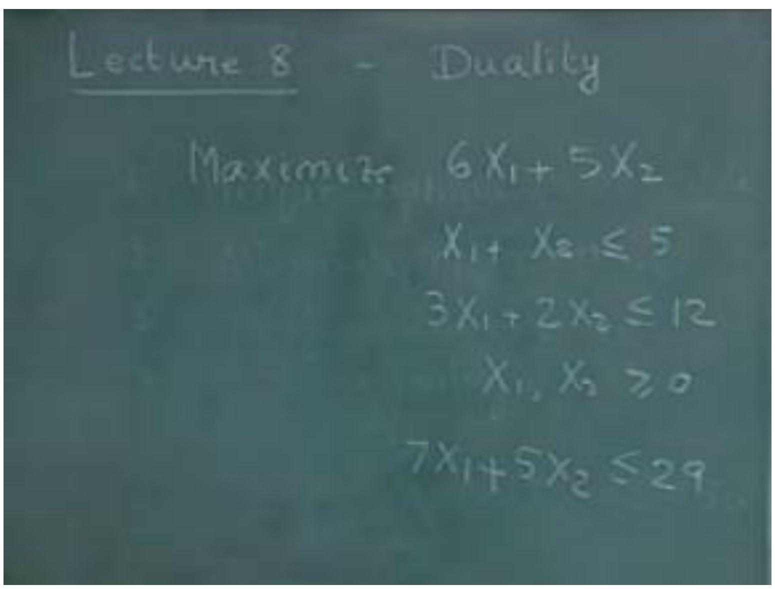

Let us multiply this by 2 and add this. So this will become 3 into 2 = 6 + 1 = $7X_1$; 2 into 2 = 4 + 1 = $5X_2$ less than or equal to 29. 12 into 2 = 24 + 5 = 29

Once again by the same logic $7X_1 + 5X_2$ should be less than or equal to 29; $6X_1 + 5X_2$ is less than $7X_1 + 5X_2$; X_1, X_2 greater than or equal to 0. $6X_1 + 5X_2$ is less than or equal to $7X_1 + 5X_2$ which in turn in less than or equal to 29.

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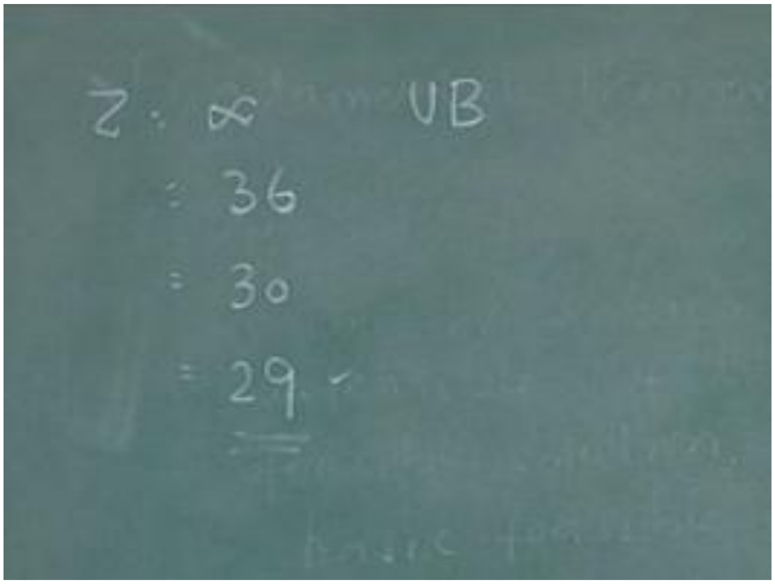

So 29 is another value of the upper bound. We are doing this in such a way we are progressively reducing this upper bound value. Now what do we understand out of this?

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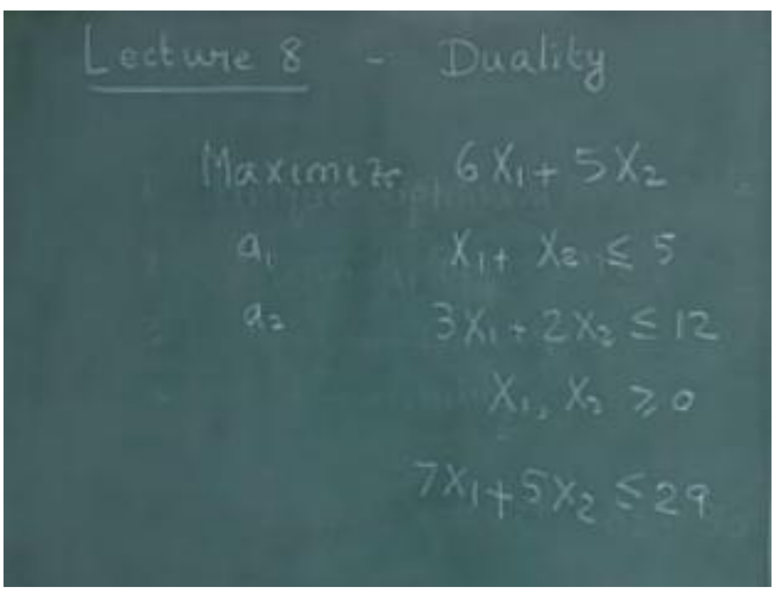

We understand that if I multiply the first inequality say by a_1 and I multiply the second one by an a_2, such that a_1, a_2 is greater than or equal to 0. Otherwise the sign of the inequality will change

and so on. We maintain that a_1, a_2 greater than or equal to 0 and if we multiply this by an a_1 and this by an a_2 such that a_1 a_2 greater than or equal to 0. In the resultant, process we get a_1 into a_1 or in the process whatever we get, we simply multiply. We get $1a_1 + 3a_2$ is greater than 6. For example I multiply this by a_1 to get $a_1X_1 + a_2X_2$ less than or equal to $5a_1$; $3a_2X_1 + 2a_2X_2$ less than or equal to$12a_2$ and then I add this so I get a $1 + 3a_2$.
So if $a_1 + 3a_2$ is greater than or equal to 6 and $a_1 + 2a_2$ is greater than or equal to 5 then $5a_1 + 12 a_2$ will be an upper bound to the objective function value at the optimum. What really have we understood from this?

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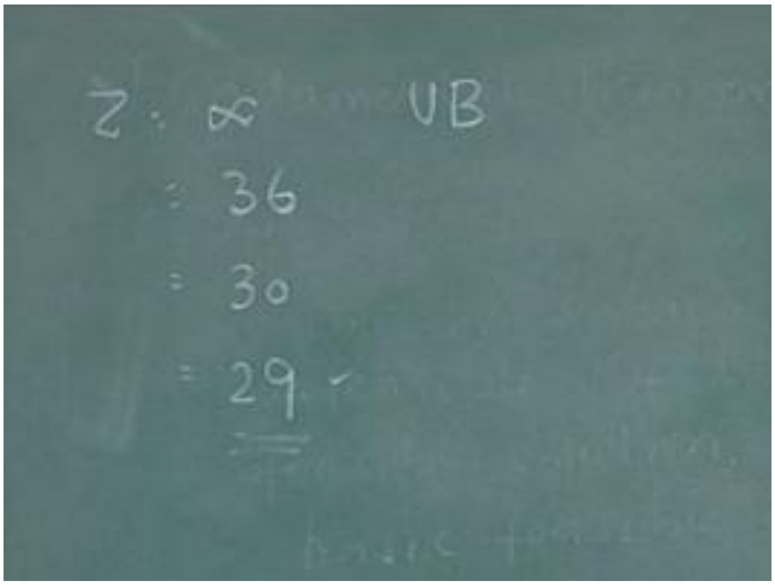

Here we multiplied this by 2 and this by 1. In that example a_1 was 1 and a_2 was 2.

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You multiply the first inequality by a_1 and second one by a_2. a_1, a_2 greater than or equal to 0 such that $1a_1 + 3a_2$ is greater than or equal to 6; $1a_1 + 2a_2$ is greater than or equal to 5.
Then $5a_1 + 12a_2$ will be an upper bound to the objective function. Why did we do all these?

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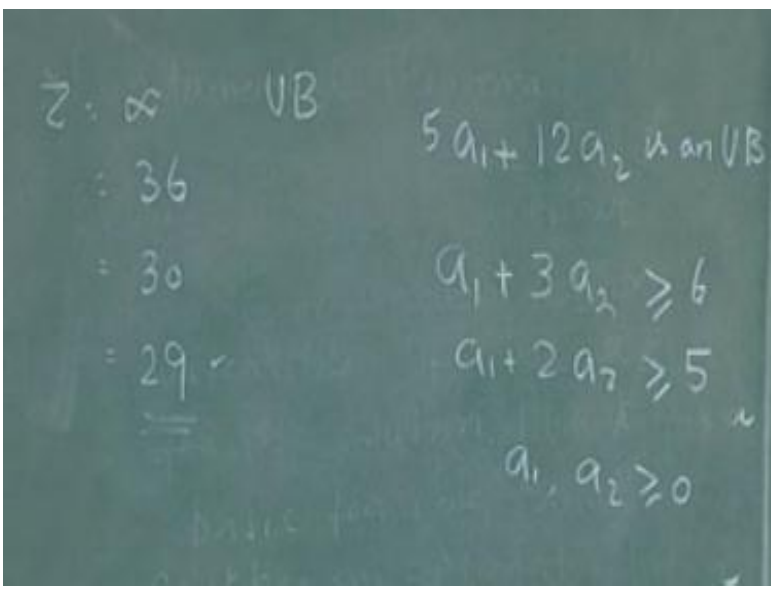

We want to find out what is a smallest value of the upper bound that we can get. In order to do that what we need to do is to multiply this by a_1 and a_2 such that $a_1 + 3a_2$ (you find a_1 and a_2) such that $a_1 + 3a_2$ greater than or equal to 6; $a_1 + 2a_2$ greater than or equal to 5. $5a_1 + 12a_2$ is upper bound, a_1, a_2 greater than or equal to 0.

Now if you are able to find out any a_1, a_2 which satisfies this and this (Refer Slide Time: 16:54) and this, then we know that $5a_1 + 12a_2$ is upper bound to the value or the objective function at the optimum. Now if we want to find out the minimum value that this upper bound will take then obviously we try to find out a_1, a_2 such that this is minimized.

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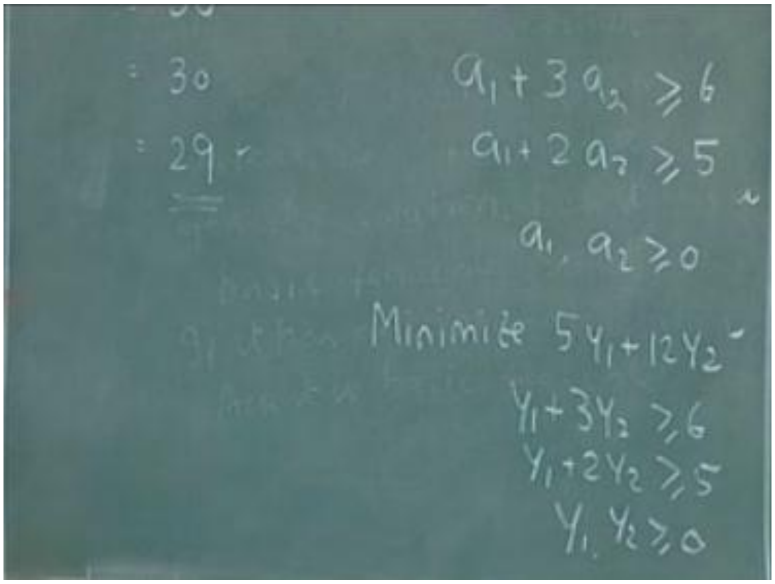

a_1, a_2 which will satisfy all these conditions and minimize $5a_1 + 12a_2$ will give the minimum possible value that this upper bound can take. This is another linear programming problem which has come out of the original problem. If we replace a_1 by y_1 and a_2 by y_2, we will get minimize $5y_1 + 12y_2$ such that $y_1 + 3y_2$ greater than or equal to 6; $y_1 + 2y_2$ greater than or equal to 5; Y_1, Y_2 greater than or equal to 0.

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This problem is called the dual of this problem. So this (Refer Slide Time: 18:08) problem is called the primal. The given problem is called the primal.

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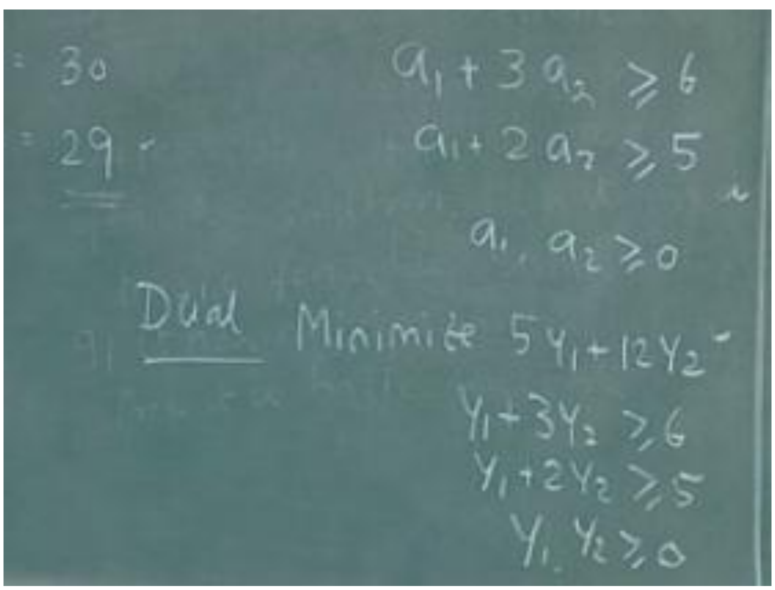

Dual provides the minimum value that this upper bound can take and dual is another linear programming problem which is written from the original one. What are some of the characteristics of the duality? To begin with, if every primal problem has a dual then by this logic we can write a dual for every primary problem. We will take another example to write the dual and learn more about how to write the dual. But before that what we need to do is this. This is example two variable, two constraint problem. But what happens is the primal is a maximization problem. Then the dual is the minimization problem. The primal has two constraints, the dual will have two variables because these Y_1 and Y_2.

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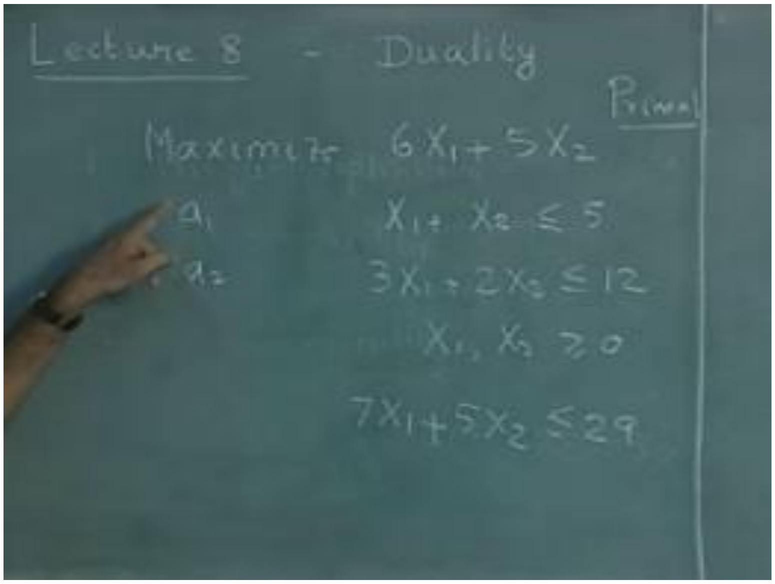

Then the dual is the minimization problem. The primal has two constraints, the dual will have two variables because these Y_1 and Y_2 represent the a_1 and a_2 and by definition, the a_1 and a_2 enter because there are two constraints. So the primal has two constraints.

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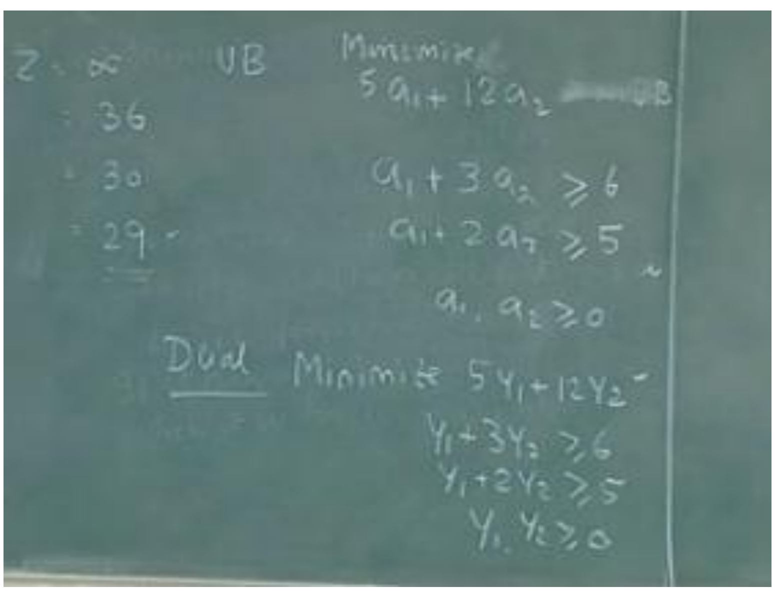

The dual has two variables Y_1 and Y_2.

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Now these two constraints come from the fact that these two variables are there in the primal.

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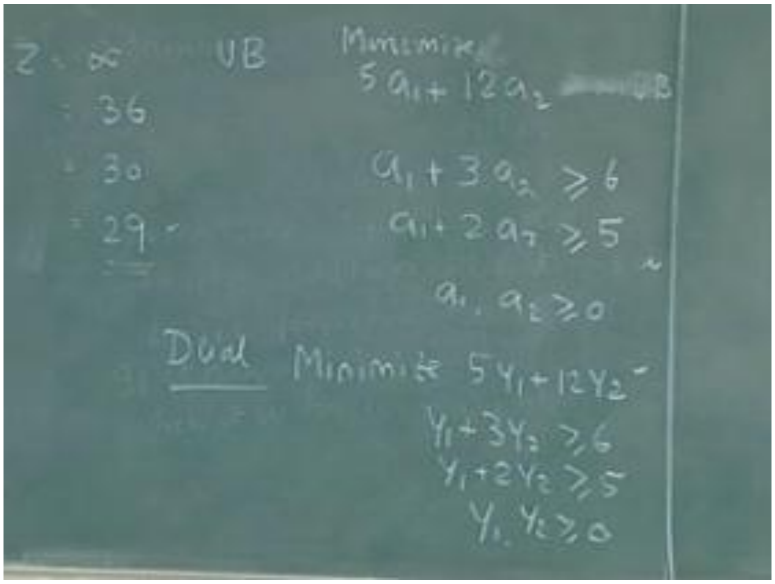

This is because each of these is written for these variables. So the dual will have as many constraints as the number of variables the primal will have and it will have as many variables as the number of constraints the primal will have.

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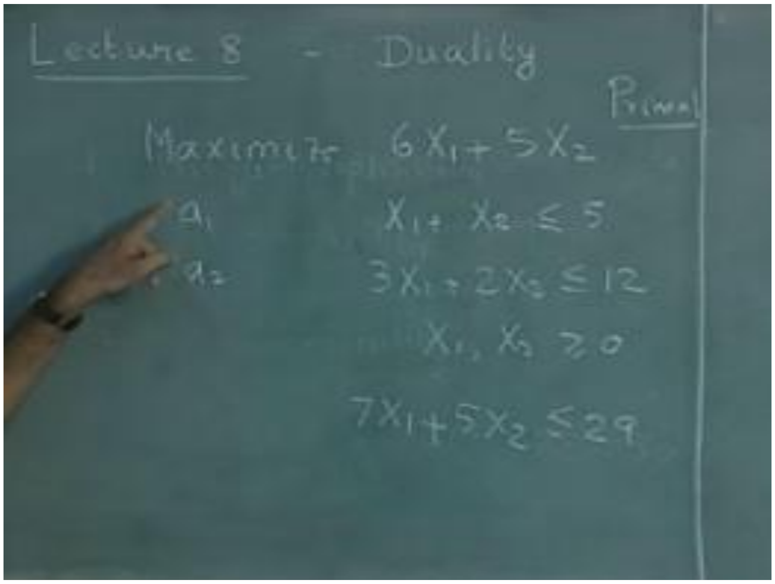

Now the right hand side values of the primal will be the objective function values of the dual

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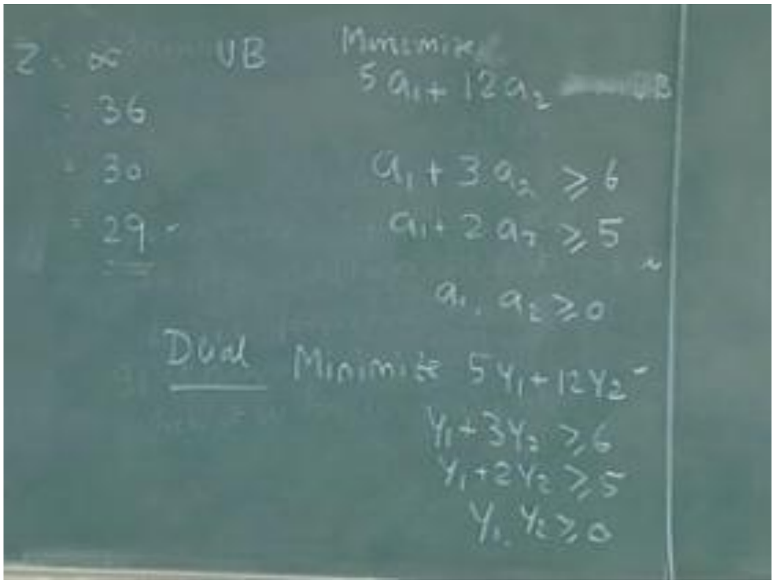

Now the right hand side values of the primal will be the objective function values of the dual

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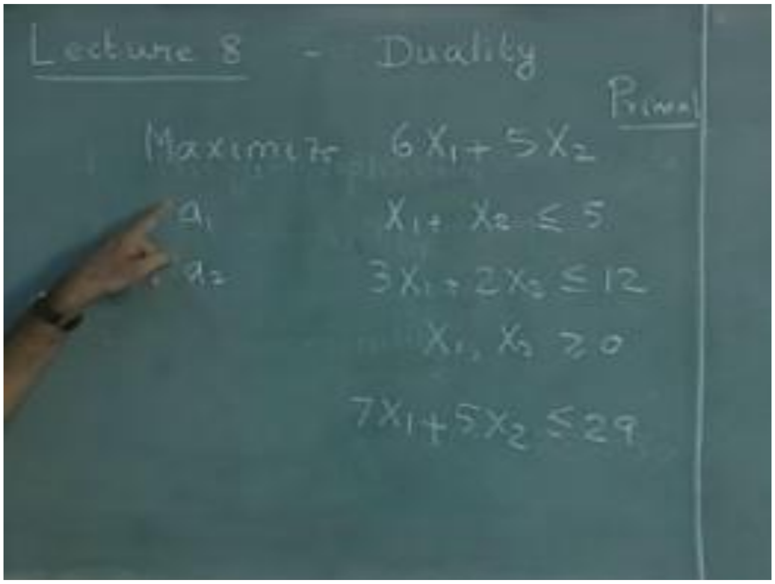

The dual and the right hand side values of the dual will be the objective function values of the primal. For example we may not be able to clearly understand the fact that two variables become two constraints. Much later will take a different example to show all these properties. In summary the properties are that the primal is a maximization problem. Dual is a minimization problem.

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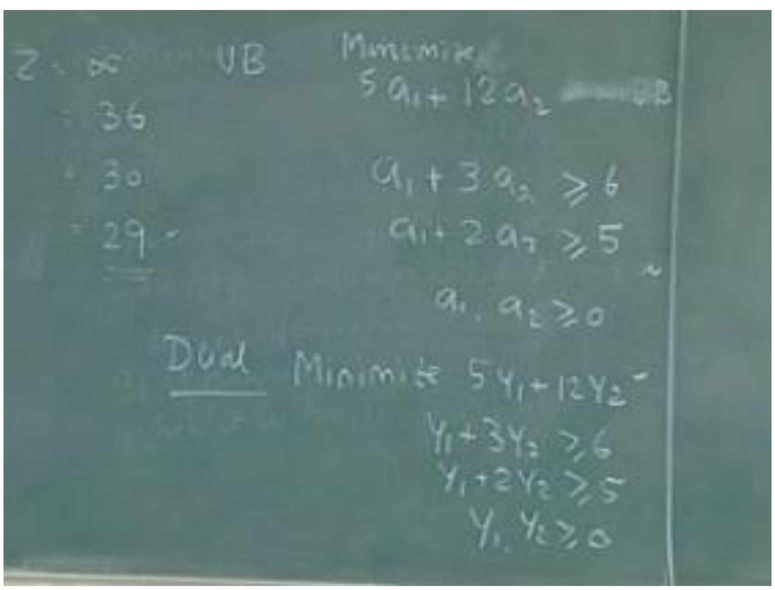

The dual will have as many constraints as the number of variables in the primal and as many variables as a number of constraints in the primal. The co efficient matrix of the primal will become a transpose here. It will get transpose and it will appear as it is.

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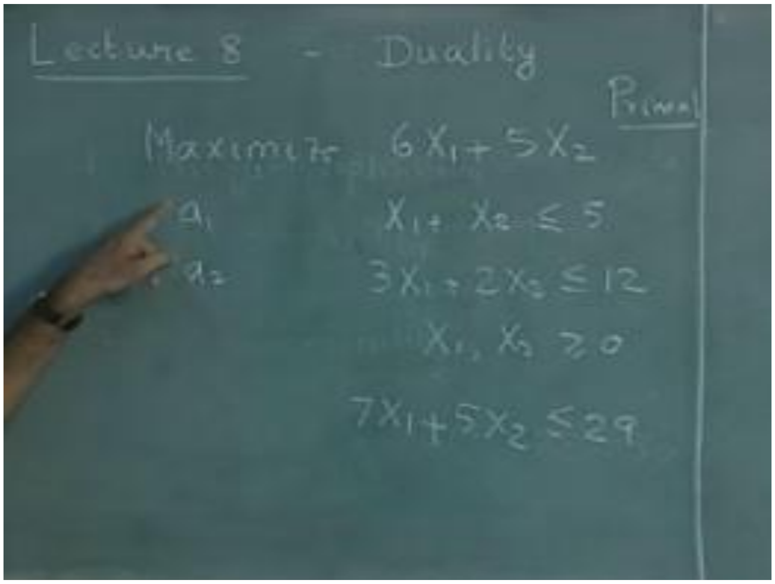

The right hand side values become the objective function co-efficient values and the objective function co-efficient become the right hand side values.

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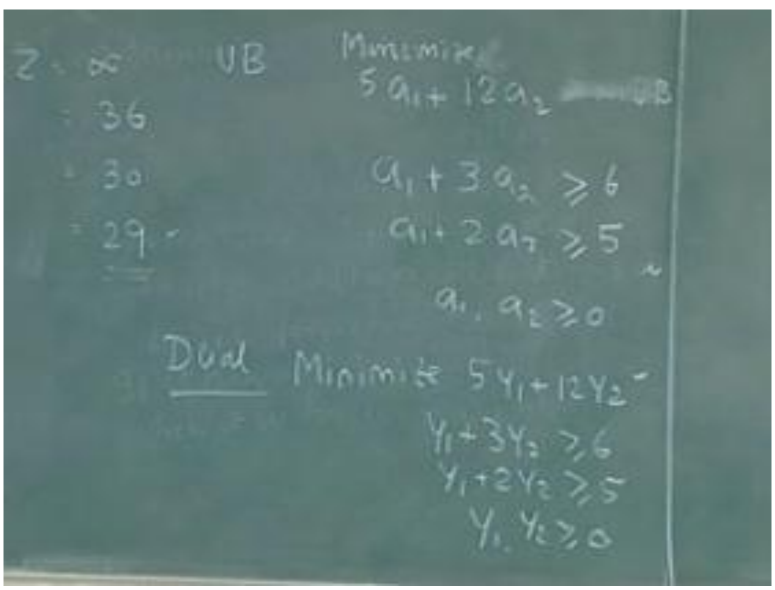

Now we will keep this as the base primal dual relationship. We will assume that the primal is a maximization problem and if the primal is a maximization problem with all less than or equal to constraints and all variables greater than or equal to 0. We know dual will be a minimization problem with all constraints greater than or equal to all variables greater than or equal to and having additional properties in terms of number of variables becoming number of constraints and number of constraints becoming number of variables, co-efficient matrix becoming the transpose, Objective function becoming the right hand side and right hand becoming the objective.

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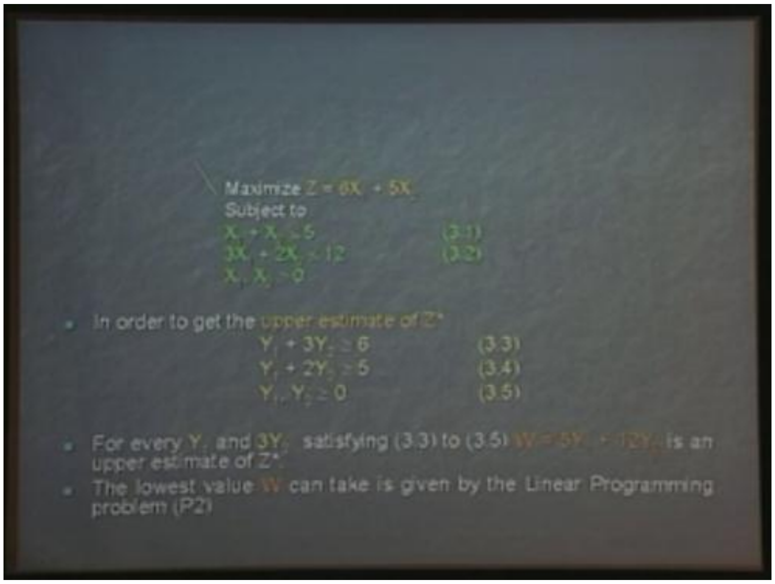

Let us go back to this and whatever we had explained, you can see again in this sheet. So these are for every Y_1 and $3Y_2$ satisfying this, we get an upper estimate of Z. So lowest value it can take is given by another linear programming problem P_2 that we have derived here. Minimize $5Y_1 + 5Y_2$; $W = 5Y_1$.

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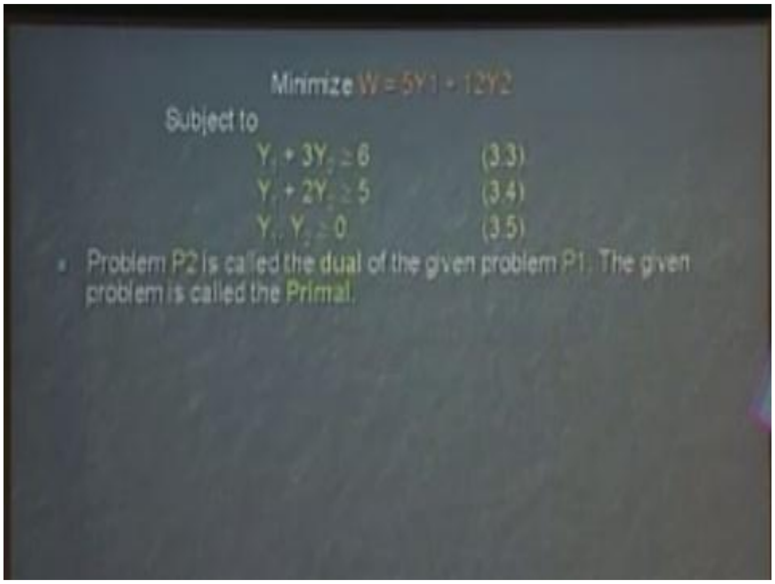

It is customary to write the dual objective function as W instead of Z.
Normally we use the vector X to represent the solution of the primal and Z to represent the objective function value of the primal.

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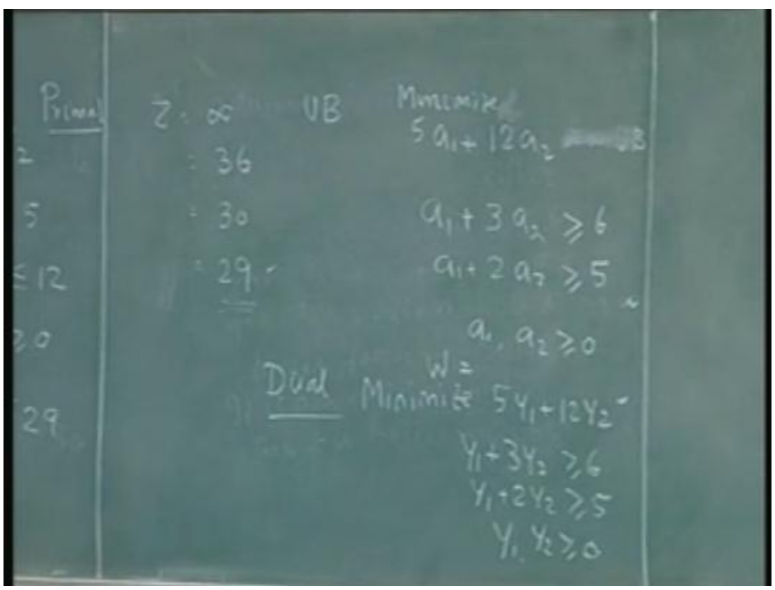

We use Y to represent the dual variables and W to represent the objective function value of the dual.

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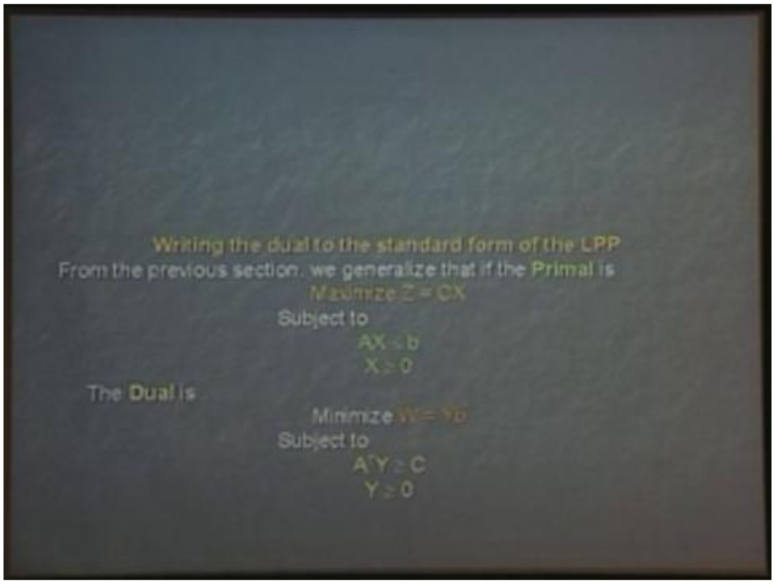

So we have minimized W = $5Y_1 + 12Y_2$ and so on. Now if the primal is maximize Z = CX subject to AX less than or equal to b; X greater than or equal to 0, the form that we have looked at, in this case C, b, X are vectors A is the matrix. The dual will be minimized W = Yb subject to A transpose Y greater than = CY greater than or equal to 0 which are the things that we have seen right now.

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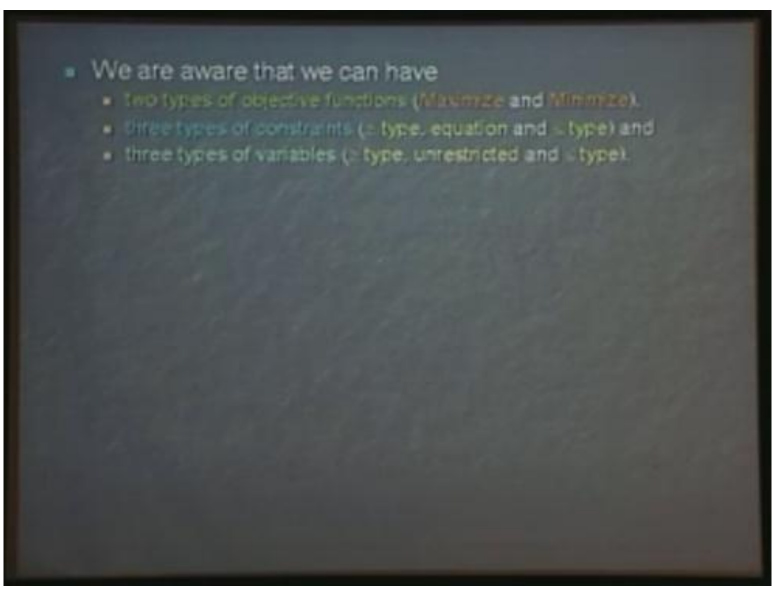

Right now the next thing that we need do is if we are given a primal which satisfies exactly this,

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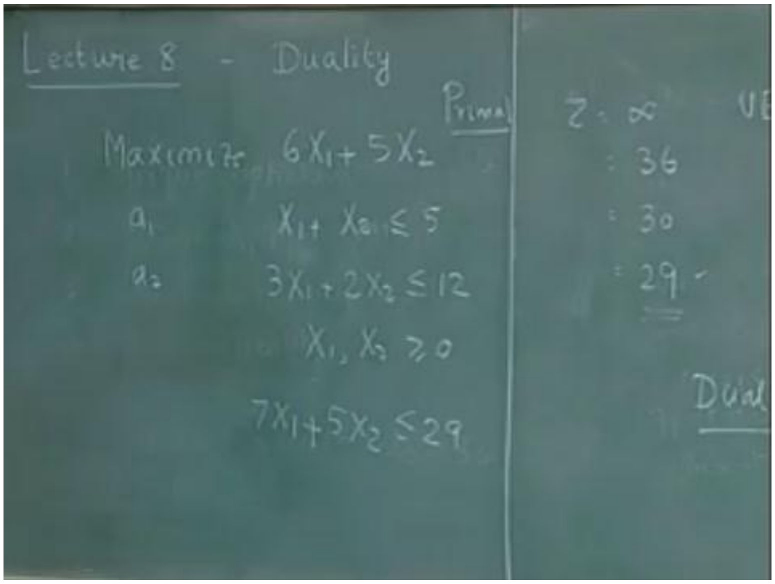

If we are given a primal which is a maximization problem with all less than or equal to constraints and all greater than or equal to variables, we assume now that we know to write the dual.

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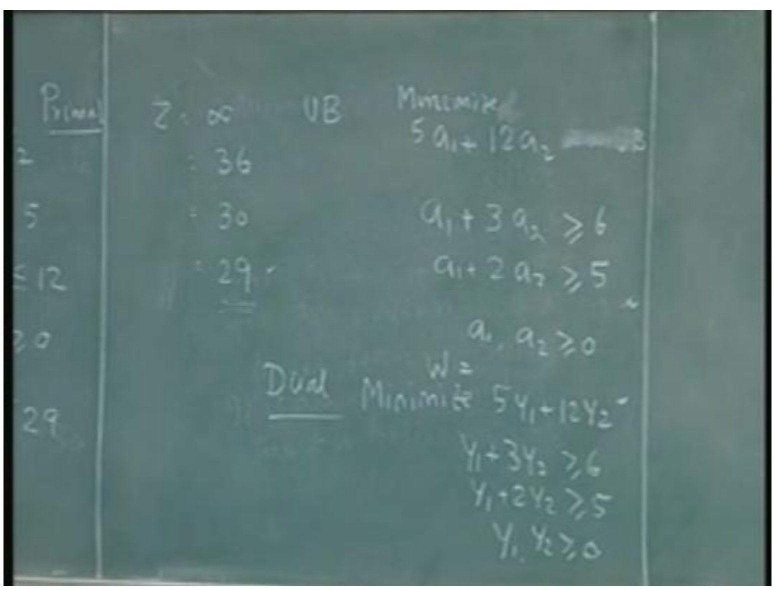

The dual will be minimization problem.

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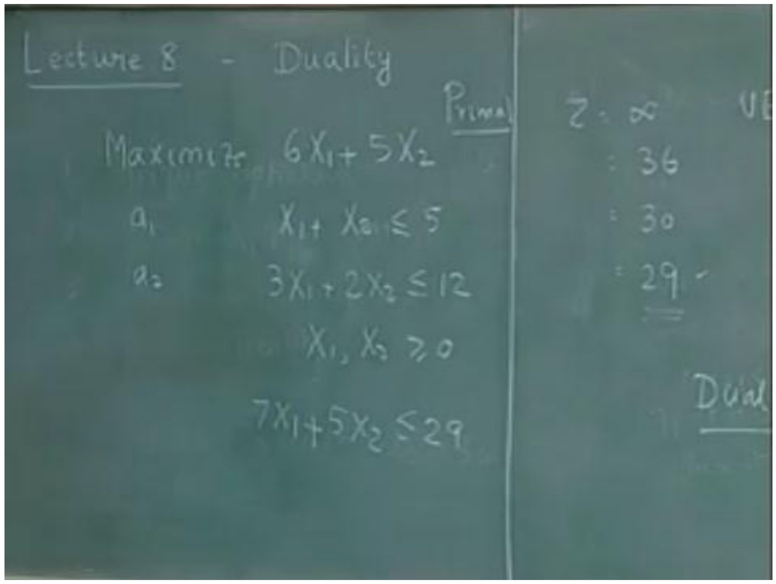

We are also aware that linear programming problems can have two types of objective functions maximization or minimization. It could have three types of constraints, greater than or equal to, less than or equal to and equation it could have three types of variables greater than or equal to less than or equal to and unrestricted.

What will happen to the dual if we have something other than this? For example if this is a minimization, if this is greater than or equal to, this is an equation or this is a less than or equal to this is unrestricted. The other things will always be satisfied. What are the other things? The number of constraints will be equal to the number of variables, number of variables equal to number of constraints,

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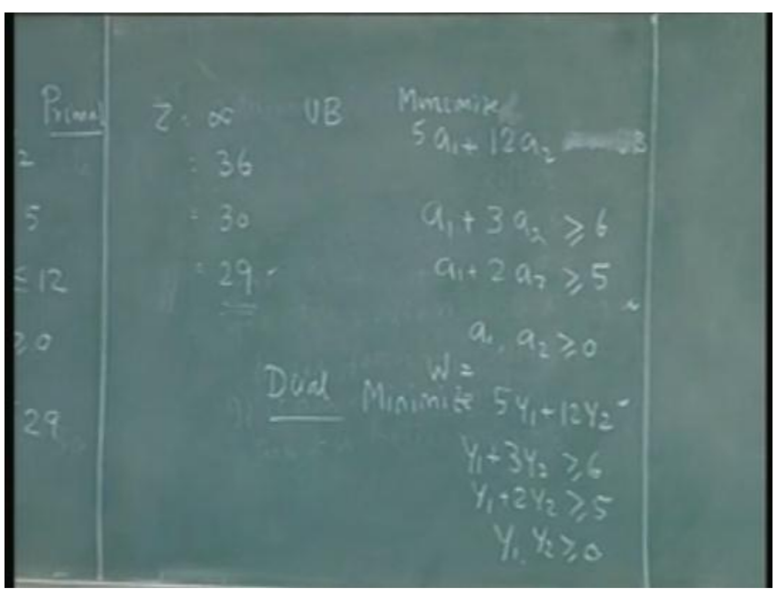

co-efficient matrix becoming transpose, right hand side becoming an objective function and objective function becoming right hand side. Those things will be always be there. They will be not change. Only these inequalities, the sign of the constraints and the sign of the variables will change depending on what we have here.

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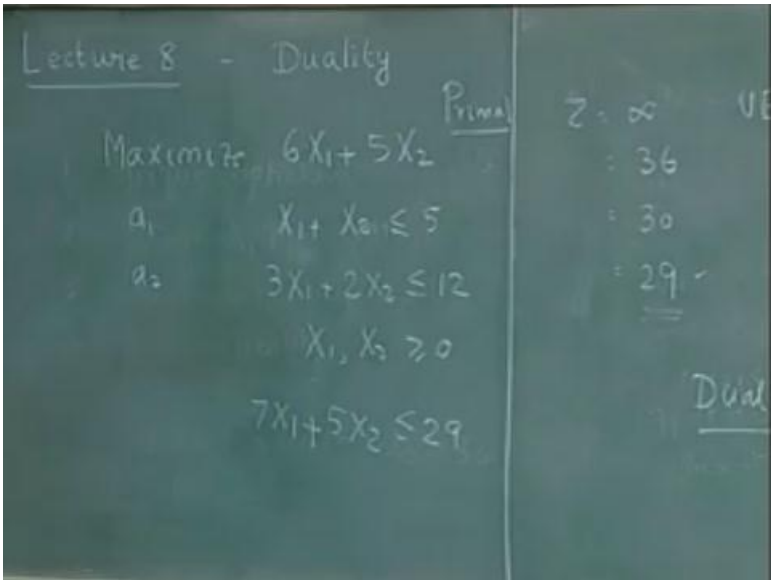

We take an example and then write the dual for that example and then understand how, for any given primal we can write the dual problem with all less than or equal to and all greater than or equal to. We learn how to write the dual whatever be the primal in terms of type of constraint and type of variable. Let us take another example.

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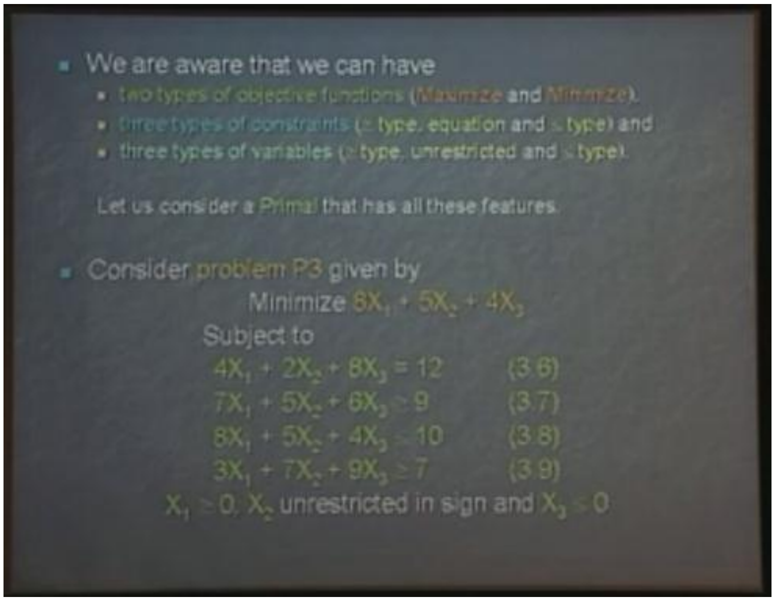

The problem that we take is as follows:

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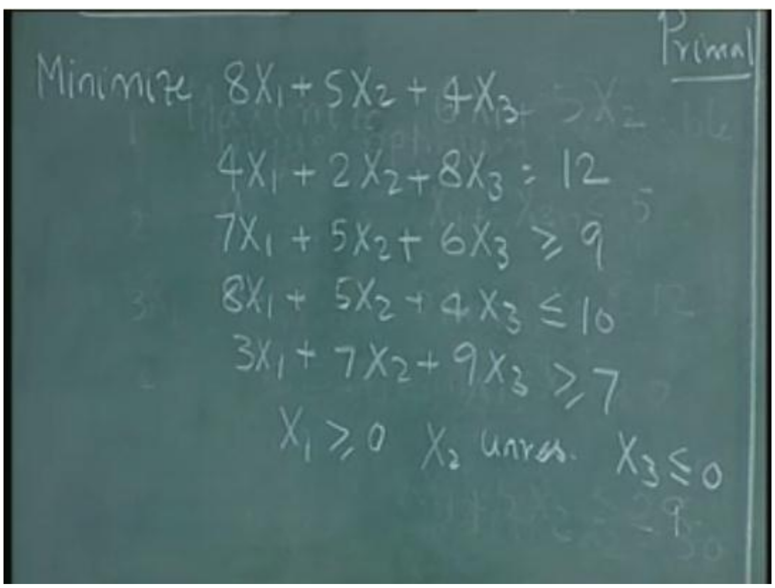

Minimize $8X_1 + 5X_2 + 4X_3$ subject to $4X_1 + 2X_2 + 8X_3 = 12$
$7X_1 + 5X_2 + 6X_3$ greater than or equal to 9
$8X_1 + 5\ X_2 + 4X_3$ less than or equal to 10
$3X_1 + 7X_2 + 9X_3$ greater than or equal to7; X_1 greater than or equal to 0; X_2 unrestricted and X_3 less than or equal to 0

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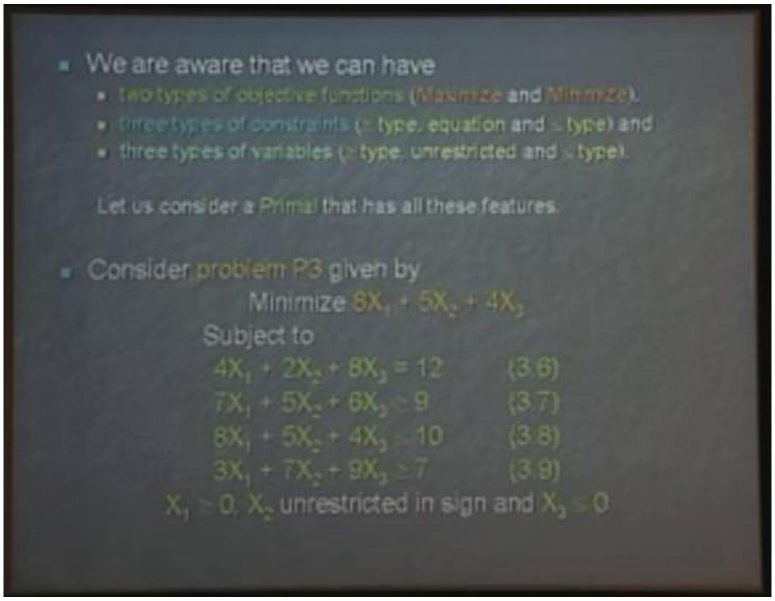

Now let us go back to this table. We realize that all possible types of objective functions constraints and variables are covered in this example. It has a minimization. We have earlier taken a maximization problem. There are four constraints and three variables; therefore we would expect the dual to have three constraints and four variables. The constraints are equation, greater than or equal to, less than or equal to types. All the three types are covered as well and we have a greater than or equal to, unrestricted and less than or equal to.

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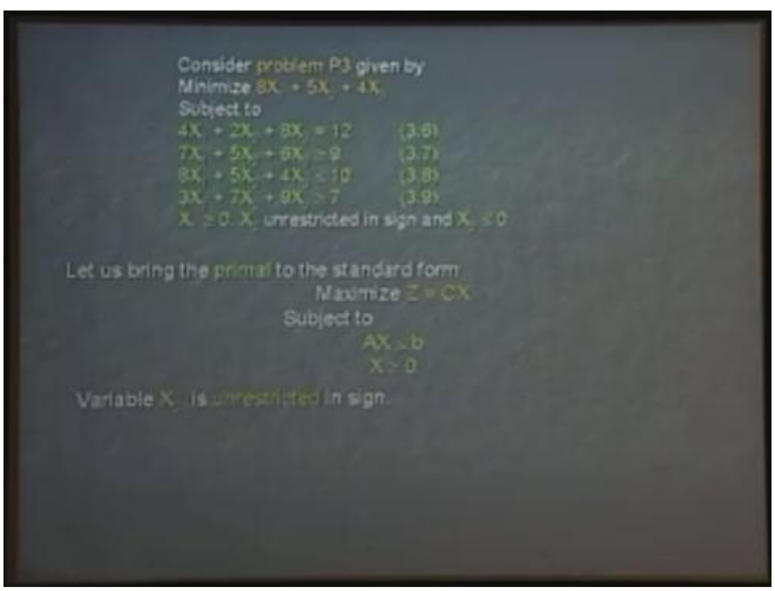

We will take you through the various steps using the power point slides. So we have written the same problem here once again. Now the first thing we need to do is we assume that we know to write the dual if the primal is in the standard form that we are aware of. We assume that the primal is the maximization problem and has all constraints less than or equal to and all variables greater than or equal to. What we do now is we convert this problem into the form that we know, which is the maximization problem with all less than or equal to and all greater than or equal to.

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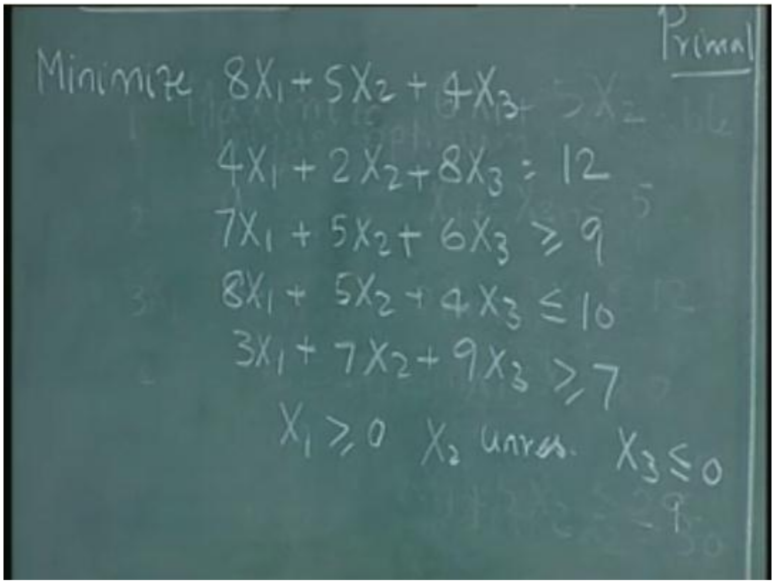

For example writing their converting objective function is not very difficult. You can multiply with the minus one and convert into a maximization problem. This constraint is desirable because it has a less than or equal to. We need to do something about this and this. We also know for example that if you multiply this with the minus one it will become a less than or equal to type. Similarly this is desirable. Thus again we can multiply with minus or redefine the variable here. We know how to convert.

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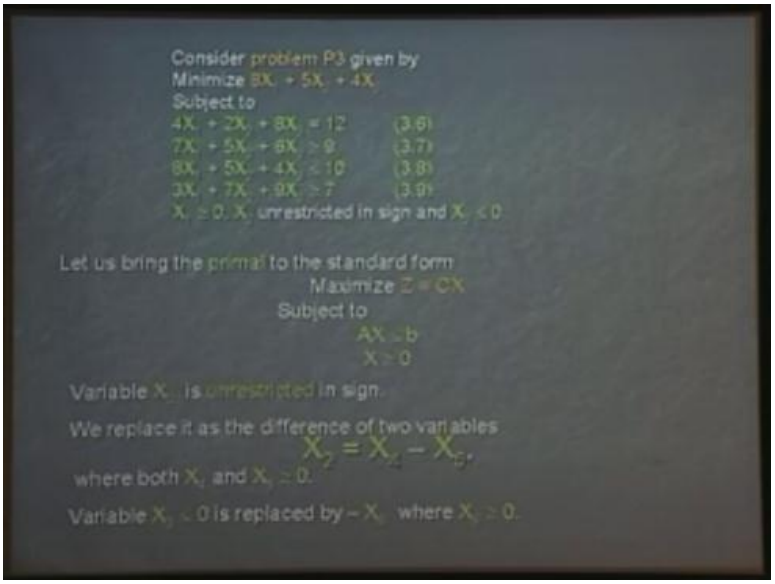

We make it a difference of two variables so we will do all this and see how this problem transforms itself. Now first thing is that we know that variable X_2 is unrestricted in sign so we replace it. As a difference of two variables $X_2 = X_4 – X_5$. This is in the correct required form and this is not.

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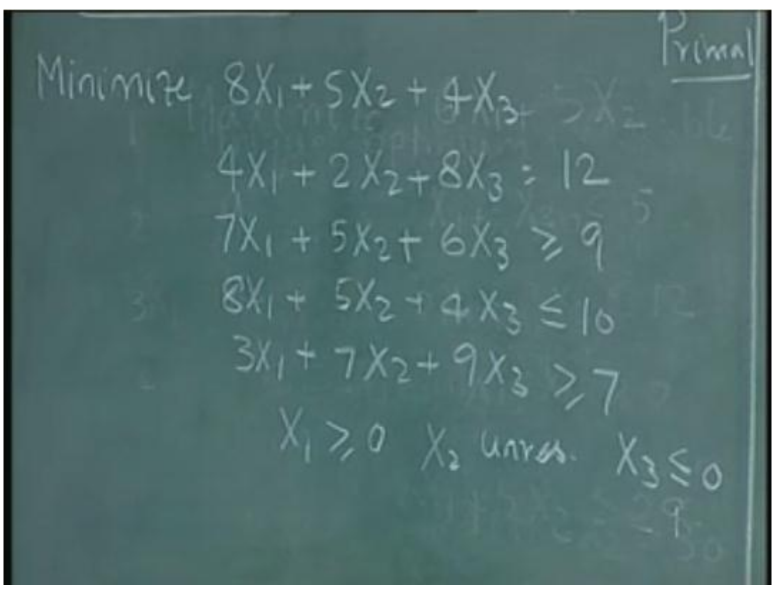

So this becomes $X_4 – X_5$ and X_3 is less than or equal to 0 so X_3 is replaced by $– X_6$.

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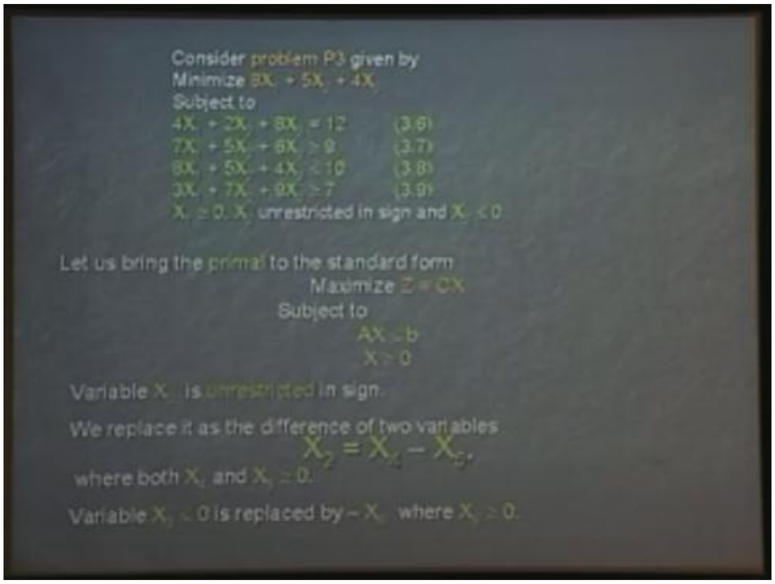

X_6 is greater than or equal to 0 so the first thing we do is we make all these variables into the form which we are comfortable with.

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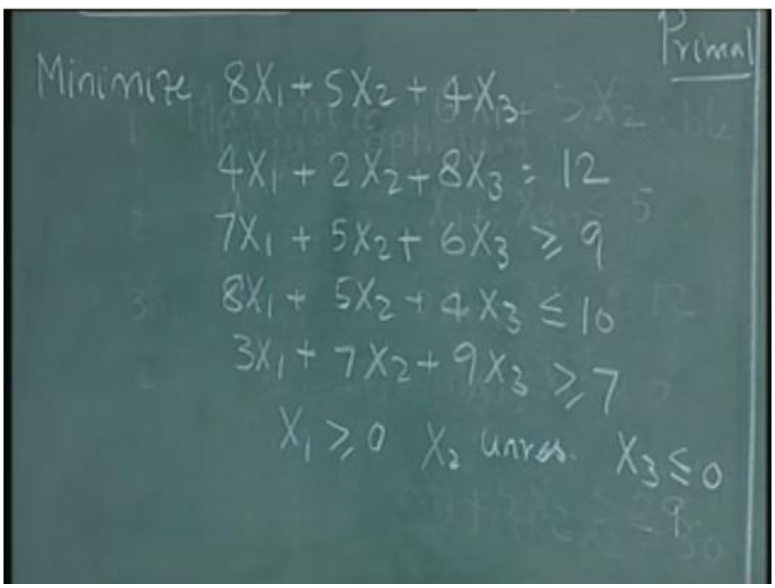

We then you go back and substitute both in the objective function as well as in the constraints.

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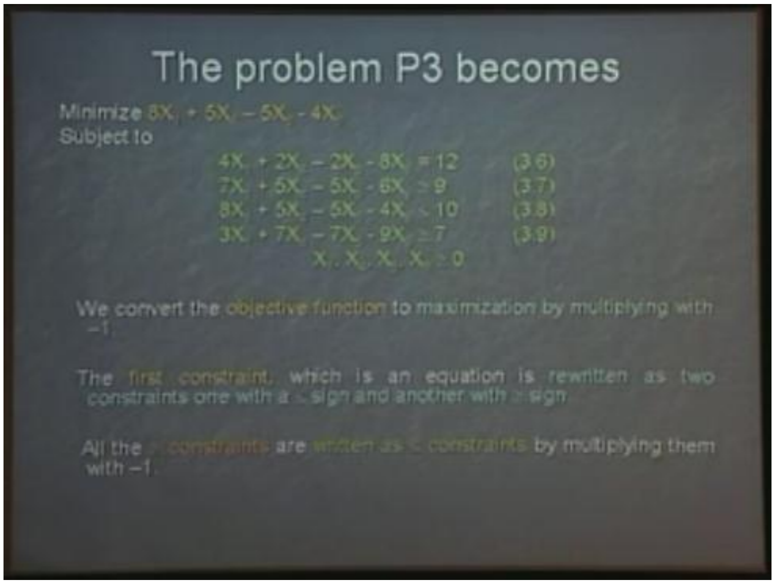

So when we do that we get this (Refer Slide Time: 28:32). Therefore X_6 becomes greater than or equal to 0 but if you look at this you have a minus coefficient. You have a – $8X_6$ here instead of the + $8X_3$.

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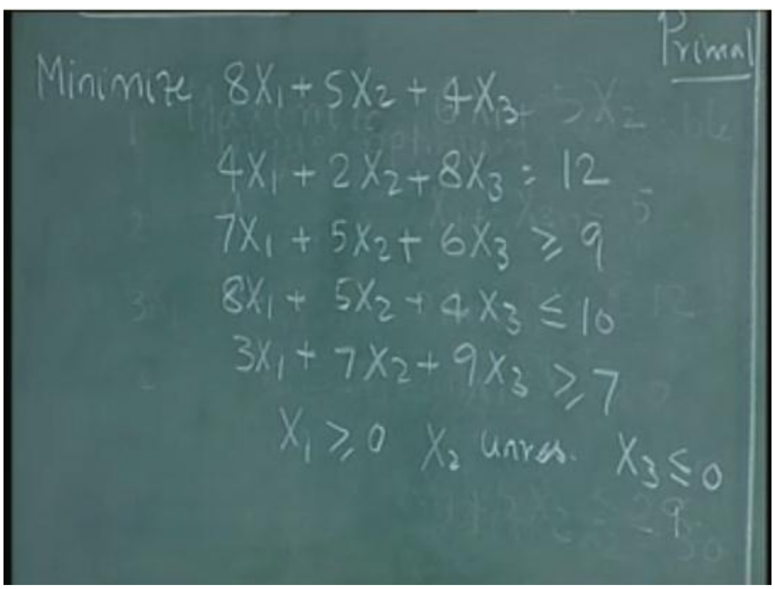

This is because this variable has been replaced by a – X_6. So you will see the change in all the 4 coefficients as well as in the objector function.

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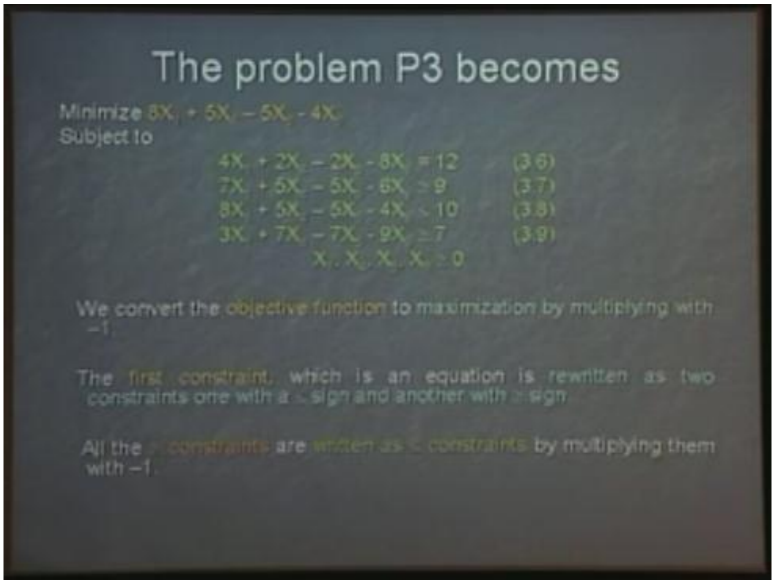

If we look at the second variable which was unrestricted and which is now replaced by $X_4 - X_5$,

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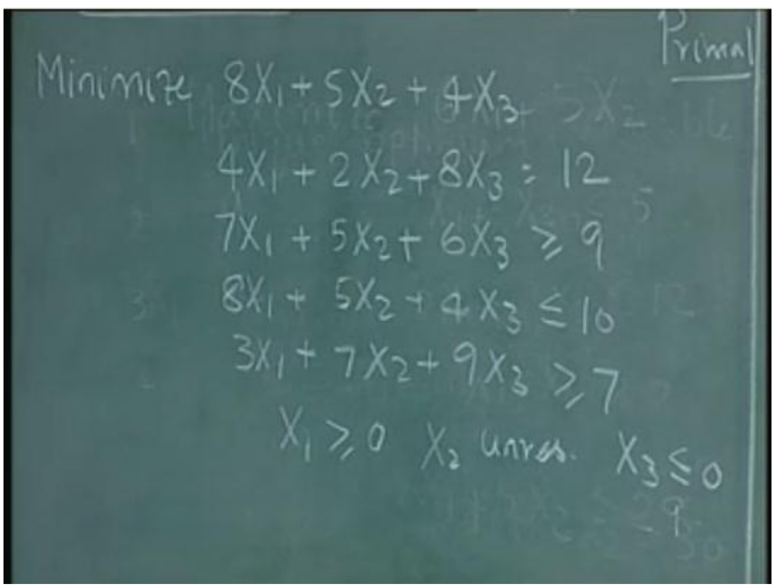

$5X_2$ becomes $5X_4 - 5X_5$.

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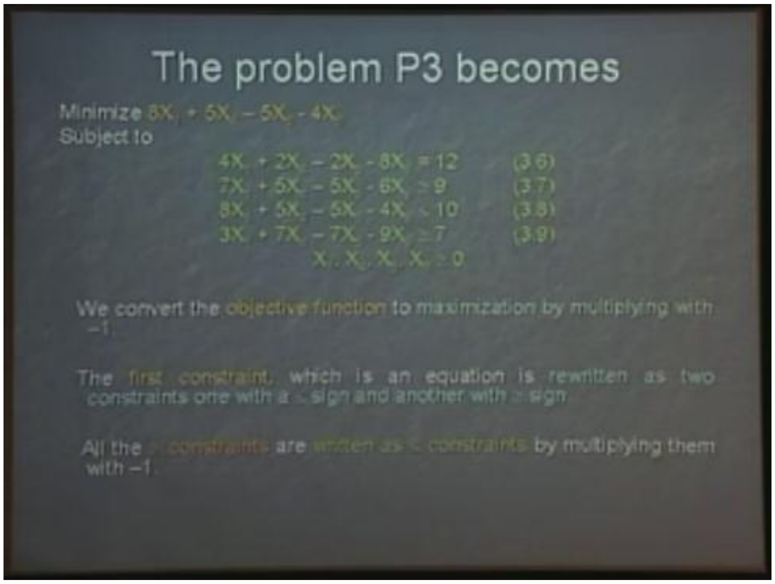

$5X_2$ becomes $5X_4 – 5X_5$. This $2X_2$ becomes $2X_4 – 2X_5$ and so on.

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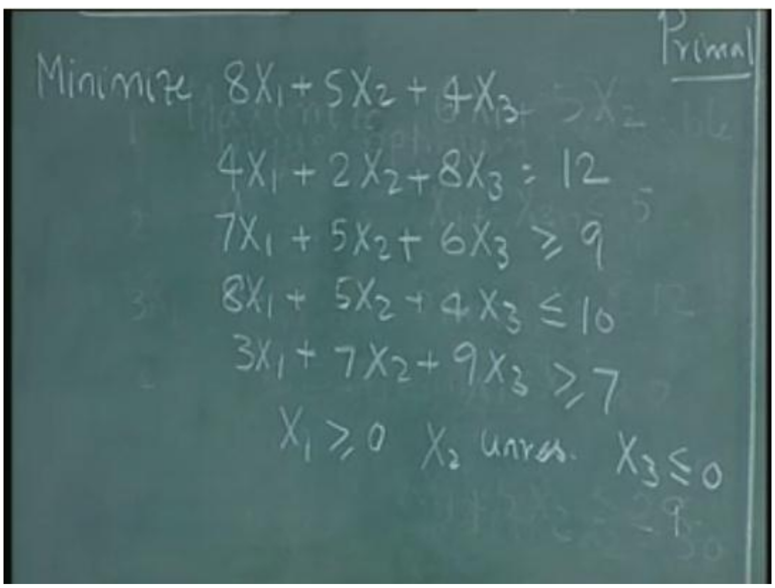

Now because we have added another variable, we now have four variables original.

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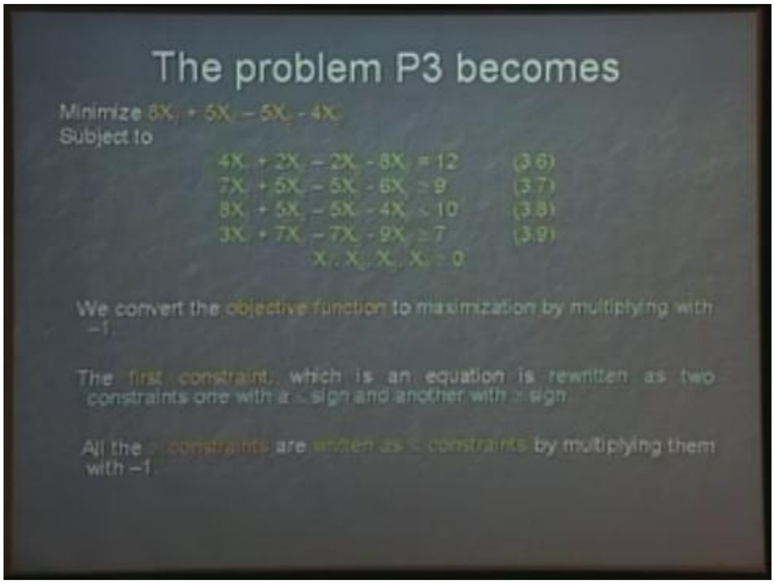

We have now four variables and we have four constraints which are given from 3.6 to 3.9. So this problem is the same as the previous one but with an additional variable.

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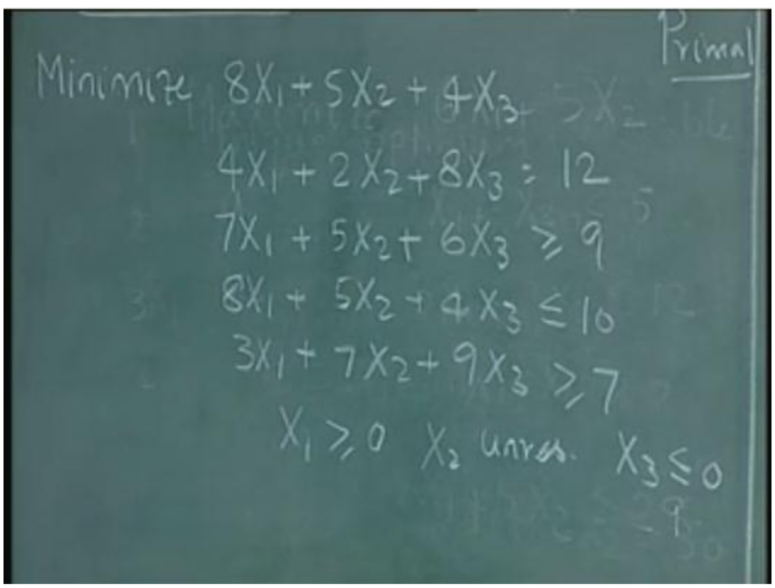

The next thing we do is we convert the objective function into the form that we know.

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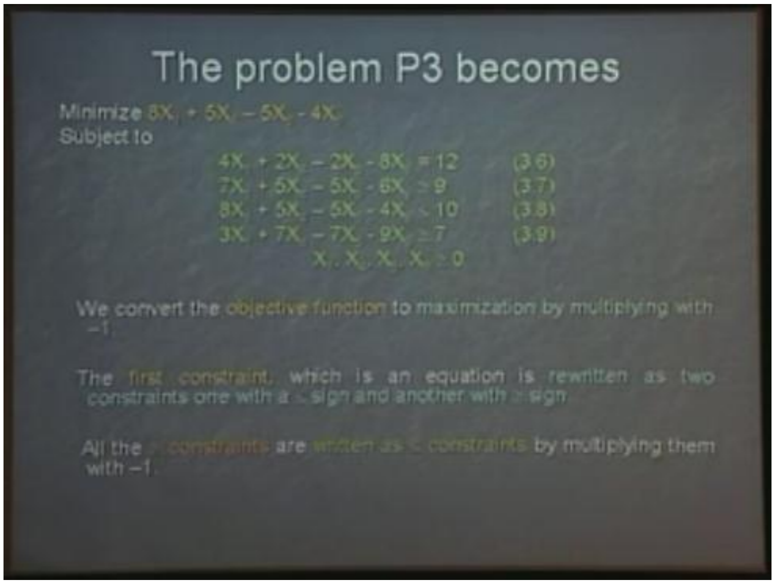

We want to bring it to maximization, so multiply the objective function by –1 to get it into a maximization form.

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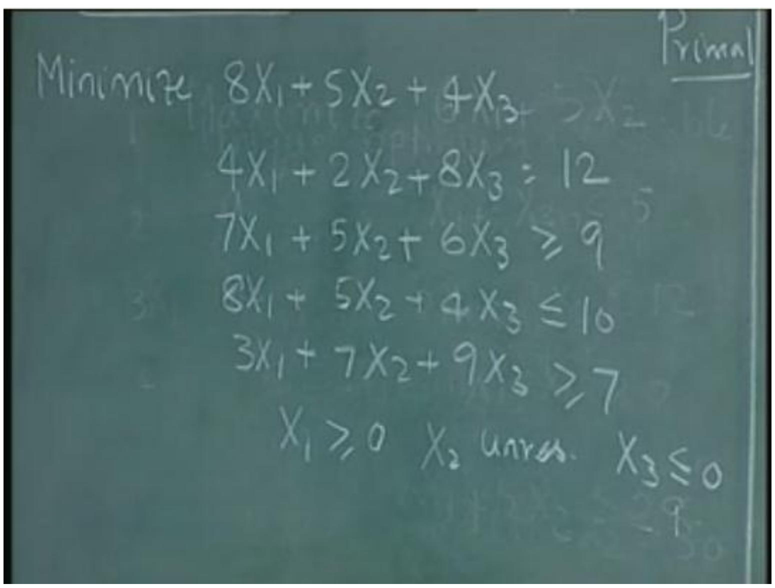

Now let us look at these 4. This is desirable. It is already less than or equal to this. The entire constraint can be multiplied by the –1 to make it less than or equal to. Here again we can multiply this with –1 and make it a less than or equal to.

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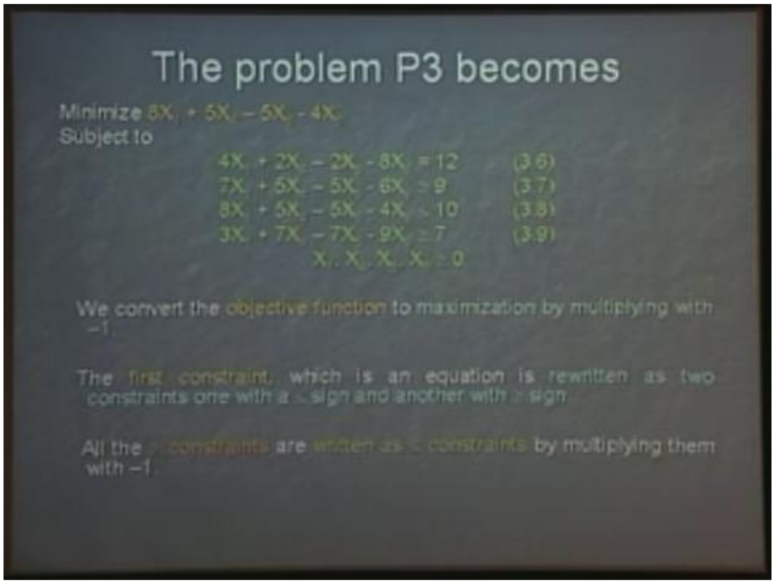

Now what do we do to the equation is, we the write it as two constraints one is $4X_1 + 2X_2 + 8X_3$ greater than or equal to12.

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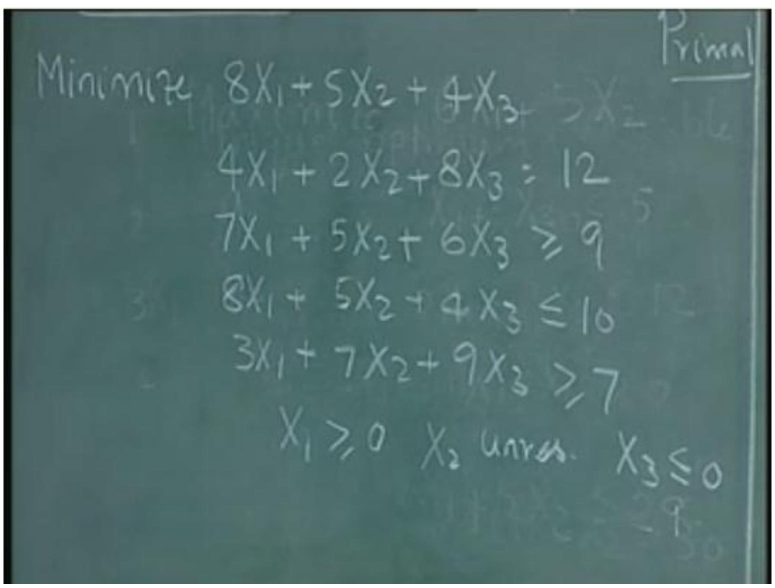

$4X_1 + 2X_2 + 8X_3$ less than or equal to 12 so the equation is written as two constraints

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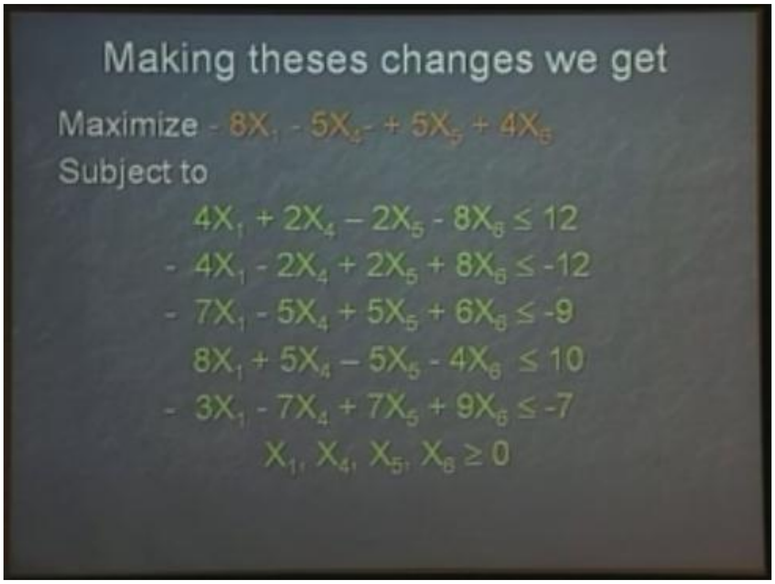

One of which is the less than or equal to straight away. The other one which is the greater than or equal to is now made a less than or equal to by multiplying with the – 1.

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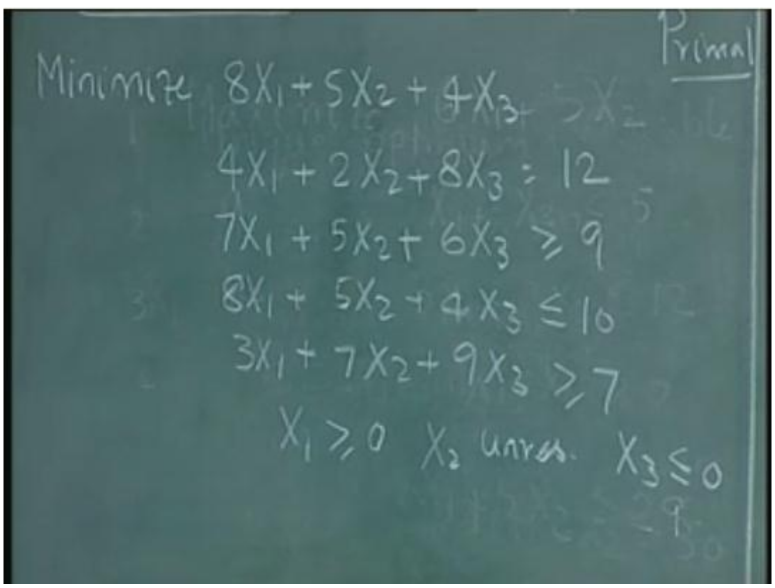

Therefore if we do that your four constraints would now become five as you can see here.

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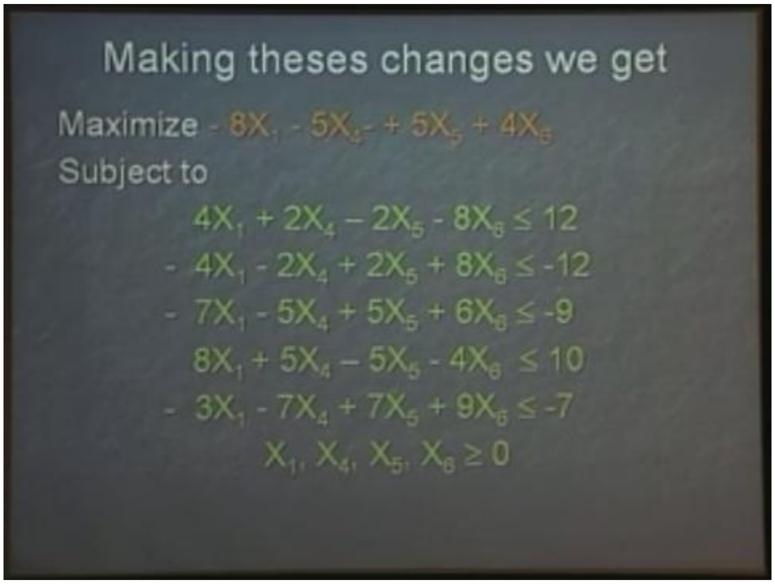

Now you realize that all the constraints are being converted into the less than or equal to form.

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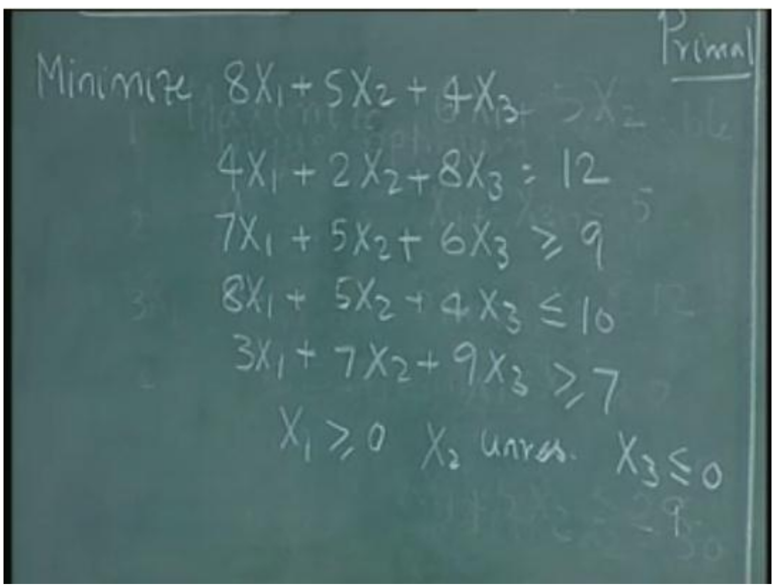

The 10 remains as 10, this greater than or equal to 7 has become less than or equal to – 7

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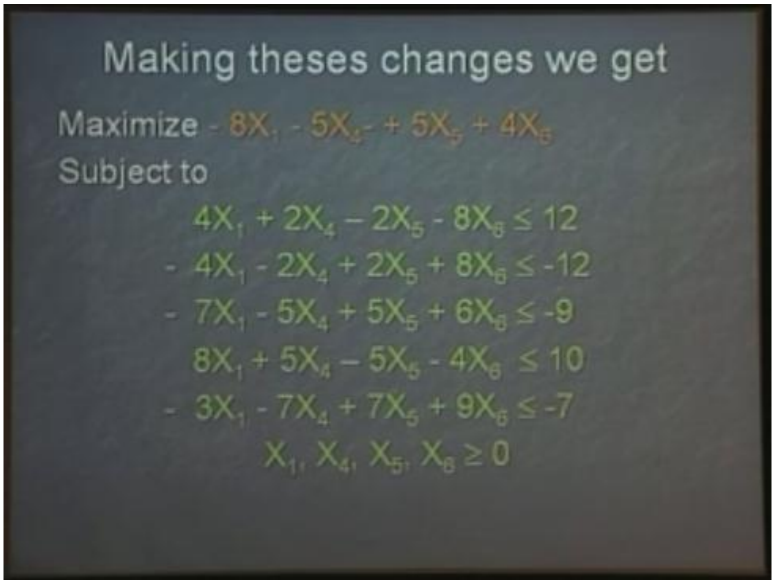

The greater than equal to 9 has become less than or equal to type.

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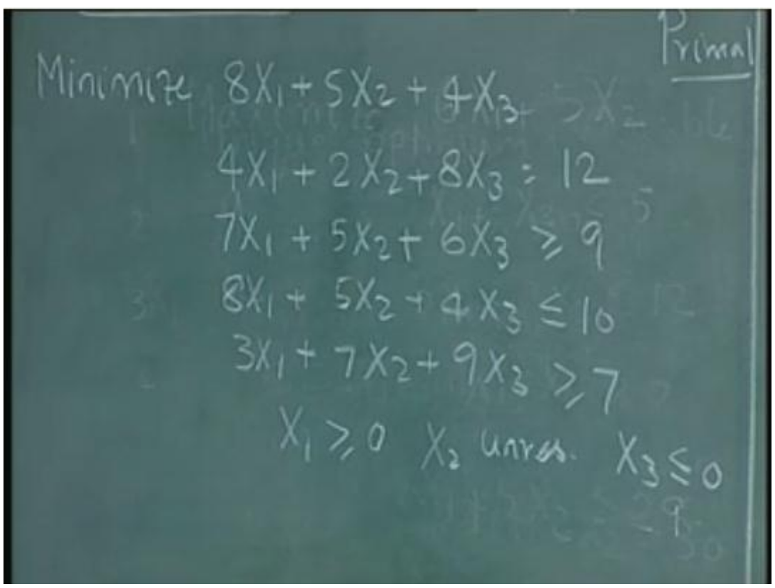

The greater than equal to 9 has become less than or equal to type.

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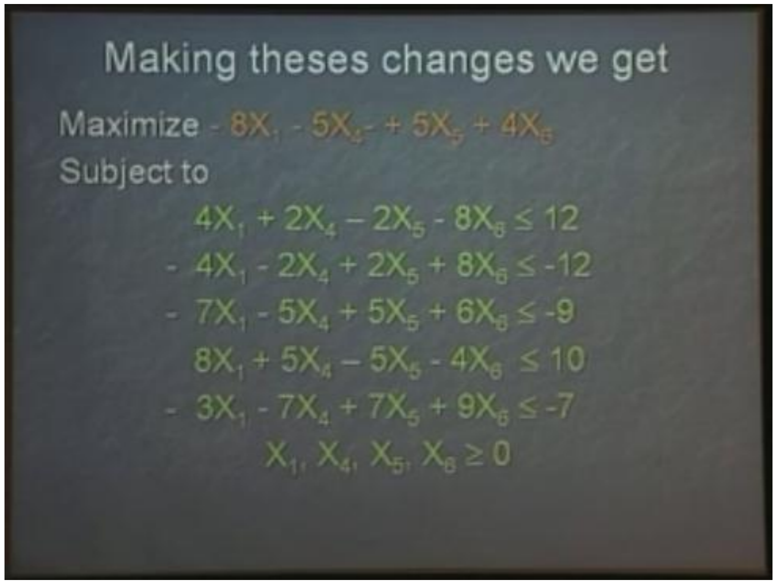

The equation has become two constraints one less than or equal to type.

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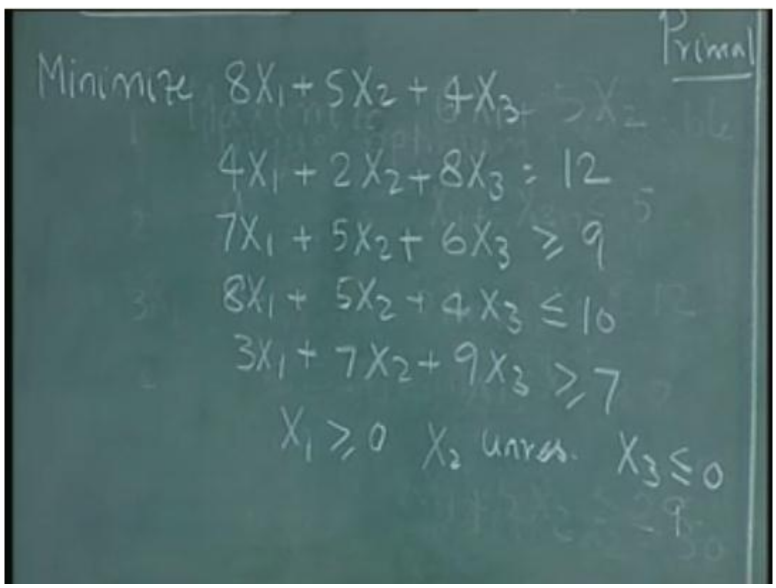

The equation has become two constraints, one less than or equal to type

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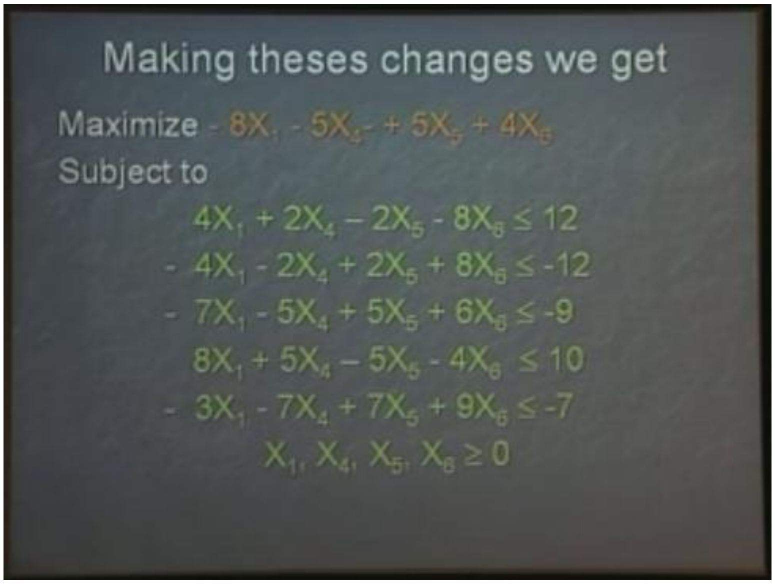

The other one greater than or equal to; so the original problem had three variables

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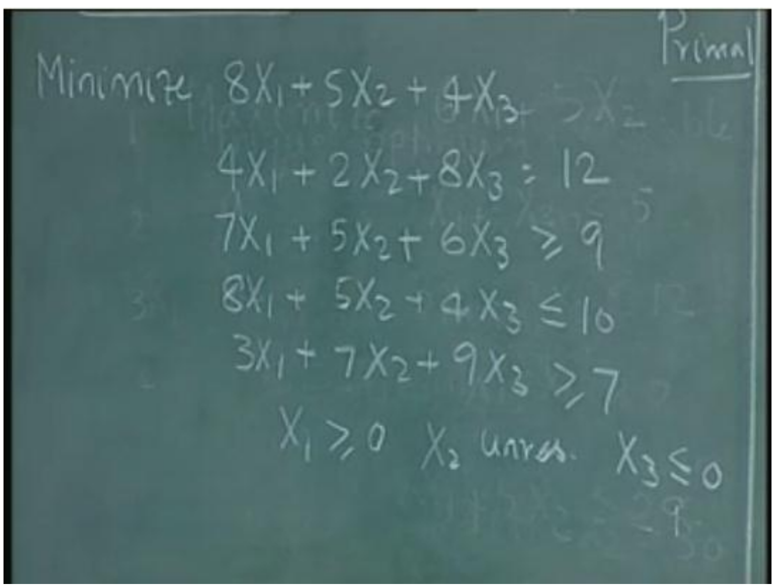

It had three variables and four constraints. The transformed problem has four variables and five constraints.

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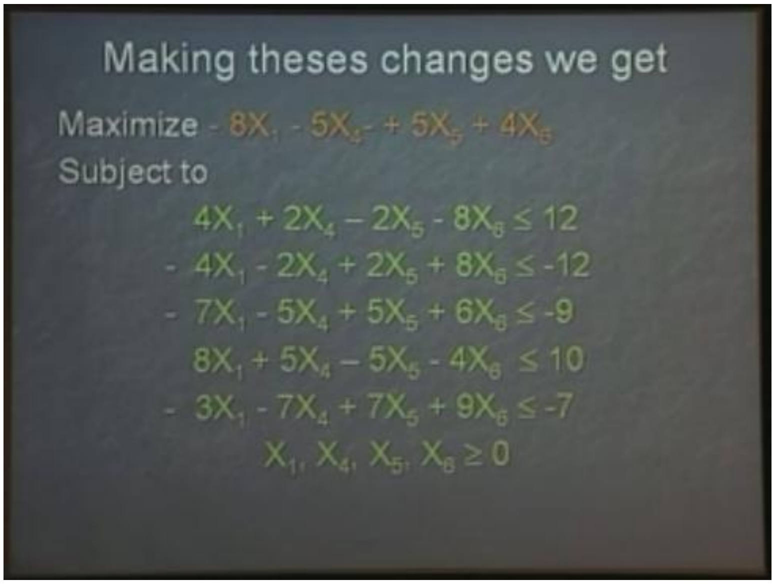

Now we have brought this to the form we are comfortable with. Remember this is still the primal. This is not the dual. We have to write the dual for this problem. The only reason we did this is because we know to write the dual for this problem so we have brought it in the form we are comfortable with and now we are going write the dual for this. So let the dual will now be a minimization problem.

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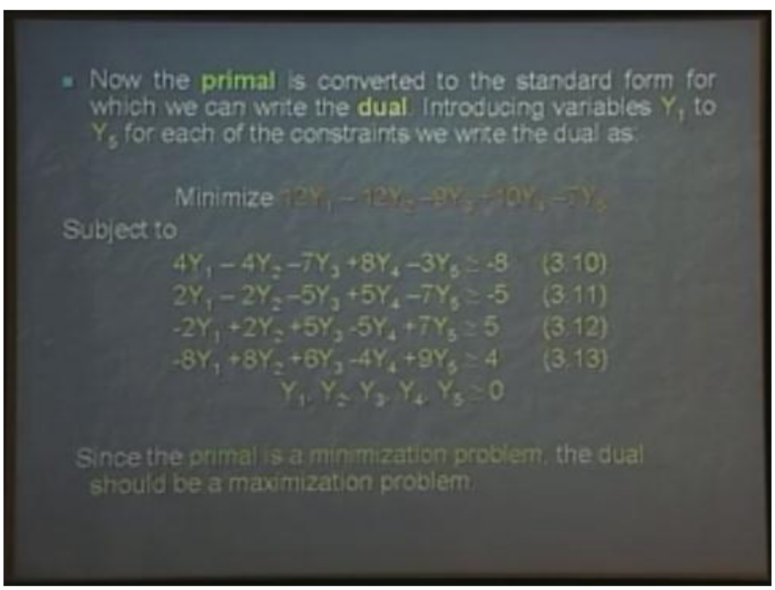

Dual will take the form, minimize go back to the previous slide

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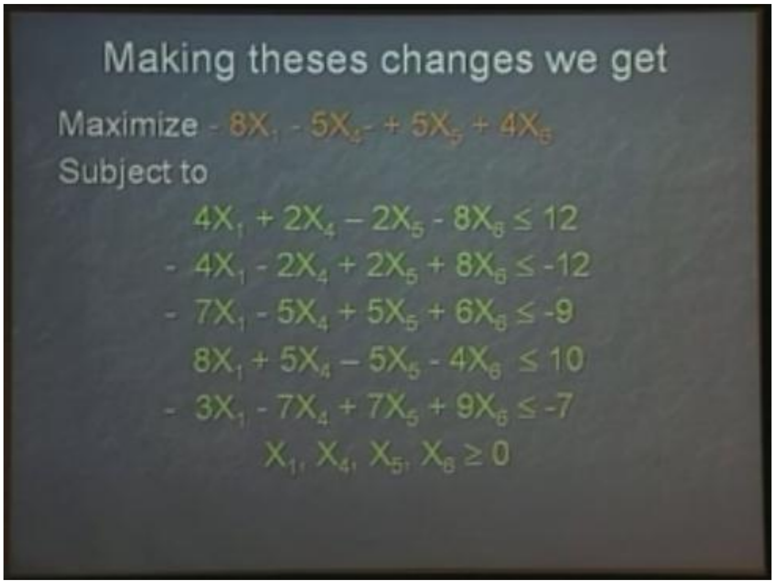

It will become minimized. Now we will introduce five variables corresponding to the five constraints we will call them Y_1, Y_2, Y_3, Y_4 and Y_5. Y_1, Y_2, Y_3, Y_4 and Y_5 corresponding to these five constraints and the dual will have four constraints one for each variable. It will be a minimization problem because this problem is a maximization problem.

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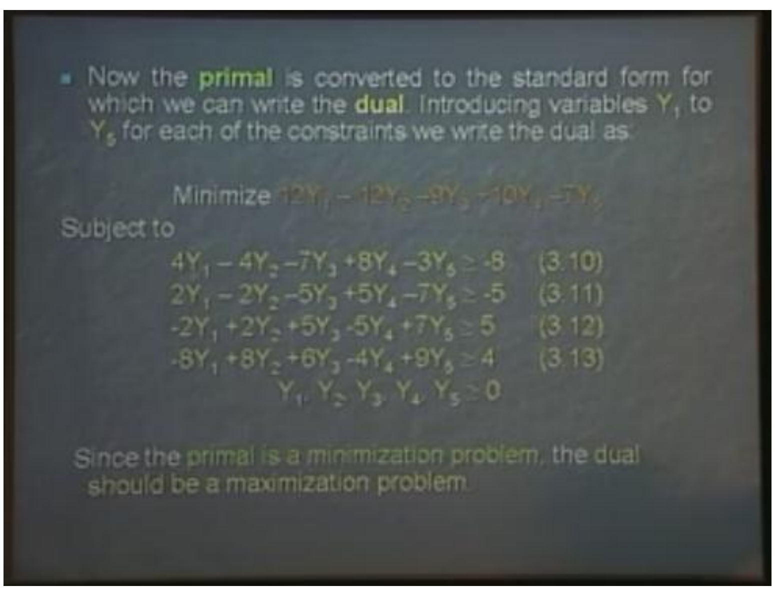

We now get the dual which is minimized $12Y_1 – 12Y_2 – 9Y_3 + 10Y_4 – 7Y_5$ subject to these 4 dual constraints corresponding to the 4 primal variables and all these 5 are greater than or equal to 0. Now this is in the form that we know because we have written the dual for what has been transformed into a maximization problem with all less than or equal to constraints and all greater than or equal to variables. So it will become a minimization problem containing all greater than or equal to constraints, all less than or equal or all greater than or equal to variables. Now this dual has four constraints corresponding to the four variables in the transformed problem and five variables corresponding to the four constraints in the transformed problem. But the original problem have has only three variables so the dual should have only three constraints. The original problem has four constraints and the dual should have only four variables. So we will try to convert this problem once again such that we are able to get the other properties satisfied.

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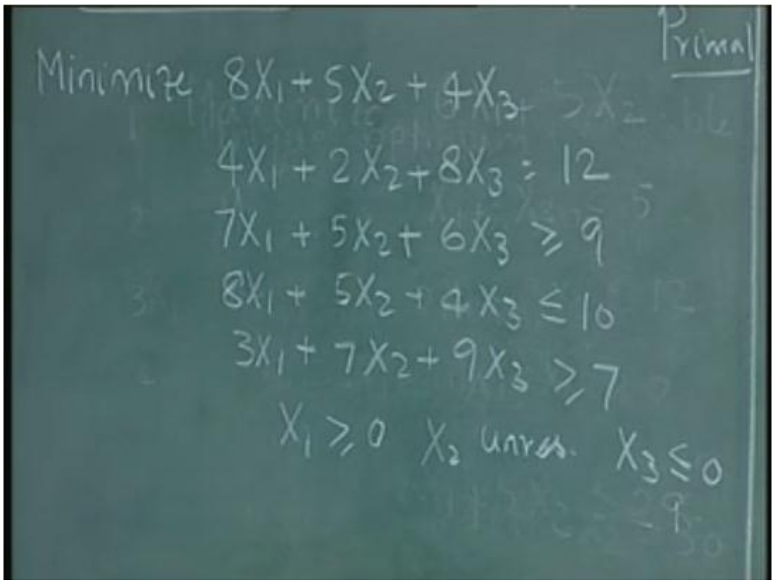

For example the dual to this problem should be of the form some $12Y_1 + 9Y_2 + 10Y_3 + 7Y_4$ where Y_1, Y_2, Y_3, Y_4 is suitably defined dual variables. The dual to this problem should have three constraints with right hand side values 8, 5 and 4 and the sign of the constraint could be anything. The dual for this problem should have a coefficient matrix which is a transpose of this matrix.

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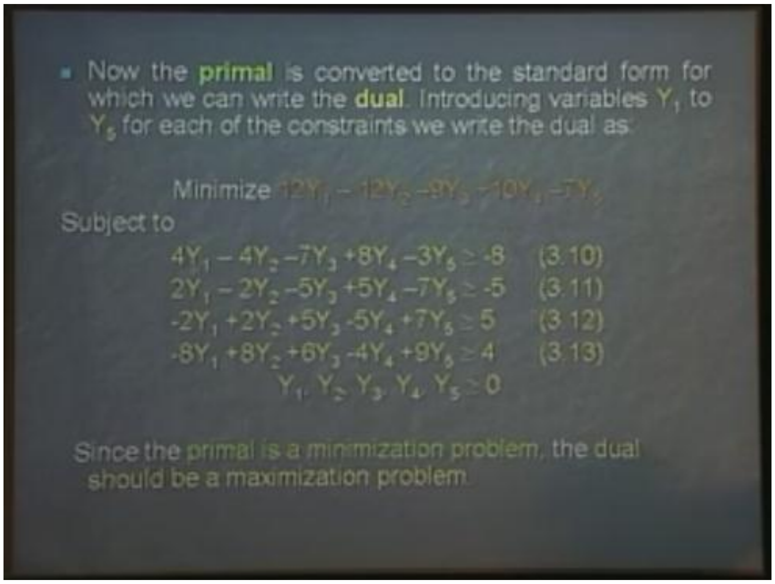

So we now bring this problem again into a form where those things are satisfied and how do we do that?

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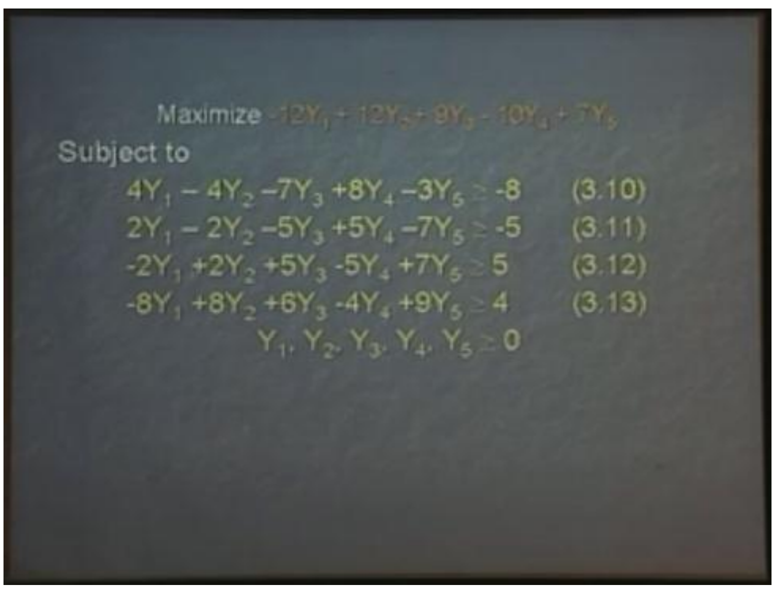

Now let us go back to the previous one. So this is the dual that we have written. We now need to bring this back into this form that we are comfortable with so let us go back to the next one.

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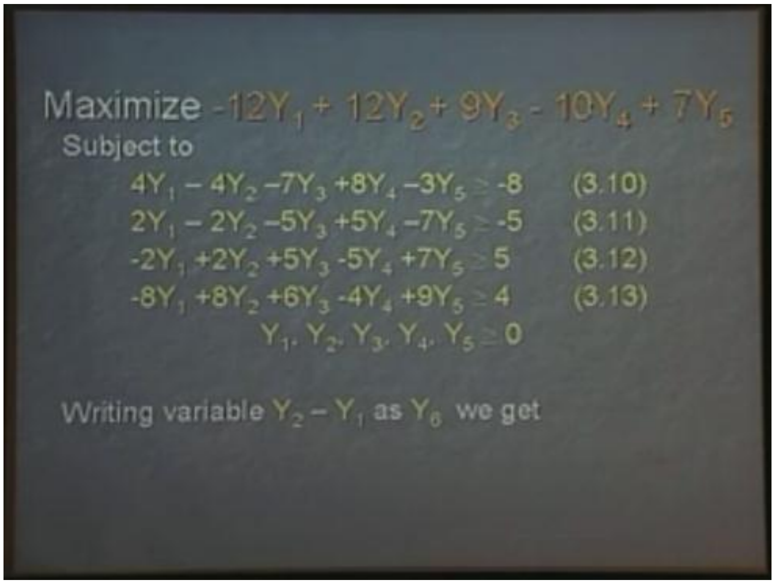

Now first thing we do is this now, look at all the four constraints. You now realize there is the $4Y_1 - 4Y_2$, $2Y_1 - 2Y_2$, $-2Y_1 + 2Y_2$, $-8Y_1 + 8Y_2$, $-12Y_1 + 12Y_2$ in the objective function. So we can now replace $Y_2 - Y_1$ as a new variable Y_6. We can do that and we reduce the number of variables now. Five variables will become four.

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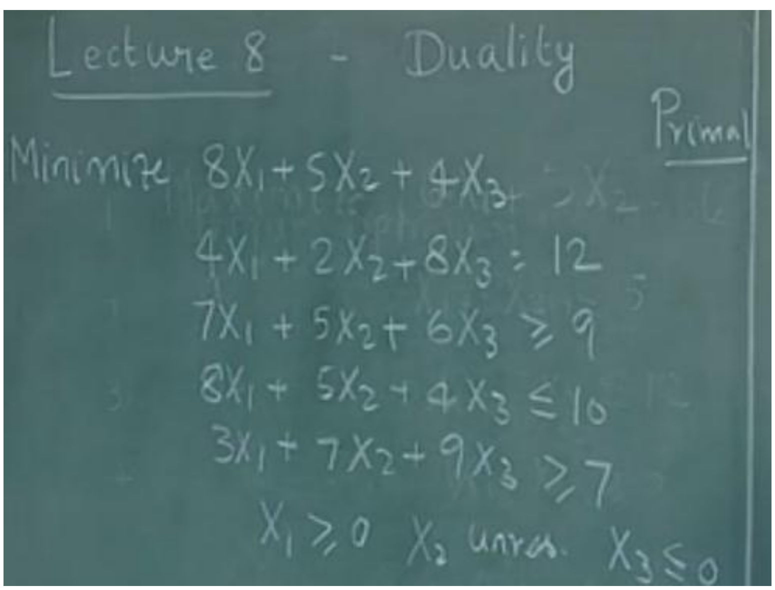

Remember again this problem has four constraints.

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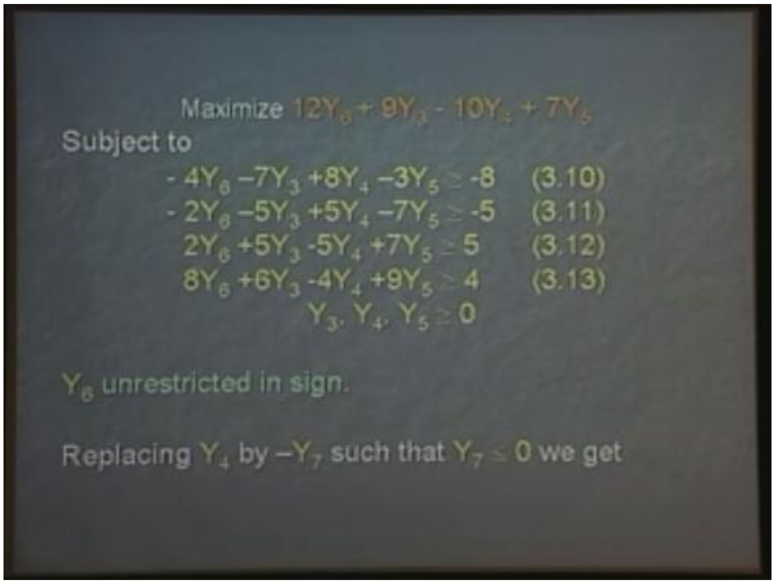

We would want the dual to have four variables so we do that and we get this we get $12Y_6 + 9Y_3 - 10Y_4 + 7Y_5$ subject to these and we have Y_3, Y_4, Y_5 greater than equal to 0. Y_6 is unrestricted in sign because Y_6 has been defined as a difference of two variables. So we get one unrestricted variable now one more thing that we can observe is there is this Y_4 which keeps coming here so, $-10Y_4$ in the objective function.

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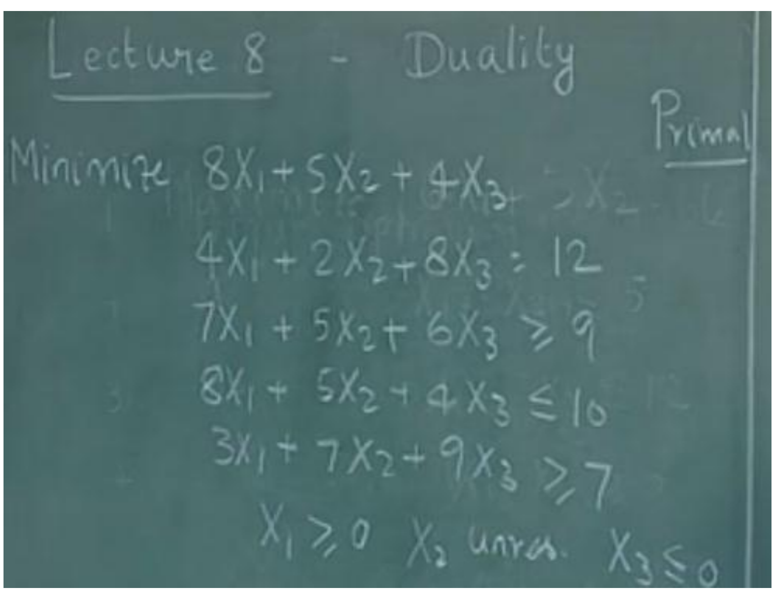

We want the final objective function to be of the form 12 into Y = $12Y_1 + 9Y_2 + 10Y_3 + 7Y_4$ where Y_1, Y_2, Y_3, and Y_4 are suitably defined variables.

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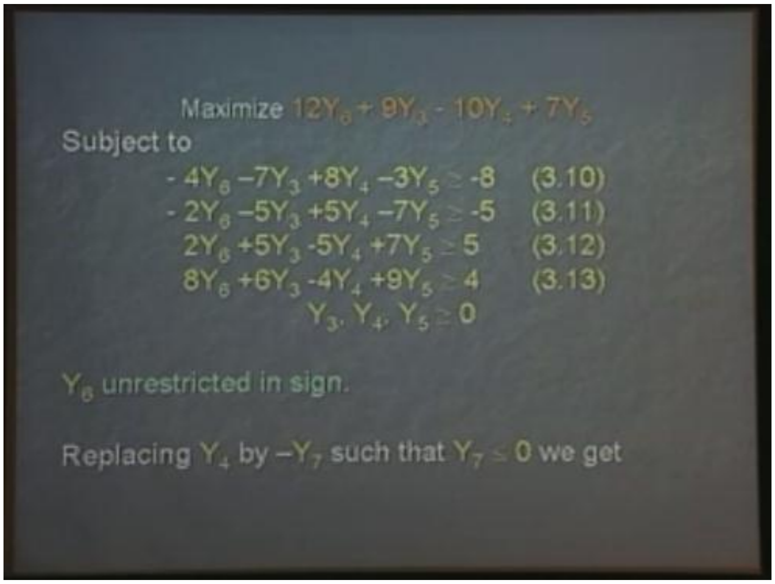

This $10Y_4$ alone has a minus sign so we replace Y_4 with a $-Y_7$ and write Y_7 less than or equal to 0. So this will become $+ 10Y_7$ in the objective function and the coefficients there will change.

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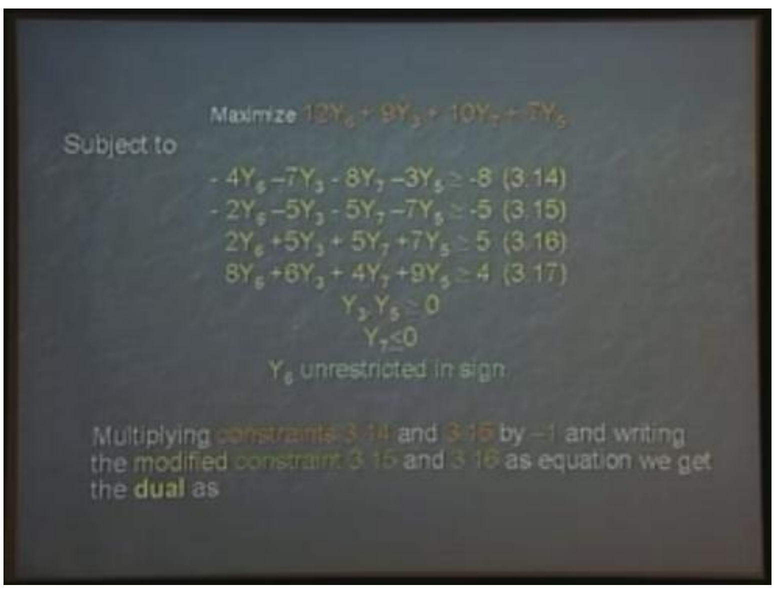

Now you realize that we have made the objective function in tune with what we want.

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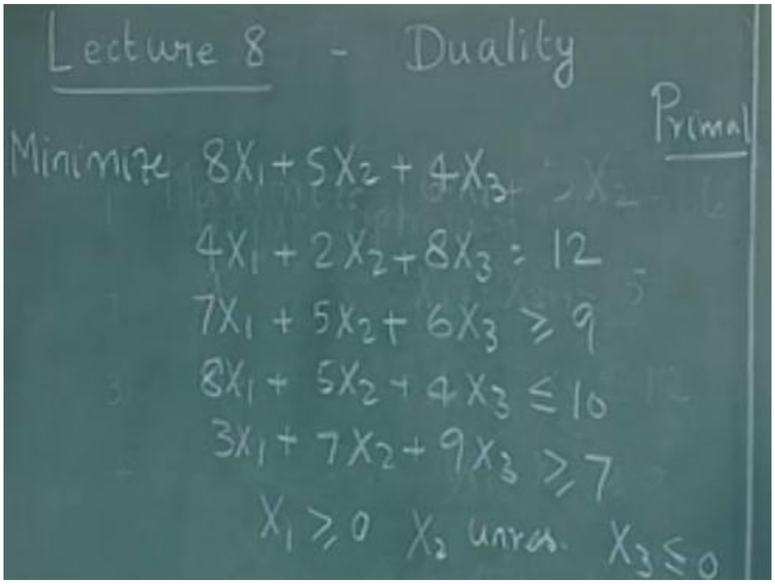

We have got this (12 9 10 and 7) here

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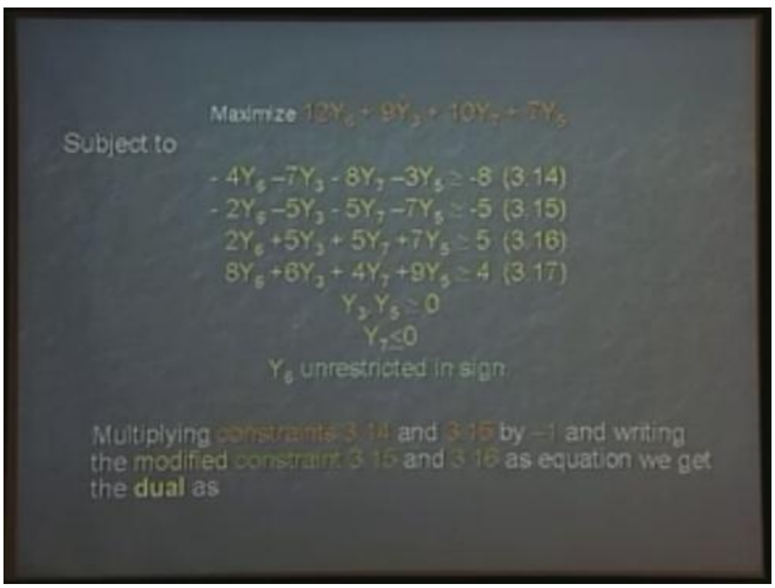

We have made those changes also in the Y_4 now all these have been converted to Y_7. Now we have taken care of the variables. Now we realize that the five variables that we had, has now become 4. One of them has become unrestricted, other has become less than or equal to type. Two of them remain as greater than or equal to type. Now we go back and try to adjust the constraints. The original problem has three variables. Dual should have three constraints. Multiplying constraints 3.14 and 3.15 by – 1 and writing the modified constraint 3.15 and 3.16 if we look at it very carefully. If we multiply 3.15 with the – 1, we would get $2Y_6 + 5Y_3 + 5Y_7 +$

$7Y_5$ less than or equal to 5, the third constraint is $2Y_6 + 5Y_3 + 5Y_7 + 7Y_5$ greater than equal to 5. Together they will become an equation so we will reduce one constraint by 1.

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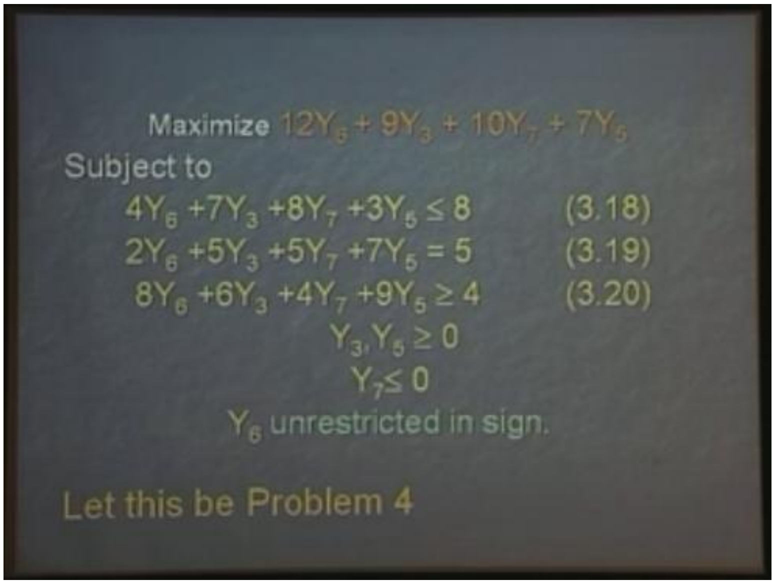

Then you go back to the other one to the previous slide.

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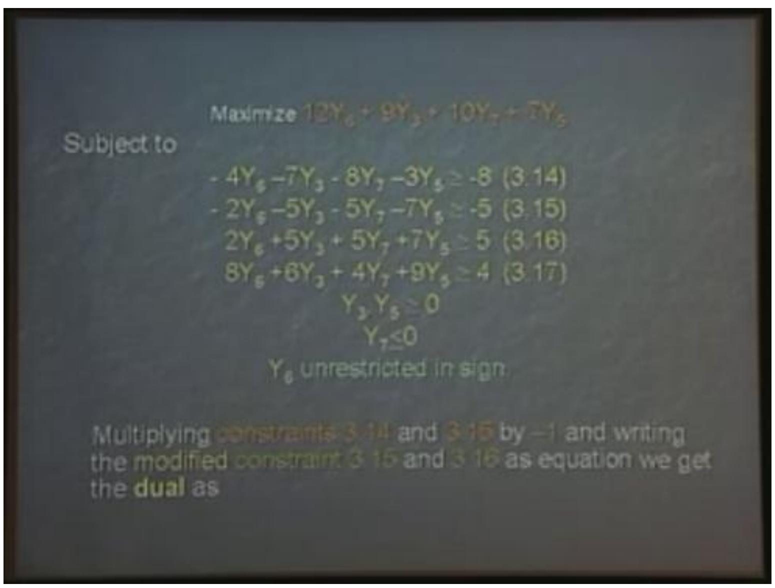

Look at the first constraint. The first constraint has all the coefficients negative including for right hand side so we can multiply the first constraint with a – one convert the inequality to a less than or equal to type and make the right hand side + 8

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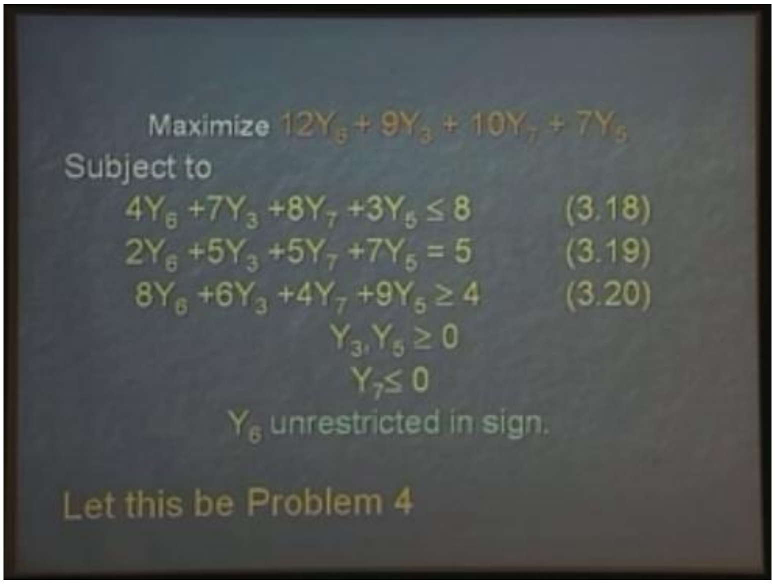

If we do that we get this we get now $12Y_6 + 9Y_3 + 10Y_7 + 7Y_5$ subject to these and we can since Y_6, Y_3, Y_7, Y_5 are very generic variable names you can rewrite them as Y_1, Y_2, Y_3, and Y_4 and replace this. Now if we keep this and compare with this problem let us just write only the dual here.

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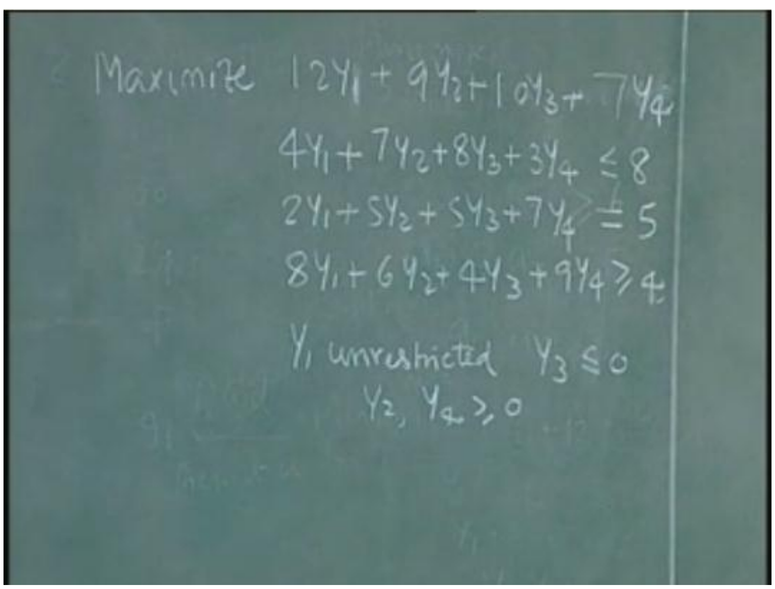

Maximize $12Y_6$ in fact let me use the term Y_1, Y_2, Y_3, and Y_4 now which means variable Y_6 has become variable Y_1 here + $9Y_2$ + $10Y_3$ + 7 Y_4 subject to $4Y_1 + 7Y_2 + 8Y_3 + 3Y_4$ less than or equal to 8; $2Y_1 + 5Y_2 + 5Y_3 + 7Y_4 = 5$; $8Y_1 + 6Y_2 + 4Y_3 + 9Y_4$ greater than or equal to 4; Y_1unrestricted.

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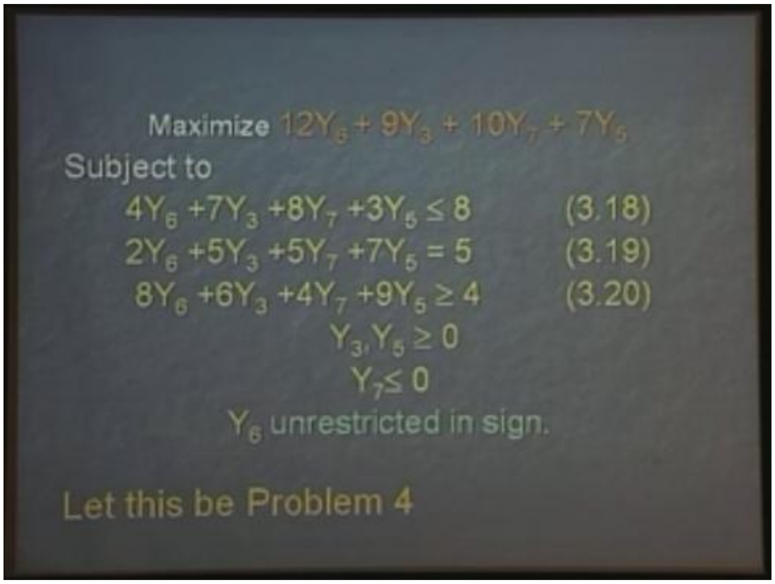

The Third variable so Y_3 less than or equal to $0Y_2$; Y_4 greater than or equal to 0 the third variable was Y_7 which is less than equal to 0.

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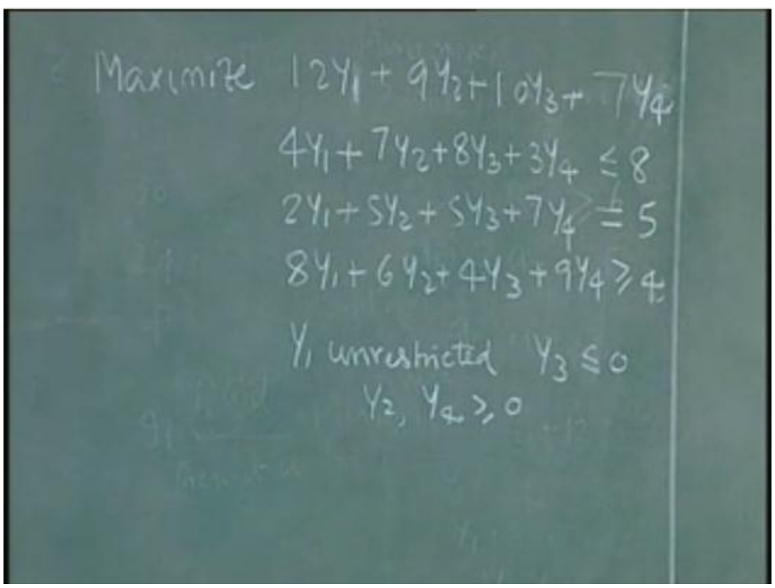

Third variable is Y_3 which is less than or equal to 0; so now this is the dual corresponding to this primal.

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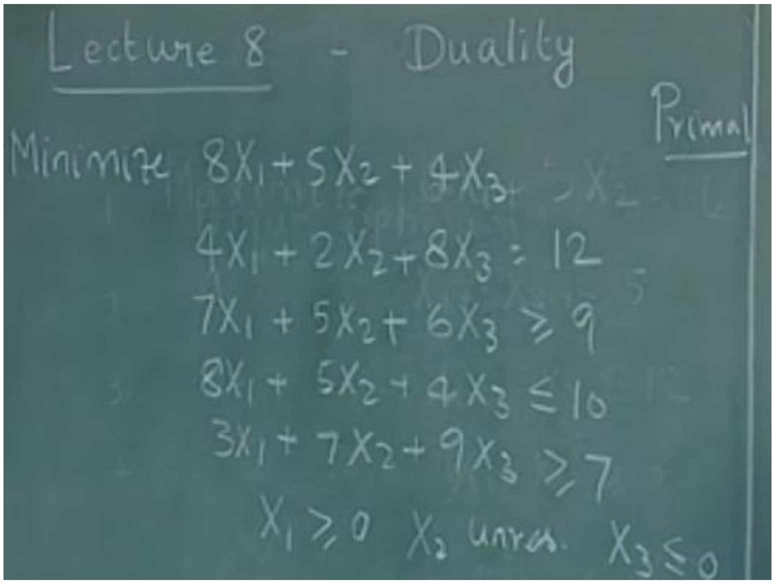

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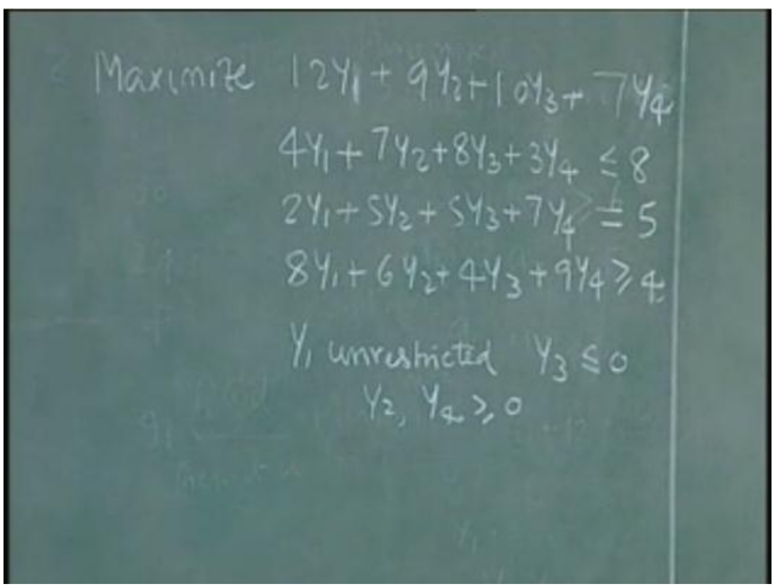

We know to write the dual to any given primal. We can convert any given primal into a standard form. The standard form that we know is the maximization problem with all less than or equal to constraints and all greater than or equal to variables. Now for that problem, we can write the dual which will be a minimization problem with all greater than or equal to constraints with all greater than or equal to variables number of constraints equal to the number of variables, number of variables in the dual equal is to number of constraints in the primal, coefficient matrix becomes transpose and so on and then we adjust the written dual in such a way that we bring it back having all the properties where the number of constraints is equal to the number of variables, number of variables is equal to the number of constraints, objective function becomes right hand side, right hand side becomes objective function, transpose and so on. We will then realize that depending only on the places we are not sure that what happens. This is an equation.

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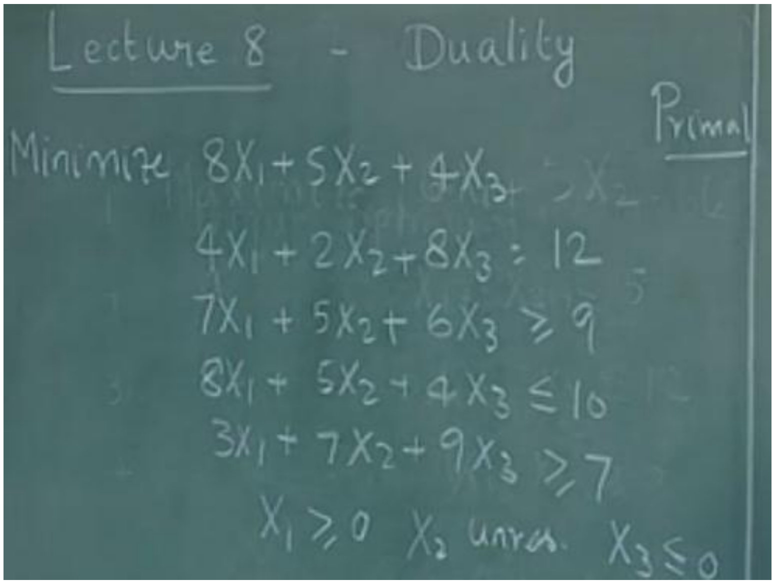

How will this equation get reflected in the dual? How are these four types equation greater than or equal to or less than equal to get reflected in the dual? Similarly how are these things getting reflected in the dual? We know the rest of the things. We know because we know all these three numbers are going to come here.

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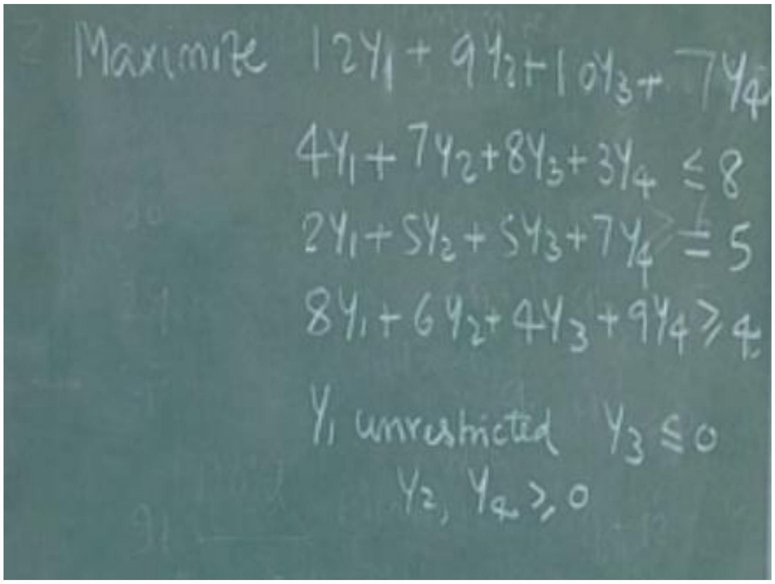

So one is to go back converted into form that we know write the dual and convert the dual back into the form where these properties are satisfied, the other is also to try understanding what exactly happened here. Let us look at this. The first constraint is an equation.

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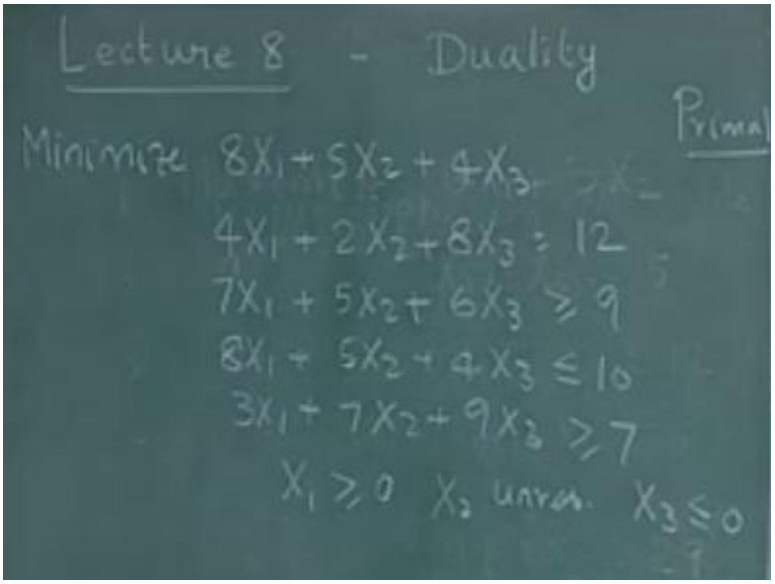

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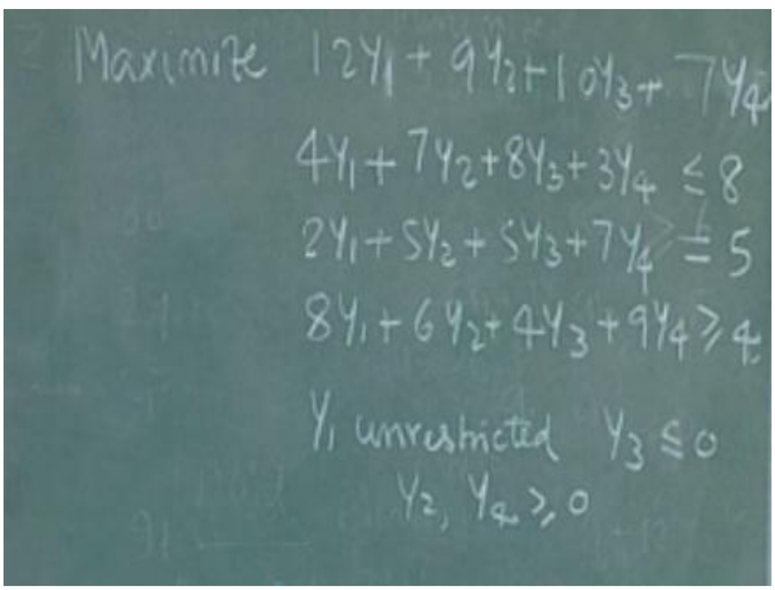

The first variable is an unrestricted variable. The second variable is an unrestricted variable

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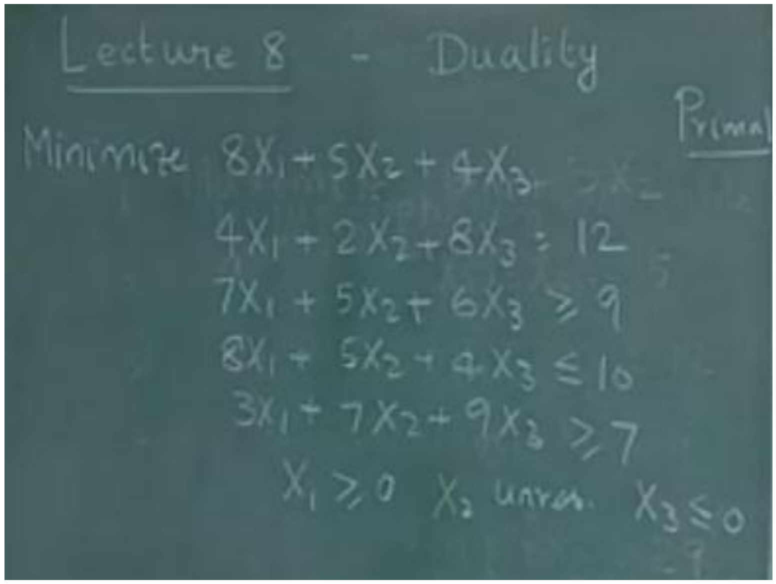

The second constraint is an equation.

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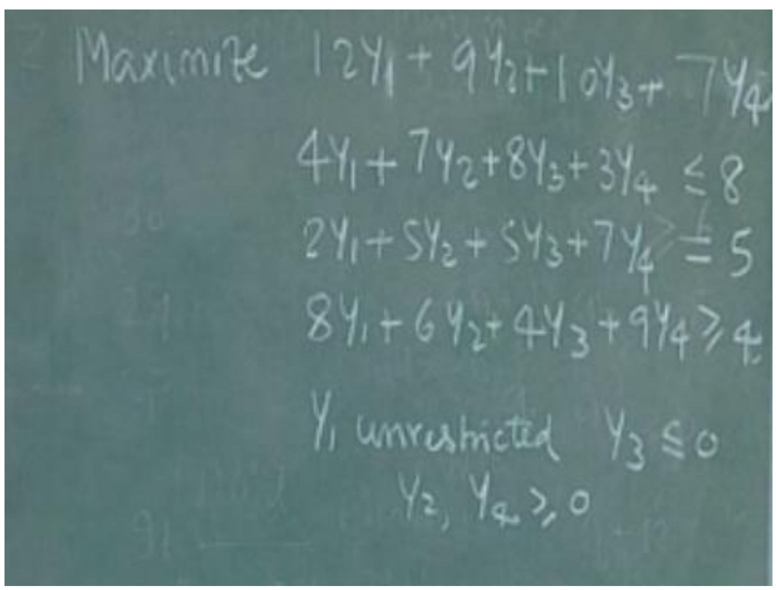

So there is a relationship between the unrestricted variable and the equation and that happens because the unrestricted variable in the original case now gets split into two variables. These two variables become two constraints which later get converted into an equation. So there is a relation between the unrestricted variable and the equation.

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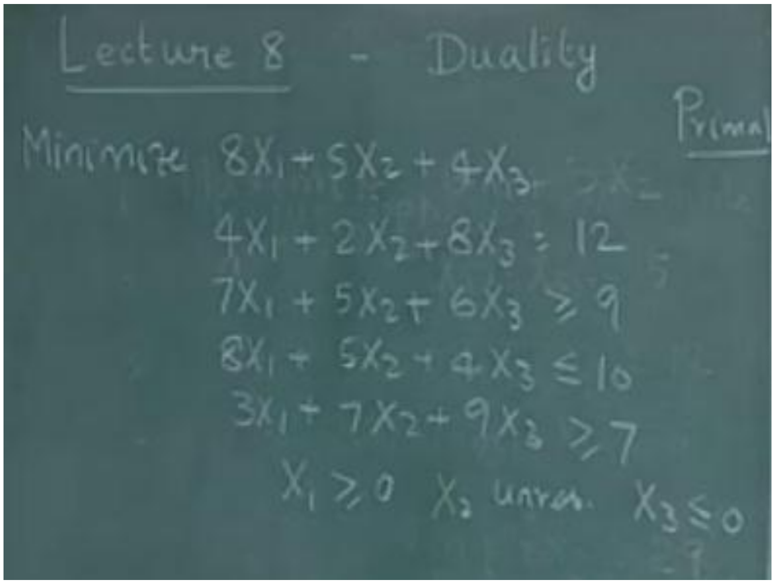

So if a primal constraint is an equation, then the dual variable will be unrestricted in sign that is an easier thing to do. The rest of the things are not so easy to understand here.
Now for example is a minimization problem. This constraint is a greater than or equal to constraint. The second constraint is a greater than or equal to constraint. There are many ways of understanding this relationship and let me propose one way which I often use to do that. In all linear programming problems, if the problem has some maximization objective then it is desirable to have a constraint of the less than or equal to type.

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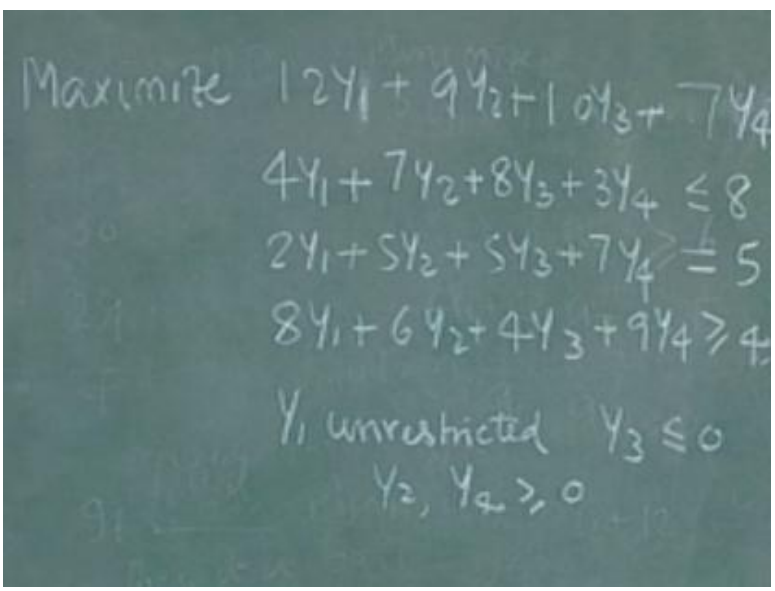

If the problem has the minimization objective it is always desirable to have a greater than or equal to type constraint. In all linear programming problems a greater than or equal to type variable is always desirable, less than or equal to type variable is undesirable.

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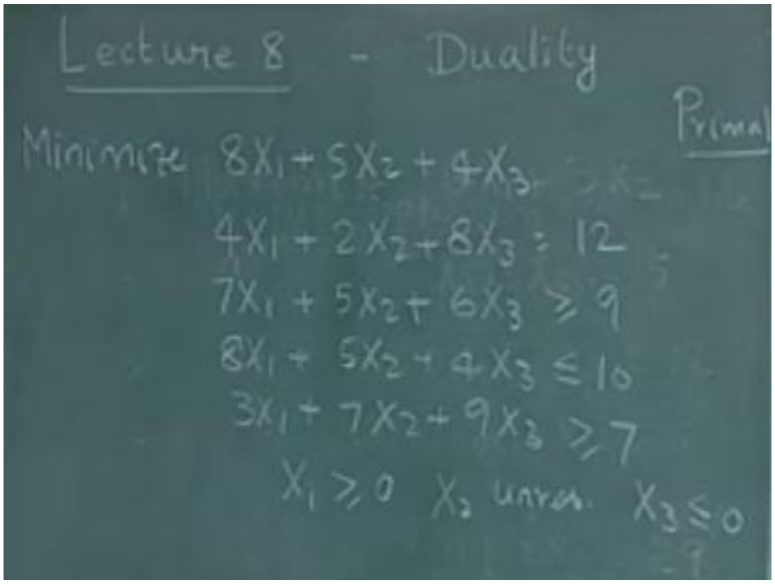

If we follow this guideline and go back and understand, it is a minimization problem.

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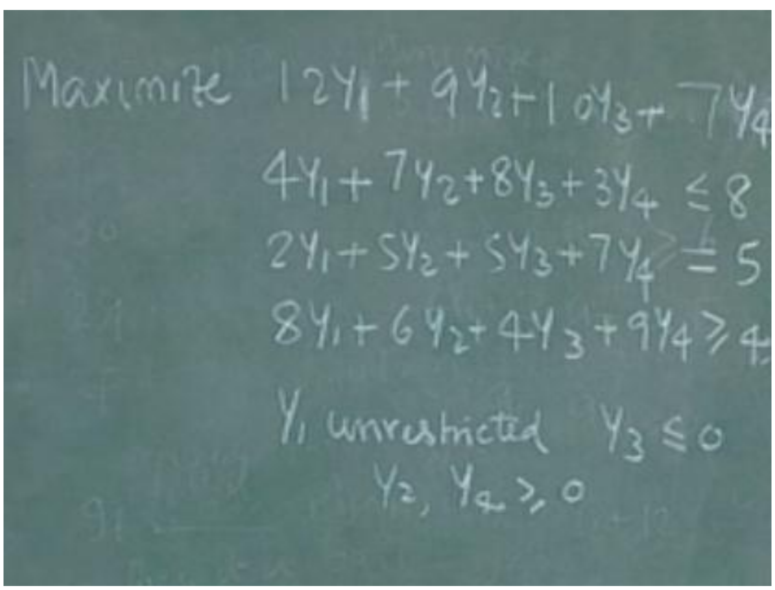

So this is a desirable kind of a constraint so the corresponding variable will be a desirable kind of a variable greater than or equal to.

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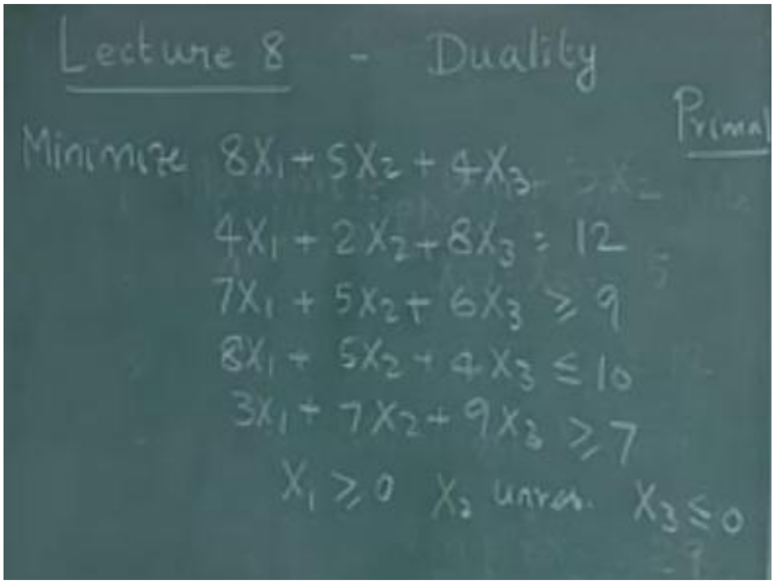

For a minimization problem this is an undesirable kind of a constraint. This is not the constraint you would like to see in a minimization problem normally. I would call it an undesirable constraint so the third constraint deserves the undesirable type.

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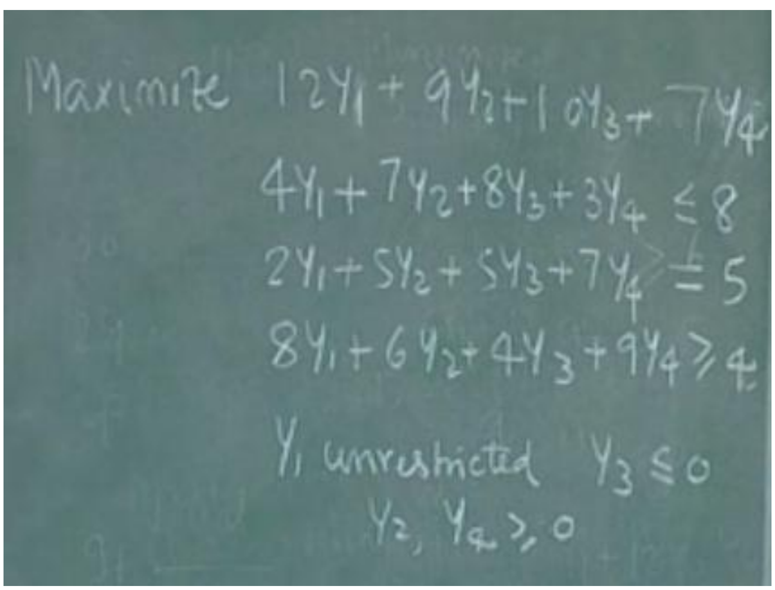

The third variable Y_3 will be an undesirable variable which is a less than or equal type. There is more than one way to understand this relationship but over a period of time I have found it convenient to interpret it this way.

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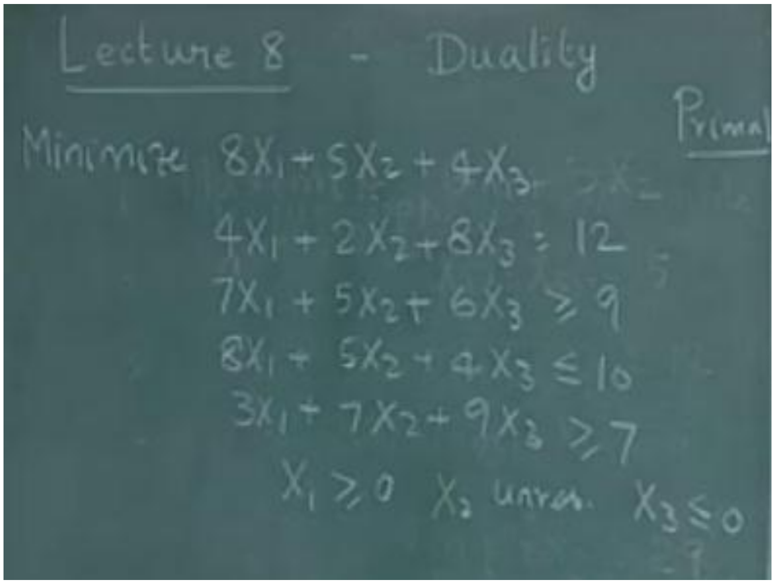

This is a desirable kind of a constraint. The 4th variable Y_4 is a desirable kind of a variable.

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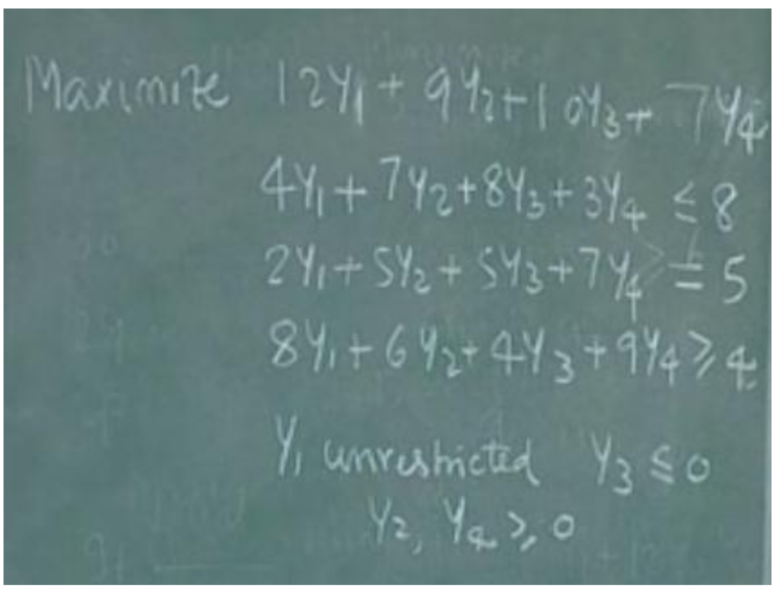

Let us go back to the variables. This is a desirable kind of a variable.

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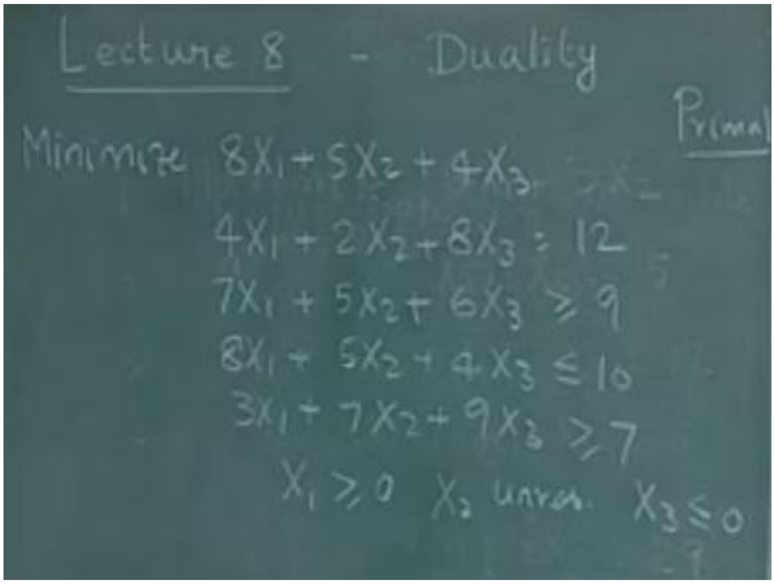

So the first variable is a desirable kind of variable. Now go back to the first constraint.

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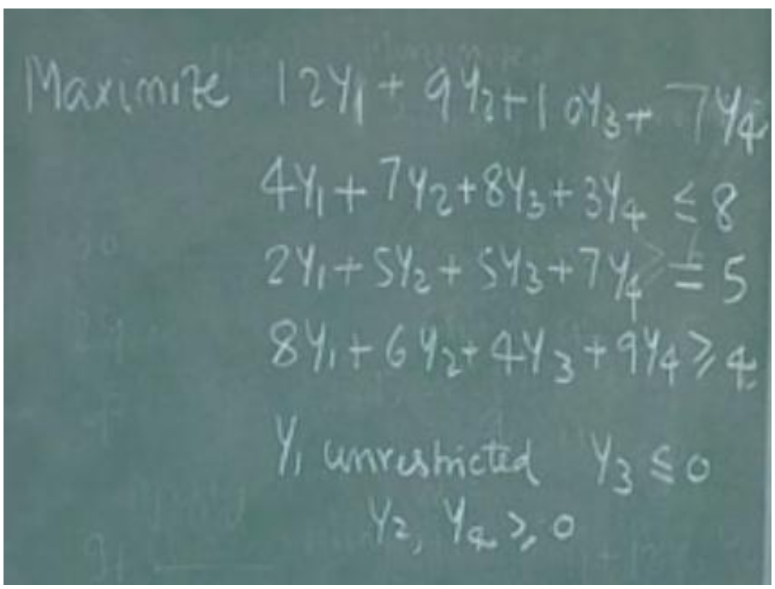

For a maximization problem you will get a less than or equal to which is a desirable type. This is an undesirable kind of a variable so for a maximization problem the third constraint will be the undesirable type.

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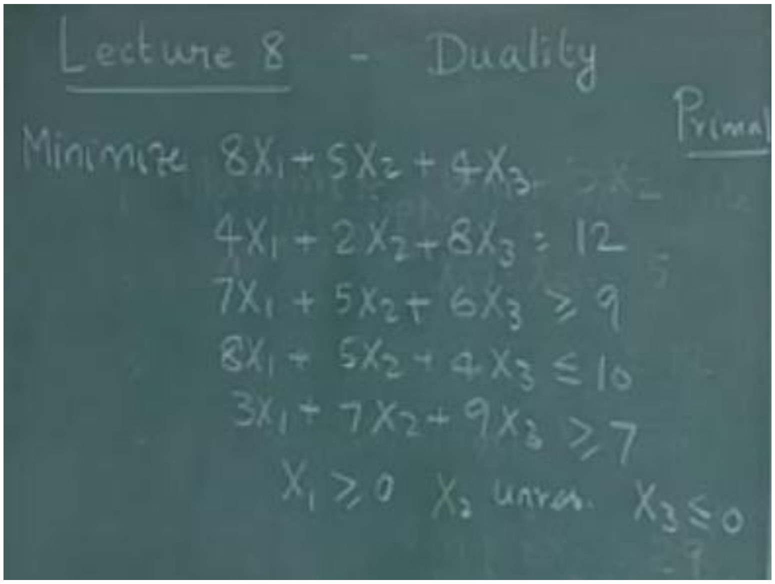

So over a period of time you can use this.

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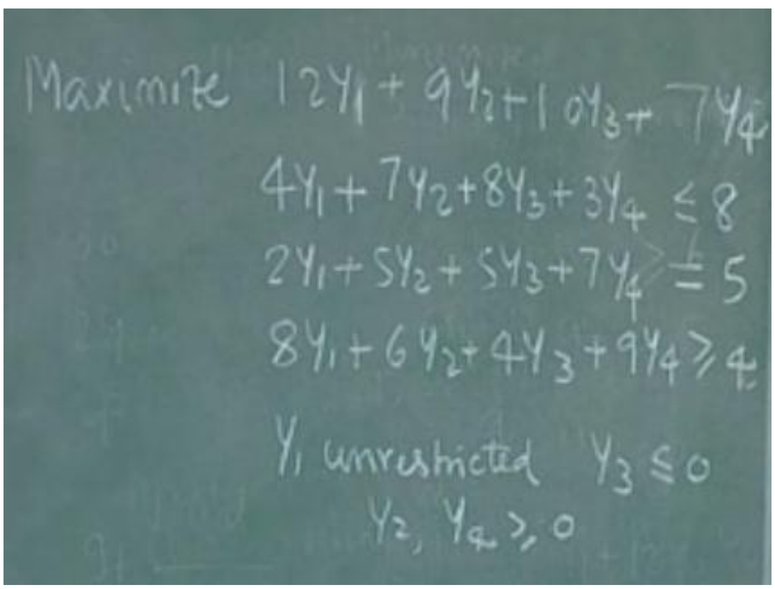

Or you can remember it in a different way. If a minimization problem has a greater than or equal to, then corresponding maximization problem will have a greater than or equal to and so on. There are different ways of remembering this. To write the dual, one could convert, can take all the labor and time to convert this into the standard form that we know and then write the dual to the standard problem and then convert back into the form that we are comfortable with or we could go back and then remember these things as we write. We create a table which would help us understand the primal dual relationships and the table is like this

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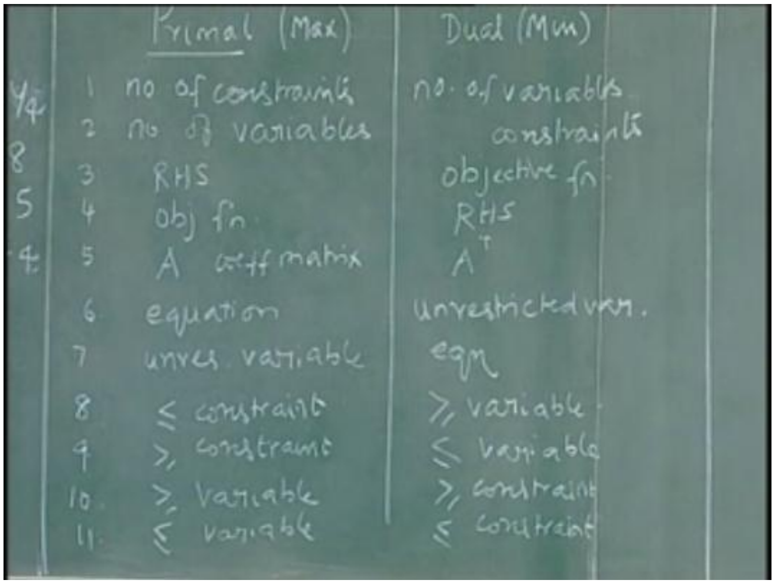

So we now call the primal as a maximization problem and the dual as minimization problem. Number of constraints is equal to number of variables, number of variables become number of constraints. Right hand side becomes objective function; Objective function becomes right hand side. Coefficient matrix A becomes a transpose. Now the next easier things are that equation becomes unrestricted variable. The unrestricted variable becomes an equation and then enters the maximization problem. The desired type of constraint is less than or equal to constraint. It will become a greater than or equal to variable because greater than or equal to is always a desired variable no matter what the objective function is.

Maximization problem greater than or equal to constraint is not the desirable one so this will convert itself into a less than or equal to variable. Greater than or equal to variable is always the desirable one so for a minimization problem, this will become greater than or equal to constraint and 11 less than or equal to variable will become less than or equal to constraint. One can remember this table also. One can do another thing. Suppose this was the primal, we had called this as $12X_1 + 9X_2 + 10X_3 + 7X_4$ and so on.
If this were the primal then what will be the dual? This will automatically become the dual.

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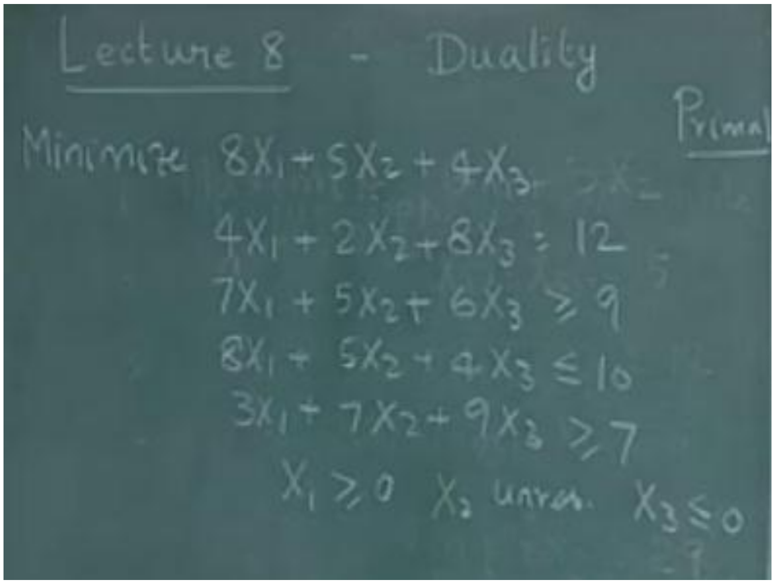

In fact whatever is applicable from this to this, is also applicable from this to this. Now anything can be the primal. The other will be the dual. For example, this is the primal.

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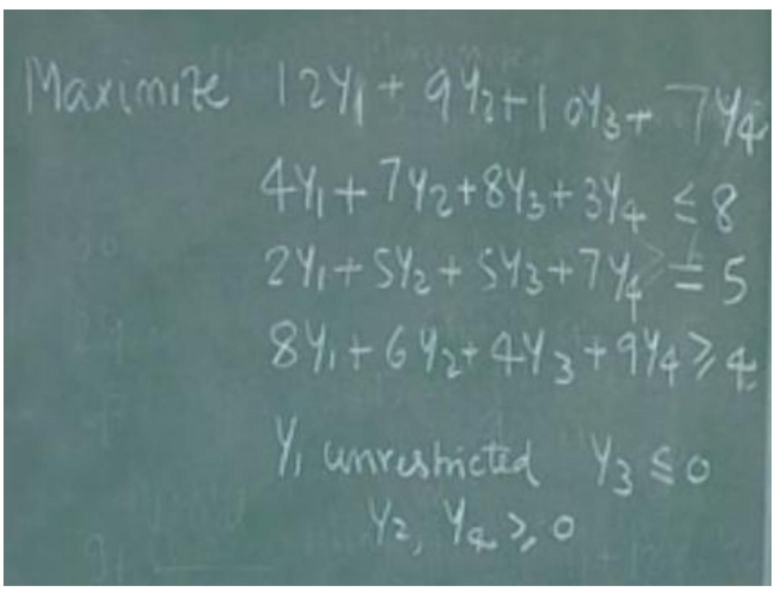

Primal maximization and dual minimization

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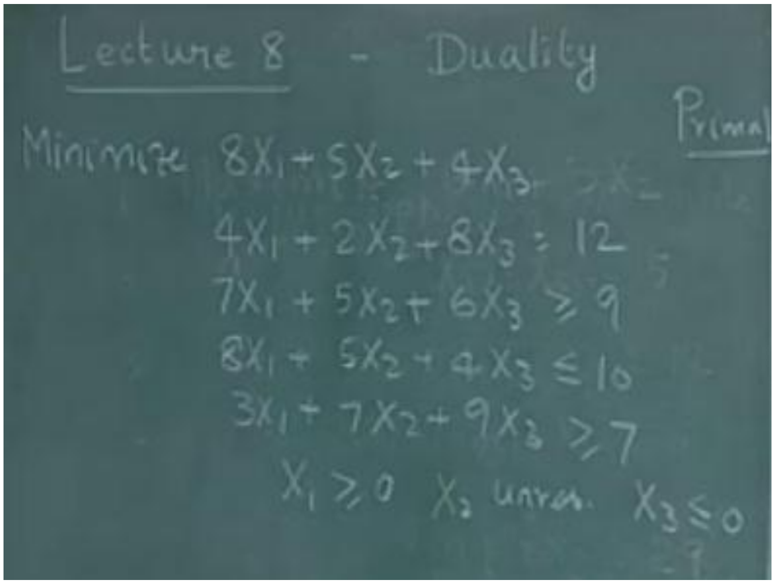

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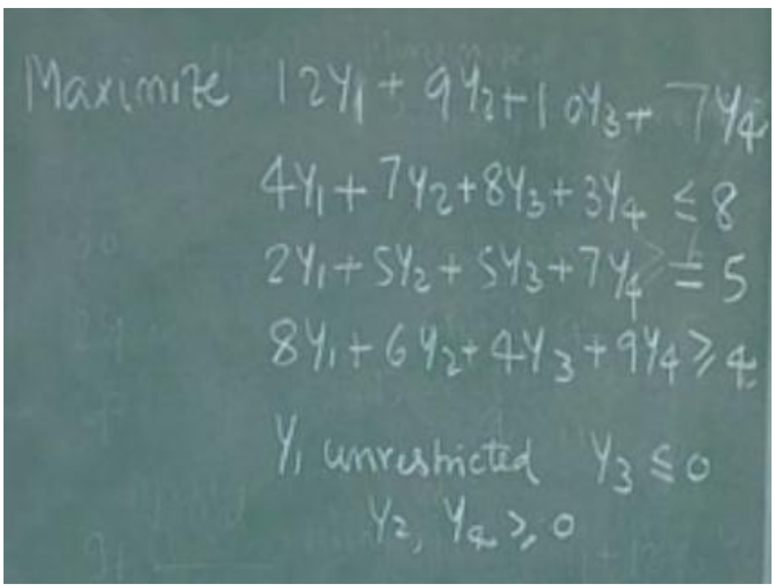

First constraint is less than or equal to maximization.

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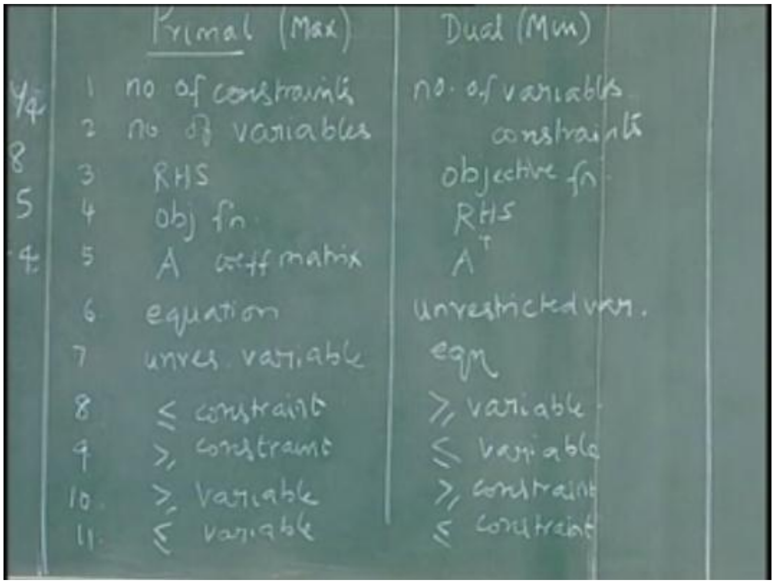

First constraint less than or equal to, first variable of the dual will be greater than or equal to, if it is greater than or equal to here, so like this you can interpret keeping this as the primal and this as the dual.

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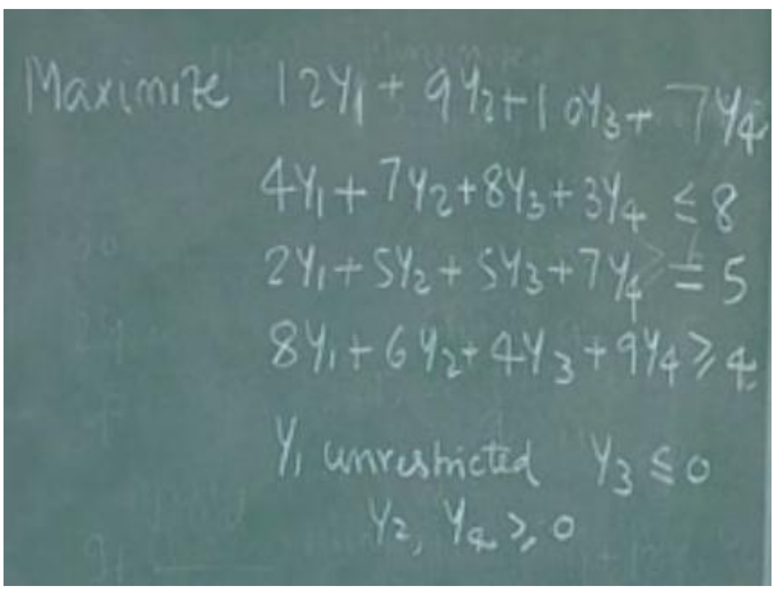

We can go back and shift the same thing. If this is the primal the only thing we need to the now is primal becomes minimization problem. This is primal.

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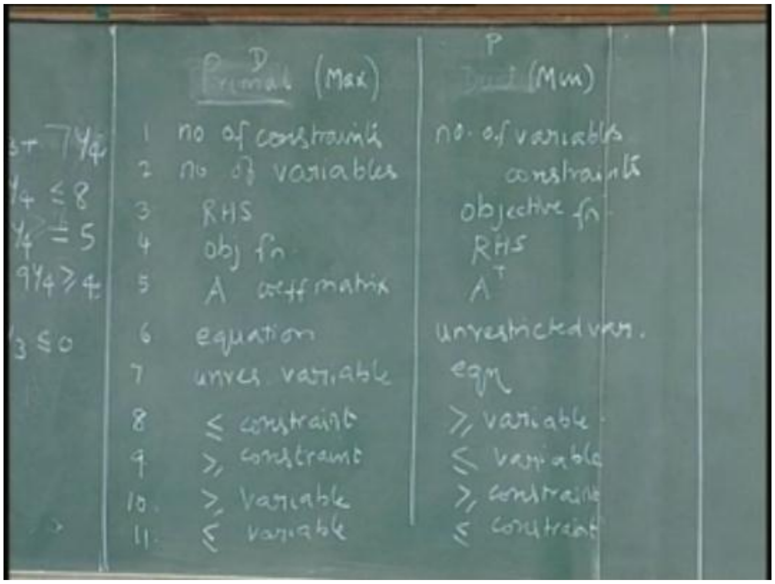

This is dual and now you can go back.

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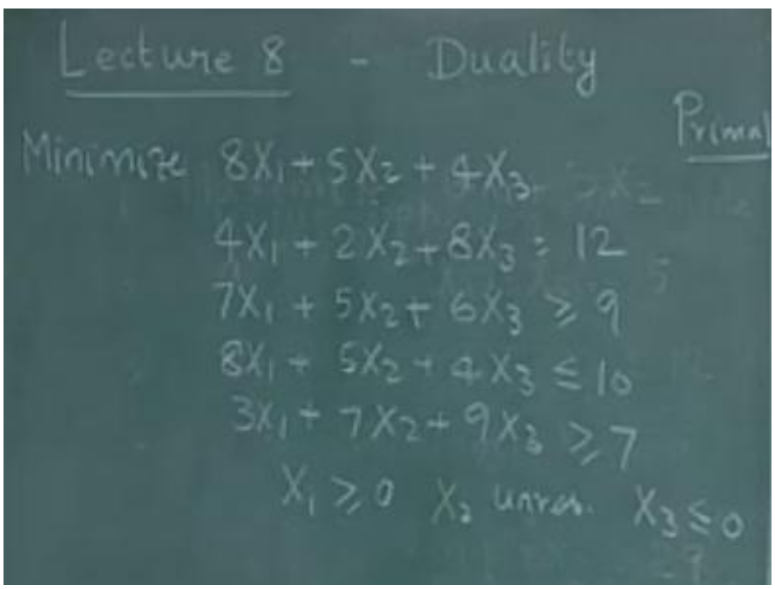

If this is the primal minimization, second constraint is greater than or equal to, so second constraint is the correct type of constraint the dual is maximization

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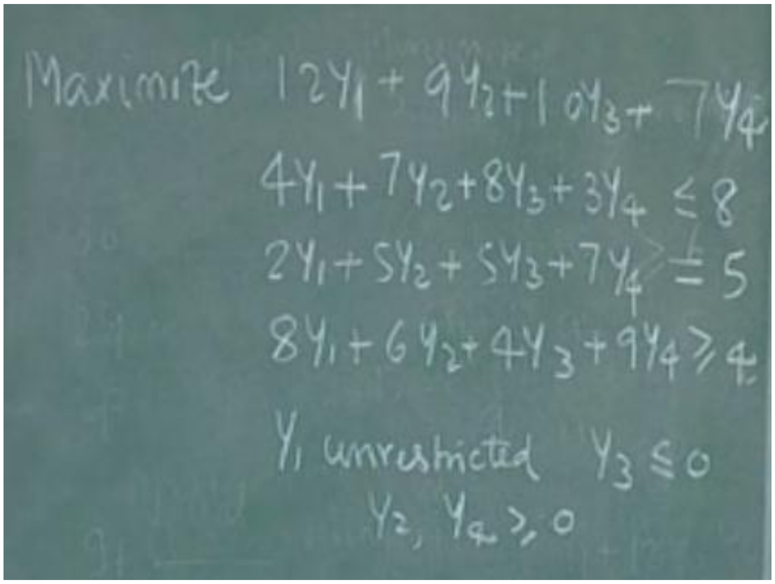

So for a minimization problem a greater than or equal to constraint will give greater than or equal to variable. The second constraint is this type. The second variable is this type.

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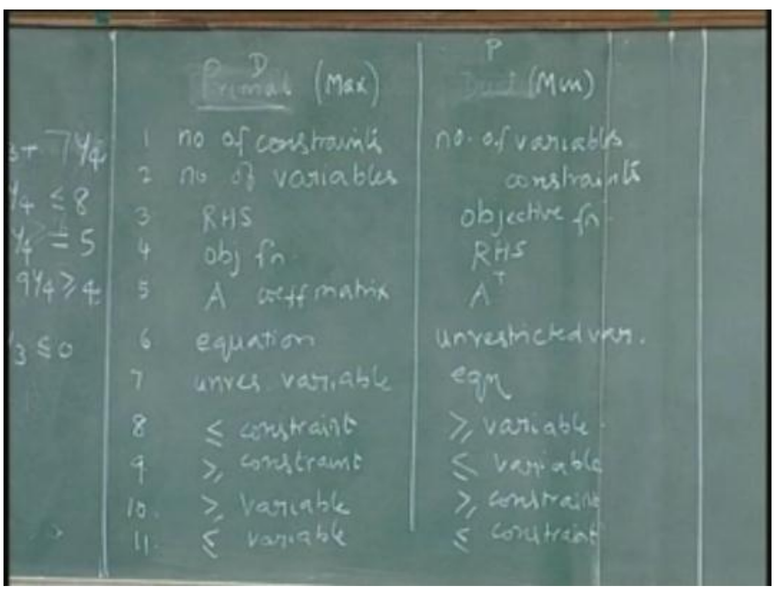

So one need not always restrict oneself to saying this will be the primal in that it will be the dual and so on. Any one of the problem can be the primal the other problem will automatically become the dual. So this table would help us write the dual for any given primal. So in summary what we have seen in this lecture is we have introduced the problem of the dual and we have also tried to explain that the dual is inherent. It comes out to primal naturally and it is inherent or hidden in the primal. We have also learnt how to write the dual for any given primal linear programming problem. In the next lecture we will see more of the primal dual relationships.