Fundamentals of Operation Research

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Lecture No. # 5

Simplex Algorithm Minimization problems
Big M and Two Phase Methods

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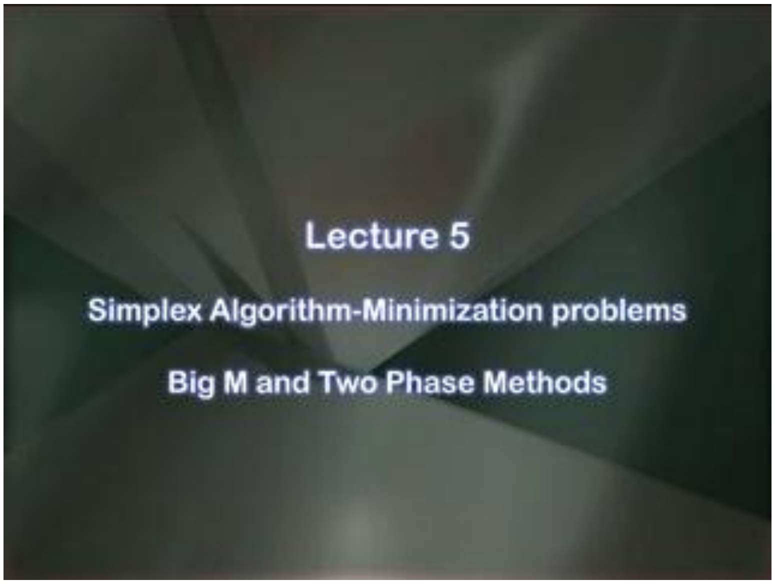

In the last lecture we looked at the Simplex algorithm and solved two examples both were maximization problems. In today's lecture we continue our discussion on the Simplex algorithm by looking at the various stages that we go through in Simplex.

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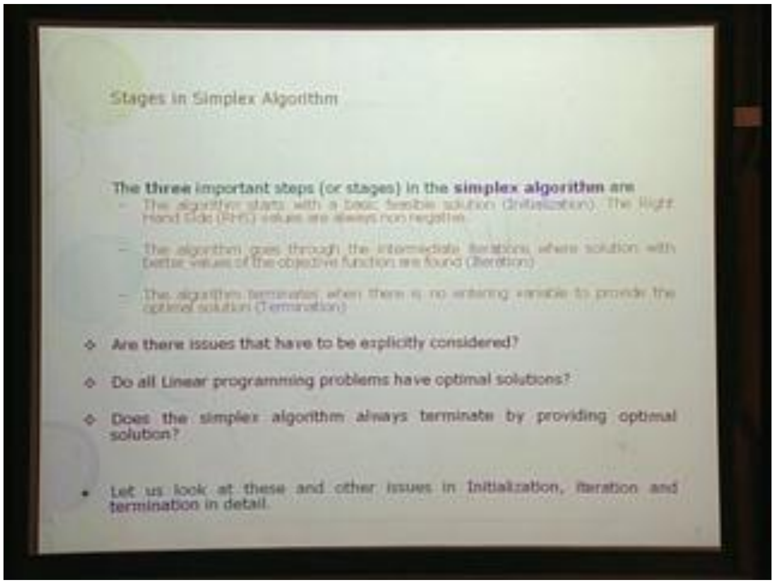

Example, the three important stages in the Simplex algorithm they are called Initialization, Iteration and Termination. Now under initialization the algorithm starts with a basic feasible solution. In the right hand side values are always non negative. So when we start solving a problem using the Simplex algorithm, the first thing we need to do is to have the right hand side values non negative and then as we construct the first Simplex table we have to make sure that the first Simplex table has a basic feasible solution. The algorithm goes through some intermediate iteration which is called Iteration and then the algorithm terminates when there is no entering variable and it provides optimal solution which is called Termination aspect of the algorithm.

Now what are the other things that we need to look at? There are other issues that have to be explicitly considered and the question arise, have you addressed all this issues in the two examples that we have seen? We have not addressed all of them and we will be addressing each one of it in detail.
Do all linear programming problems have optimal solutions? Can we have a linear programming problem that does not have an optimal solution at all?

So we will be looking at that. Does the Simplex algorithm always terminate by providing the optimal solution? Can it terminate otherwise? We will have to look at those aspects.
Now let us look at all these 3 aspects, Initialization, Iteration and Termination through different examples in detail.

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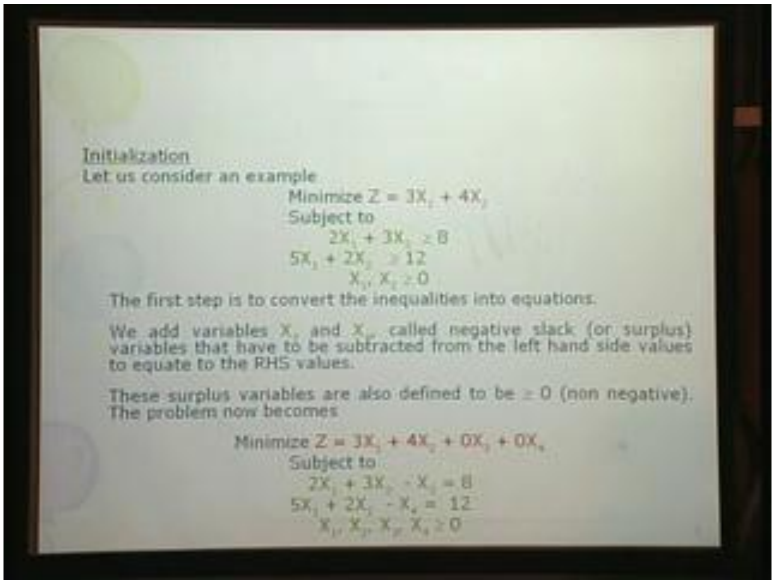

Now let us consider a different type of a problem. We now consider a problem with a minimization objective.

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So the problem is to minimize $Z = 3X_1 + 4X_2$ subject to $2X_1 + 3X_2$ greater than or equal to 8 and $5X_1 + 2X_2$ greater than or equal to 12 and X_1, X_2 greater than or equal to 0. Now this is also a linear programming problem but is different from the two examples that we have considered. Both the examples had a maximization objective. More importantly both the examples that we considered had constraints that were of the less than or equal to type. Now we are looking at a problem which has constraints of the greater than or equal to type. Now as we did in the previous example, we will convert these inequalities to equations. So this will become $2X_1 + 3X_2 – X_3 = 8$ and $5X_1 + 2X_2 – X_4 = 12$. X_1, X_2, X_3, and X_4 greater than or equal to 0. Now let us explain this now. $2X_1 + 3X_2$ can be exactly equal to 8 or greater than 8. So we now add a variable X_3 which has a negative sign so when $2X_1 + 3X_2$ becomes greater than 8 then X_3 will take a positive value and then this inequality will become an equation. Similarly X_4 is introduced with a negative sign to take care of the fact that when $5X_1 + 2X_2$ exceed 12, X_4 takes a strictly positive value. X_4 will not take a negative value because $5X_1 + 2X_2$ cannot be less than 1. Now we have introduced two more variables $X_3$and X_4 into the problem. Now these are also slack variables like those introduced in the earlier two examples. The only difference here is these are called negative slack variables or surplus variables.

The fact that we have introduced negative slack variables to convert a greater than or equal to inequality into an equation also makes these negative slack variables greater than or equal to 0 in addition to the explicit condition where the decision variables should be a non negative.

So we stick to the fact that all variables in the Simplex algorithm will have to be strictly non negative whether they are decision variables X_1, X_2 or slack variables X_3, X_4 negative slacks in this case. Now we have converted these two inequalities to equations. We have to find out what is the contribution of these slack variables to the objective function. As discussed in the earlier example these slack variables do not contribute anything to the objective function. So the problem now becomes minimize $Z = 3X_1 + 4X_2 + 0X_3 + 0X_4$ subject to these two and the condition that X_1, X_2, X_3, and X_4 are greater than or equal to 0. Now let us try to set off the first Simplex table.

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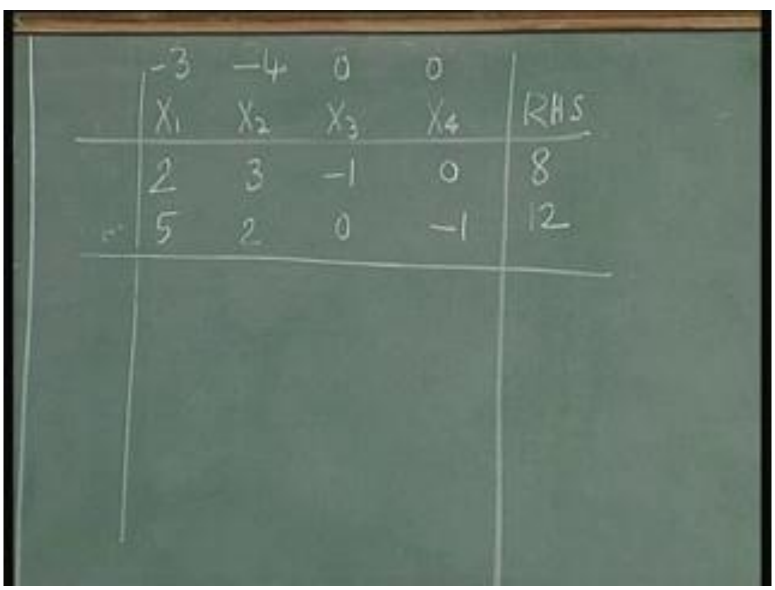

Now what we do here is, we set up the table with X_1, X_2, X_3, and X_4 and we write this as minimization. So we write $2X_1 + 3X_2 – X_3 + X_4 = 8$; $5X_1 + 2X_2$, 0, – 1, 12. The objective function coefficient is a minimization problem. Since we are used to solving a maximization problem we convert this minimization objective to a maximization objective and the problem

would become maximize – $3X_1$ – $4X_2$ + $0X_3$ or – $0X_3$ – $0X_4$. So we would get here (– 3, – 4, 0 and 0).
Normally we would have liked to start with X_3 and X_4 as basic variables as we did in the two previous examples. Now let us see whether we can actually do that. If we fix X_1, X_2 as non basic and X_3, X_4 as basic as we did in the earlier two examples, then we have a solution $X_1 = 0$; $X_2 = 0$; $X_3 = -8$; $X_4 = -12$ which is not a basic feasible solution.

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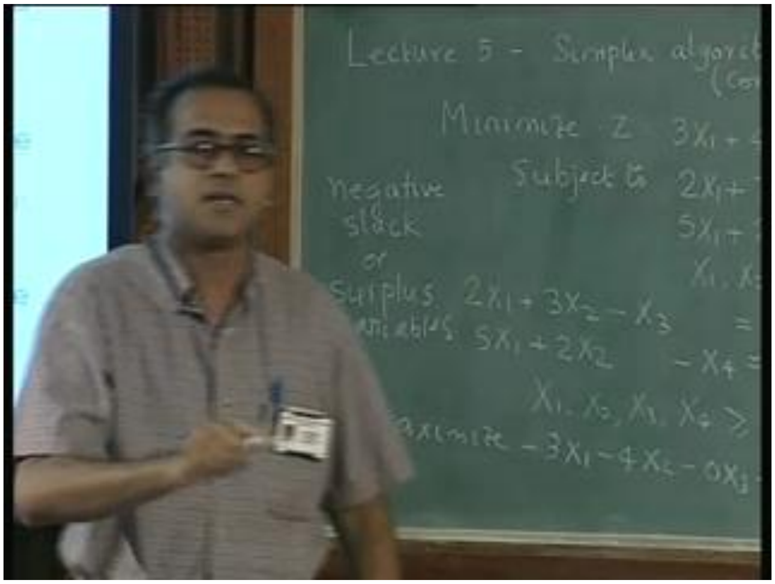

So in all, problems that involve greater than or equal to type, slack variables automatically do not qualify to be the initial basic variables because they end up giving an infeasible solution. So we cannot start the Simplex table with X_3 and X_4 as we did in the earlier case. So we should not be doing this. Now we need to do something else. What is required to be done is we have to somehow try and identify a basic feasible solution out of this. Once again this as a system has four variables and two equations. If we go back with algebraic idea then we can evaluate six basic solutions for this out of which one or more may be basic feasible. The only thing we have to do is we need to go back and individually evaluate all these six which we do not want to because we have already moved from the algebraic method to the Simplex, realizing that the Simplex method is better than the algebraic method simply because it has better features. So we do not want go back to the algebraic method and then try to get a basic feasible solution. So what we do now is something different.

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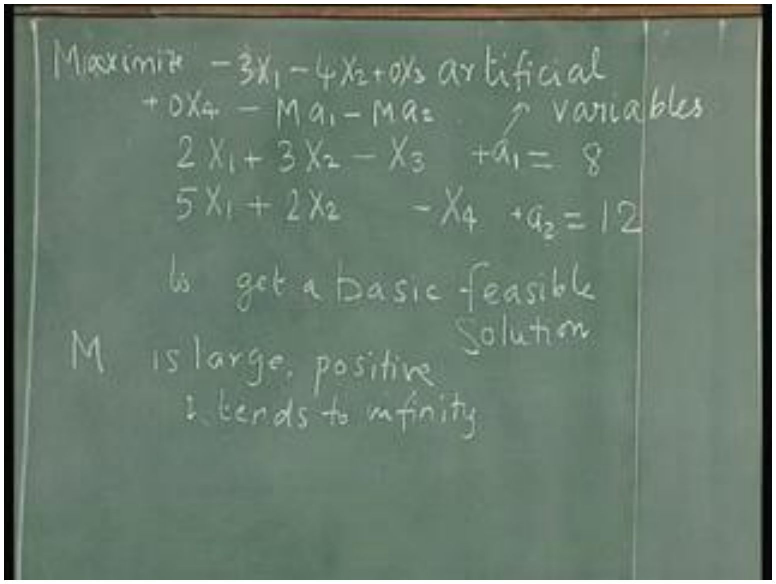

The two equations can be written again as, $2X_1 + 3X_2 – X_3 = 8$ and $5X_1 + 2X_2 – X_4 = 12$
So these are the first two equations. Now by merely looking at these equations we are not able get a basic feasible solution as we did in the two previous examples. So in order to get a first simple basic feasible solution we add two more variables to this set. We call them, for example $+ a_1 = 8$; $+ a_2 = 12$. If we simply add these two variables, a_1 and a_2 into these (Refer Slide Time: 11:28) equations and rewrite the equations in this form then we now we have six variables and two equations. So two of these will be basic and the remaining four will be non basic.

Now we can keep a_1 and a_2 basic and treat X_1, X_2, X_3, and X_4 as non basic and then start with a basic feasible solution with $a_1 = 8$ and $a_2 = 12$. So we add these two variables with a very specific objective of identifying a basic feasible solution to get a basic feasible solution. So these two variables that are added are called artificial variables. Most important thing about these artificial variables is that they are not a part of the problem. For example, X_1 could represent some amount or quantity of some product that is made. X_3 could represent the amount of a certain resource that is not utilized. So likewise, we will be able to have a clear definition of a_1 and a_2 or a_1 and a_2 do not have any physical significance with respect to the problem simply because they are not part of the problem.

We have been introduced to a very specific objective of getting a basic feasible solution that is why they are called artificial variables; because they do not belong to the problem and in order to clearly separate them and identify them we use a rotation a_1, a_2 to represent these as division variables. Now what are the other things that we need to do? Do these artificial variables have a contribution to the objective function number 1? The objective function is now minimized or maximized to $– 3X_1 – 4X_2 + 0X_3 + 0X_4$. We now need to find out whether these artificial variables have any contribution in the objective function. To do that let us go back and look at this problem again. Do we want to actually solve this problem?

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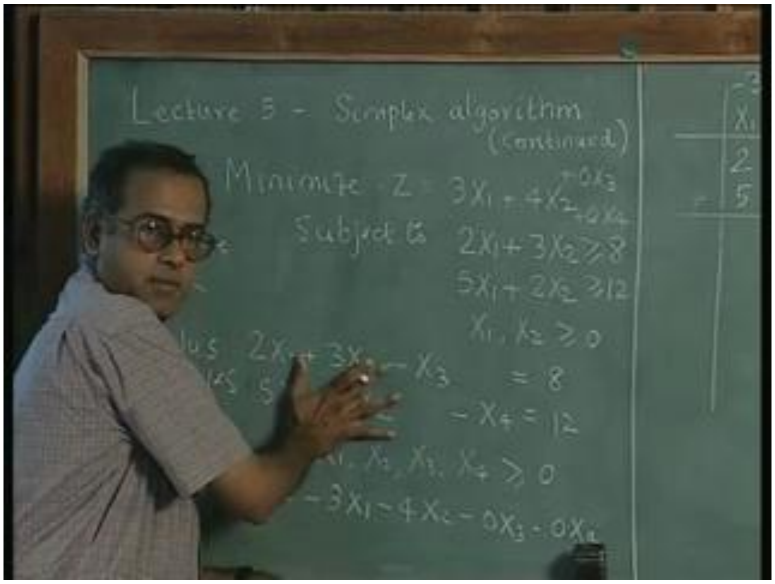

But we have ended up creating a new set of equations with two additional variables that we have introduced into the problem which the problem does not have. If this problem has to adequately represent this (Refer Slide Time: 14:17) problem, then we have to ensure that these two variables will not appear in the final solution because if they appear in the final solution then it means that this problem has not adequately represented this problem.

Remember that we wanted to solve this and thus we converted this set of inequalities to equations by adding a negative slack variable then for the purpose of getting a basic feasible solution we have added two more variables into this problem which are the artificial variables. Since these variables are not part of the problem they should not be part of this solution.

So we have to ensure or we have to make sure that these artificial variables are not part of the solution and they will not appear in the optimal solution. If one such optimal solution exists and how do we do that? What we do here is for a maximization problem we give a very small contribution to the objective function.

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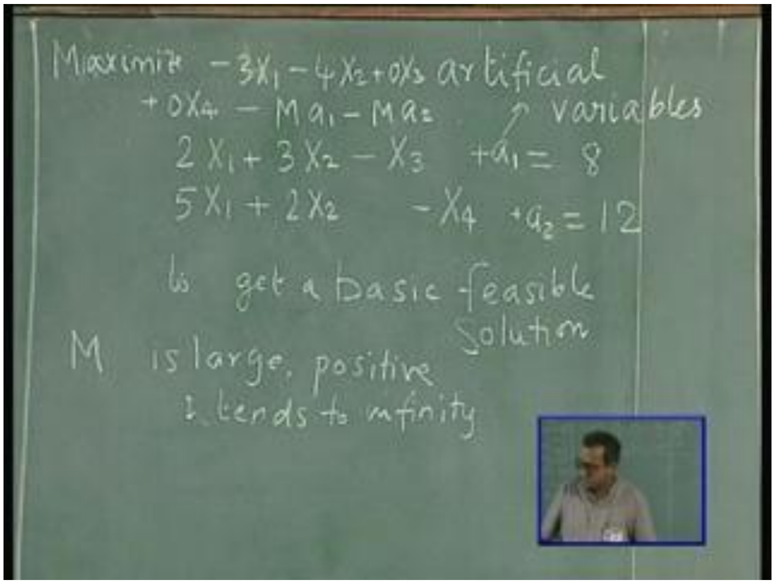

We for example write – Ma_1 – Ma_2 where this M is called big M. This M is large, positive and tends to infinity which means – M is a very small number and the contribution of these artificial variables to the objective function is a very small. So why are we doing this? What Simplex would do is out of these six variables that we have, it is going to identify that basis which optimizes the objective function. Therefore what we want to make sure is that these artificial variables do not appear in this solution and so they should not be in the solution.

Now if there is a feasible solution which does not have these artificial variables and if there is another feasible solution which has these artificial variables, then we have to make sure that the feasible solution that does not have these artificial variables has a higher value of the objective function than every feasible solution that has one or more of these artificial variables because we are solving a maximization problem. Because the contribution of these artificial variables is very small if a_1 or a_2 appears in a basic feasible solution then because of these – M, (– M being close to – infinity), the value of the objective function will be very less and it will be minus infinity. For every basic feasible solution which does not have either a_1 or a_2 the contribution will be a number which will be far higher than minus infinity.

Therefore indirectly what we try to do is we prevent this a_1 and a_2 from coming into the solution by making sure that for every basic feasible solution that has a_1 or a_2. If there is a basic feasible solution that does not have any of the artificial variables then that solution will have a higher value of the objective function and since we are maximizing, we have converted a minimization problem into a maximization problem.

Since we are maximizing, if there is even one basic feasible solution that does not have these artificial variables then that solution will dominate every basic feasible solution that has at least one artificial variable therefore if the problem has a solution that does not have $a_1$and a_2 or alternately if this problem as an optimal solution then obviously that will not have $a_1$and a_2.

Therefore if this problem has a basic feasible solution without a_1 and a_2 then that basic feasible solution will be better than every basic feasible solution with a_1 and a_2. We are now convinced that by putting this – Ma_1 and – Ma_2 here. We are preventing this a_1 and a_2 from appearing in this solution. So now we go back and construct the Simplex stable.

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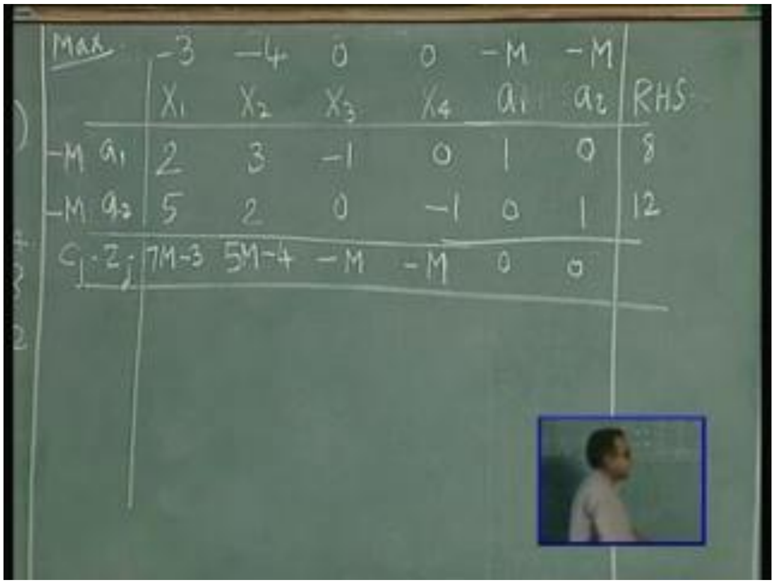

We add this a_1 and a_2 here. We have + a_1 and 0, 0, 1. The right hand side values are 8 and 12. Now a_1 and a_2, the artificial variables are the starting basic variable where contribution to the objective function will be – M for a maximization problem. So we maximize and we have a – M here (Refer Slide Time: 19:29) and a – M here. Now we write our $C_j – Z_j$.

Please remember that we will use this M very judiciously and you will see how we use this M. We know this M is large, positive and tends to infinity. We will see how we use this, why we do not put infinity but we put an M. It will be apparent once we start computing this Cj – Zj. Now let us keep this as – M. So Cj – Zj, this is (– M into 2) + (– M into 5) which is = – 7M. This is – 3 – (– 7M) which is 7M – 3. This is – 3M – 2M is = – 5M – 4 – (– 5M) is = + 5M – 4 and this is + M, 0 and you get a – M here. This is 0 + M so you get a – M here and this is – M – M – (– M) = 0. In any case a_1, a_2 being the basic variable will have a 0 in $C_j – Z_j$. In this example we do not compute this. For example we do not put a –20M here.

Normally we would have put – M into 8 is = – M into 12 which will be = – 20M. We do not put any number there simply because we still have artificial variables appearing in the basis. This (Refer Slide Time: 21:00) number does not convey anything to us so we just leave it as it is. Now we want to enter the variable which has the largest positive $C_j – Z_j$ so 7M – 3; 5M – 4. Now M is large, positive not exactly infinity but close to infinity 7M – 3 will be bigger than 5M – 4.

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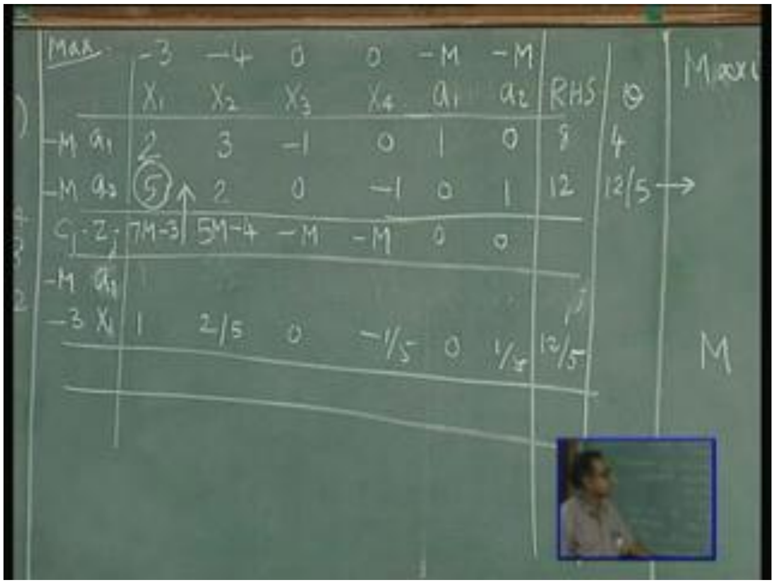

So we enter variable X_1. so precisely for this reason, we keep it as big M and we do not make it explicitly as equal to M. We realize that 7M – 3 will be bigger than 5M – 4, so variable X_1 will enter now to find out the leaving variable. We compute the theta. Therefore this is entering column. 8 divided by 2 is = 4. 12 divided by 5 = 12/5. The minimum value is 12/5. Variable a_2 leaves the basis. Variable X_1 enters and this is the pivot element. We now have the next iteration which is X_1 or with a_1 and X_1 coming here. As usual we do the row operation.

So the pivot row is rewritten first by dividing every element of the pivot row by the pivot element. In this case dividing it by 5, we get 1, 2/5, 0, – 1/5, 0, 1/5, 12/5. Incidentally these values are – 1 and 3. It is the pivot row, so divide every element of the pivot row by the pivot element to get 1, 2/5, 0, – 1/5, 0, 1/5, 12/5. Now we need a 1, 0 and a 0, 1. We need to get a 0 here, so this – 2 times this row will give a 0.

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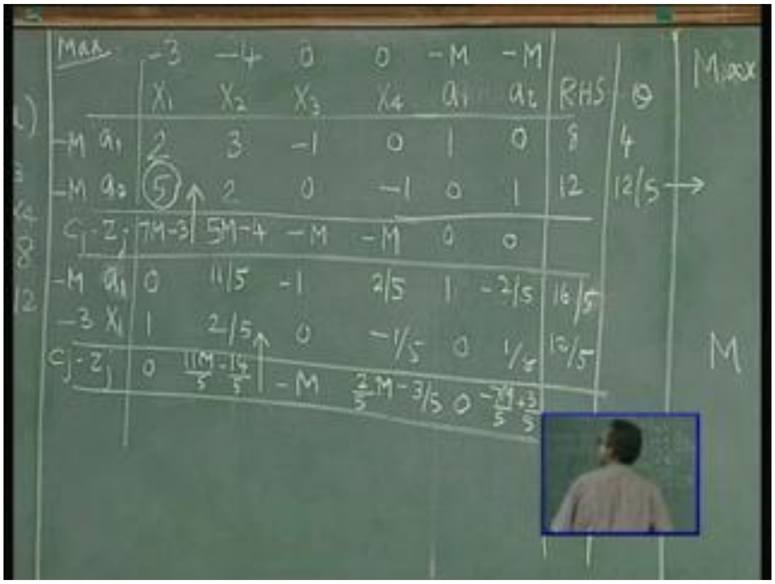

So 2 – (2 times 1) is = 0; 3 – (2 into 2)/5 = 3 – 4/5 which is 11/5 – 1 – (2 into 0) is = – 1; 0 – 2 into – 1/5 is = + 2/5; 1 – (2 into 0) = 1; 0 – (2 into 1/5) is = – 2/5 and 8 – (2 into 12/5) = 8 – 24/5 is 16/5

Again we compute $C_j – Z_j$, a_1 and X_1 being basic variables. We will have 0 now, this is – 11M/5 – 6/5. This (Refer Slide Time: 24:30) is – 4 + 11M /5 + 6/5

So 11M/5 + 6/5 – 4 is = – 14/5. This is + M so this is = 0. Now this will give – M. This is – 2M/5 + 3/5 therefore 0 – (– 2M/5) is = + 2M /5 – 3/5. This will be + 2M/5 – 3/5, so – M – 2M/5 is = – 7M/5 + 3/5.

Again we do not write anything here because we have an artificial variable lying in the basis now. We enter the variable which has the most largest positive $C_j – Z_j$. 11M/5 is bigger than 2M/5 – so variable X_2 will enter. For example, since M is very large, the other constant term whether it is positive or negative does not have any impact. We only look at the terms that involve M so we look at 11M/5 here and 2M/5 here so variable X_2 enters the basis.

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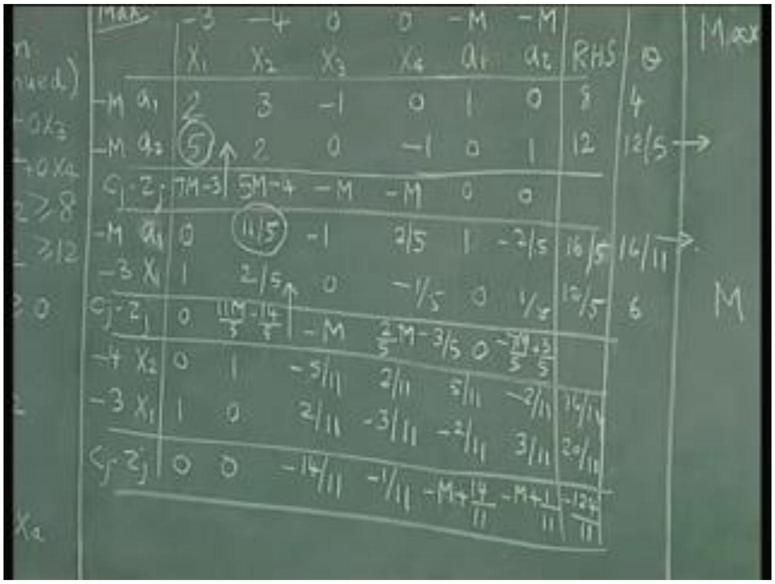

In order to find out the leaving variable we have to compute theta. So this is 16/5 divided by 11/ 5 which is 16/11. 12/5 divided by 2/5 which is 6.

16/11 is smaller. This variable leaves. This (Refer Slide Time: 26:23) is the pivot element. Now variable X_2 replaces variable a_1 so we have X_2 here (Refer Slide Time: 26:40) and we have X_1 here and the coefficients have – 4 and – 3. This is the pivot row. This is the pivot element.

Divide every element of the pivot row by the pivot element. This case divided by 11/5 or multiplied by 5/11. So (0, 1, – 5/11, 2/11, 5/11, – 2/11, 16/11) and then X_2 and X_1 are the basic variables. We need a (1 0 0 1). We need to get a 0 here, so this row – 2/5 times this will give a 0. So do the row operation for every element. (1 – 2/5 into 0) is 1; 2/5 – 2/5 into 1 is 0;
0 – 2/5 into – 5/11 is + 2/11; – 1/5 – 2/5 into 2/11 from – 1/5 – 4/55 would give us – 15/55 which is – 3/11. 0 – 2/5 into 5/11 would give – 2/11, 1/5 – 2/5 into – 2/11 is 1/5 + 4/55 which is 15/55 which is 3/11. 12/5 – 2/5 into 16/11
So 2/5 into 16/11 is = 32/55 this is 12 into 11 = 132 –32.
So you get 100/55 which is 20/11. X_1 and X_2 are the basic variables so you get 0 here. Now this is – 4 into – 5/11 is = 20/11; 20/11 – 6/11 is 14/11; 0 – 14/11 is = – 14/11.
– 8/11 + 9/11 is = 1/11; 0 – 1/11 is = – 1/11.
This is – 20/11 + 6/11 is – 14/11
So this is – M + 14/11 and this is 8/11 – 9/11 is = – 1/11
So – M + 1/11; (16 into 4 is 64) + (20 into 360) = 124. So – 124/11

Now we look at the $C_j – Z_j$'s and we realize that all these are negative because even though these are positive numbers, we have a – M; M being very large M positive. So both these terms are negative. The algorithm terminates and we have reached the optimal solution with X_1 = 20/11; X_2 = 16/11; Z = – 124/11 for the maximization problem that we have solved. We had earlier multiplied this subjective with a – 1 to convert it to maximization. In order to get the objective function for the original problem we need to multiply this again with a – 1.

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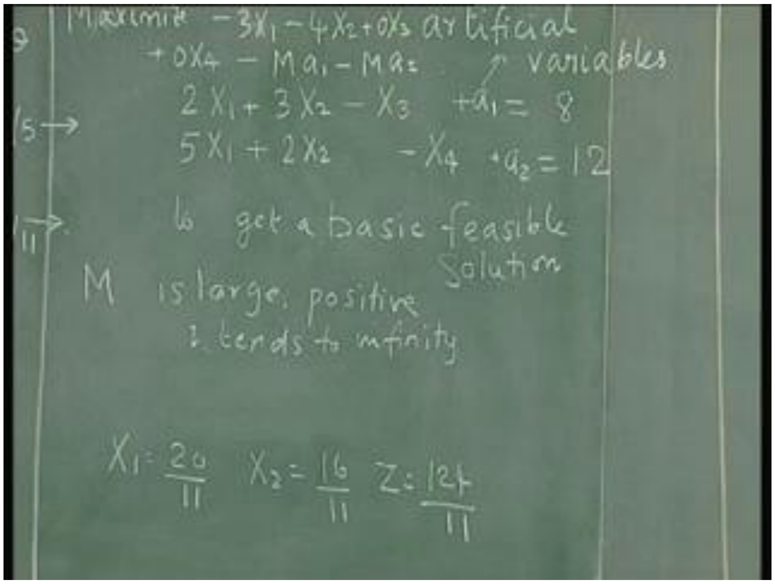

Therefore, the optimal solution is X_1= 20/11; X_2= 16/11; Z for the original problem will be 124 /11. So this algorithm terminates this way. Now what are the other issues? For example a_1 and$_2$ being artificial variables, we do not want them in the basics that is how we have defined them you do not want them in the basis. So we actually need not explicitly compute these numbers. These numbers will always be negative. It will never be positive artificial variable and will never try to enter the basis at all. So we need not have actually calculated these two numbers these two which we were obvious and these two numbers but then for the sake of completion we have done all this and this is how we solve a minimization problem. The moment we have all C_j – Z_j's negative, the termination condition that we know, the algorithm will start. In this case we have the optimal solution with two decision variables X_1 and X_2 coming in to the solution. Now we have also seen that by introducing these artificial variables and more importantly, by giving this – M here, we have prevented them from appearing in the optimal solution.

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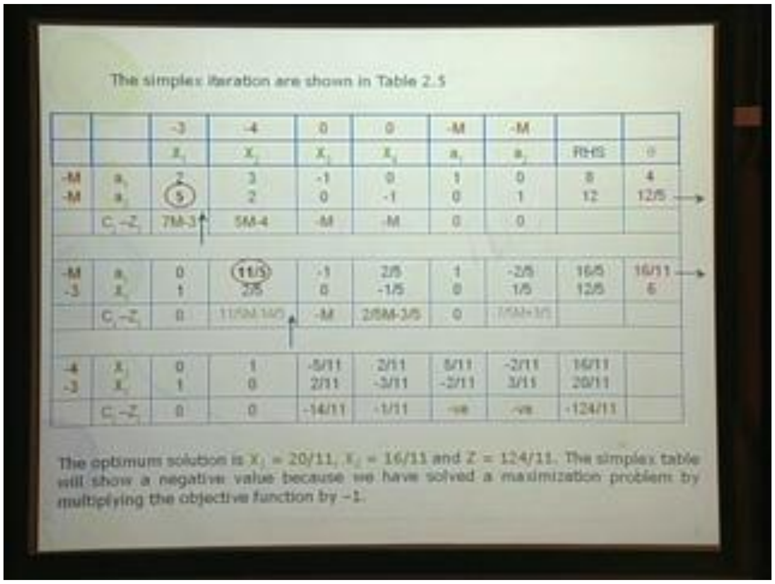

Now let us go back to this. We show the same thing in this table so Simplex table show negative value because we have we solved a maximization problem. So we need to multiply the objective function again a – 1 to get the + 124/11 that we have here. Next now what are the other observations that we have?

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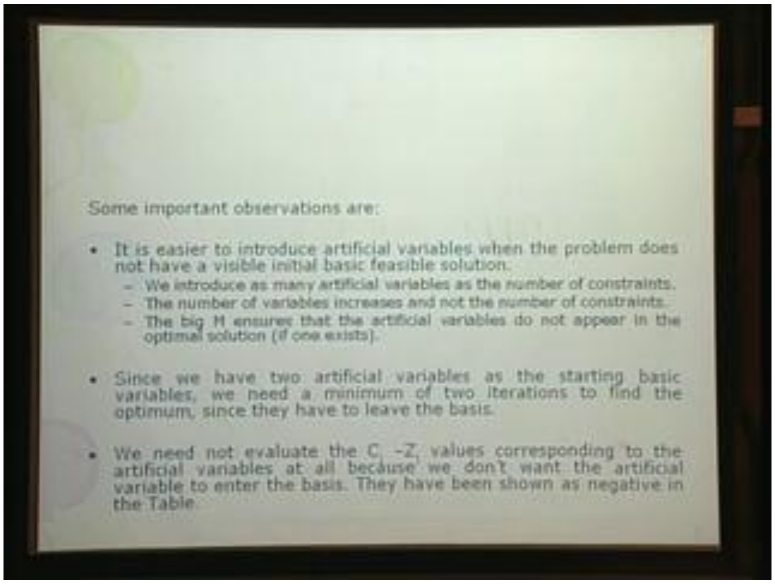

As it is easier to introduce artificial variables when the problem does not have a visible initial basic feasible solution, like the one we have had here, now this problem does not have a very clear basic feasible solution here because this the basic feasible solution by fixing X_1, X_2 to 0 would give us $X_3 = -8$; $X_4 = -12$. So this is not a basic feasible solution. We might ask ourselves another question. What would have happened had we defined these X_3 and X_4 with a + sign and then force them here to less than or equal to 0? Mathematically it is correct.

We do not do that because we are very keen about the fact that all these variables should be greater than or equal to 0. This is actually going to get reflected in the minimum theta rule or the entering rule so if we have some variables greater than equal to 0 and some less than or equal to 0 then we cannot blindly as well as uniformly implement the minimum theta row.

So we are extremely concerned about the fact that the Simplex algorithm should not end up evaluating any infeasible solution in a middle iterations. We do not want that to happen and that is ensured by the minimum theta rule and we would like to keep the minimum rule intact and not disturb it. Therefore we would rather introduce X_3 and X_4 in such a way that X_3 and X_4 are always greater than or equal to 0 even though they end up having ah negative term here. Now that we have learnt to use artificial variables to get the initial basic feasible solution and it becomes easy now to solve problems with greater than or equal to.

Assuming that right hand side values are non negative any greater than or equal to constraint will only give us a negative slot. A negative slot will not qualify to be a starting basic feasible solution. So when it does not qualify to be starting basic feasible solutions, we will have to introduce an artificial variable. Since we have two of artificial variables in this problem, we need a minimum of two iterations in this example.

It turned out that after these two iterations the third one gave this solution. Now such small problems usually the moment all the artificial variables leave the basis we will get the optimal solution. As we had indicated, we need not explicitly evaluate the $C_j - Z_j$ values. It any way turns out to be negative.

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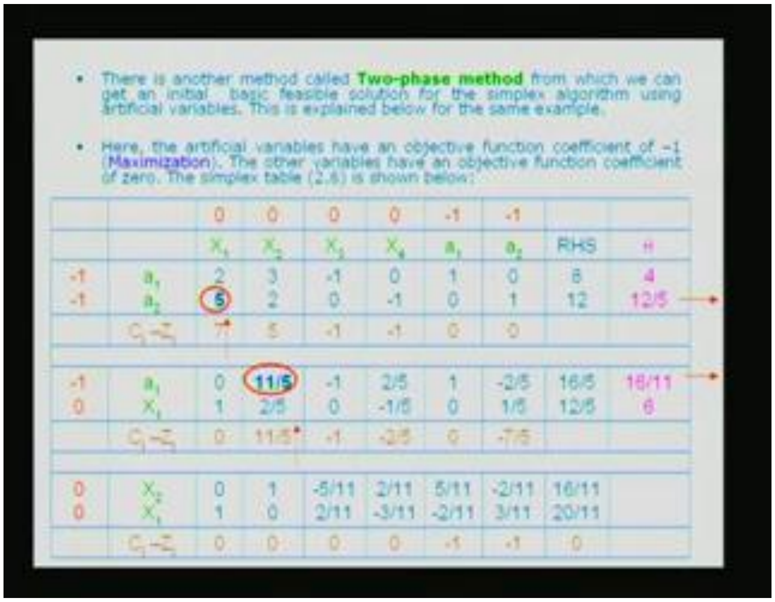

Now let us look at another way to solve minimization problems. What we did here is we introduced artificial variables straight away. We then make sure that they do not appear in the solution by giving them a large negative value for a maximization problem.

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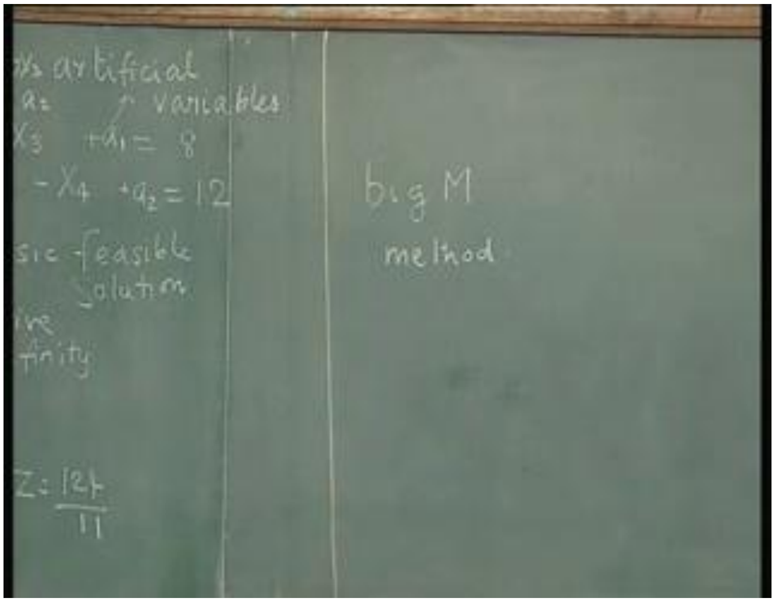

Now this method that we have used is called the big M method where the big M is introduced. We also have a very similar method called the two phase method which we will see now in order to handle problems of this type. Let us workout the two phase method with the same example to understand what now what we are going to do in the two phase method is we are going to solve the problems in two phases. In the first phase our idea will be to send all the artificial variables out and then in the second phase we solve the problem without the artificial variables.

Remember again artificial variables were introduced because we wanted an initial basic feasible solution. So at the end of the two phase method when the artificial variables leave we will have a basic feasible solution without any artificial variable. From that point onwards we can carry on with this Simplex and solve. Let us explain the two phase method again with the same example that we have now.

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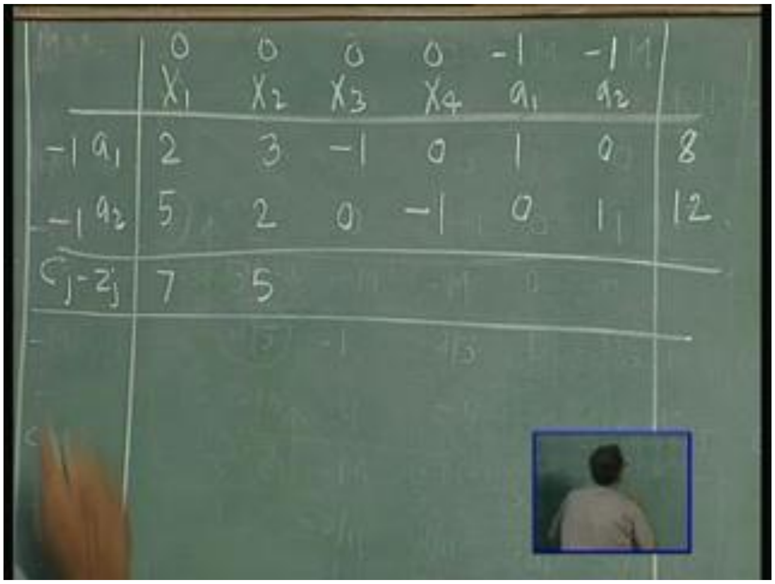

What we do in this two phase method is we have X_1, $X_{2,}$ X_3, X_4, $a_1$and $a_{2.}$
We again start with a_1 and a_2.
We have $2X_1 + 3X_2 – X_3 + a_1, a_2 = 8$
And we have $(5X_1, 2X_2, 0X_3, – X_4, 0a_1, 1a_2) = 12$

What we are going to do now is we are going to change the objective function coefficients in the first phase of the two phase method. We are now going to separate the problem to artificial variables and other variables. All the artificial variables will have an objective function coefficient of 1 for a minimization problem a – 1 for maximization problem. All other decision variables and slack variables uniformly will have a 0 contribution in the objective function. Now we start with the – 1 here and a – 1 here (Refer Slide Time: 37:52).

Please remember that we are not using this $3X_1 + 4X_2$ in the first phase of the two phase method. We are not doing that. You would put all 0s in this phase. Now we solve this problem again to try and get and a basic feasible solution without the artificial variables. We do that like this. So we again write $C_j – Z_j$. Now this will be – 2 – 5 so you get a +7. You get a – 3 – 2 you get a 5.

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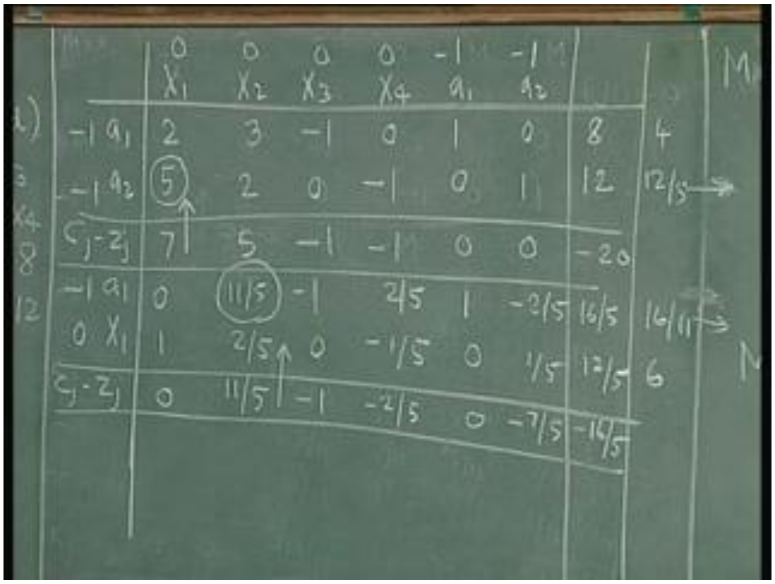

You have a 0 and so (– 1 0 – 1). You have a – 1. You have a 0. You have a 0 and the value you have here is – 20. Now variable X_1 has the largest $C_j – Z_j$ so variable X_1 enters. To find out the leaving variable, we again compute theta 8/2 is 4, 12/5 and 12/5 leaves the basis. This is the pivot element. Now we have a_1 and X_1 – 1, 0. This is the pivot element. So divide every element of the pivot row by the pivot element to get (1, 2/5, 0, – 1/5, 0, 1/5, 12/5).

We need a 0 here so this – 2 times this row will give 0; 3 – (2 into 2/5) = 3 – 4/5 is = 11/5, – 1.

0 – (2 into – 1/5) which is 2/5 into – 1= – 2/5; 8 – 24/5 is = 16/5.

Now $C_j – Z_j$ values will be, a_1 and X_1 will have 0.

Now this will be – 11/5; 0 – 11/5 i.e; 0 – (– 11/5) is = + 11/5; (– 1 into – 1) is + 1

So 0 – 1 is – 1.

– 1 into 2/5 is = – 2/5, 0; 0 – 2/5 is – 2/5. This is – 1 into – 2/5which is + 2/5

So – 1 – 2/5 is –7/5 and this will be – 16/5. Now this variable has a positive $C_j – Z_j$ the largest and so variable X_2 enters the basis. Now I have to find out the leaving variable. 16/5 divided/11/5 is 16/11. 12/5 divided/2/5 is 6. So this leaves. This (Refer Slide Time: 41:57) is the pivot.

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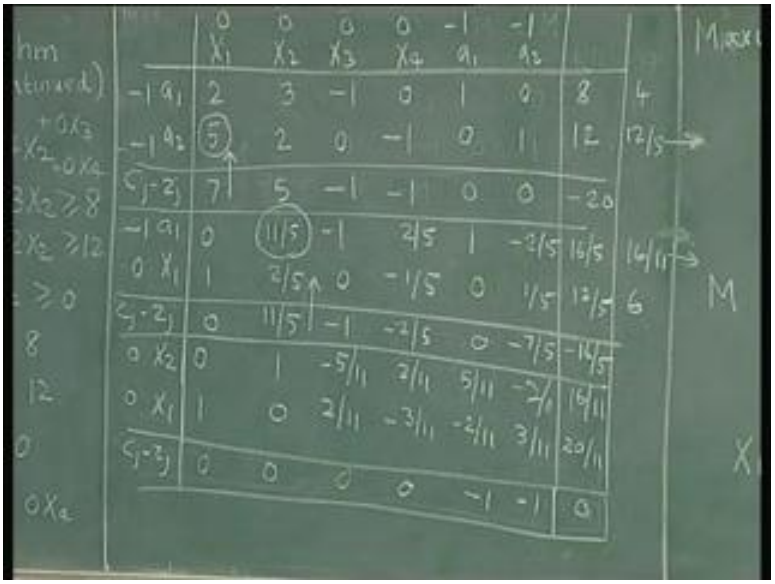

Now you have X_2, X_1 so X_2 has 0; X_1 has 0. This is the pivot element. Divide every element of the pivot row by the pivot element so (0, 1, – 5/11, 2/11, 5/11, – 2/11, 16/11). You need a 0 here so this – 2/5 times this would give us a 0 so 1 – (2/5 into 0) = 1. (2/5 – 2/5) into 1 is 0; 0 – 2/5 into – 5/11 is 2/11; – 1/5 – 2/5 into 2/11 – 1/5 – 4/55

That would give us –15/55 which is – 3/11

0 – (2/5 into 5/11) is – 2/11; 1/5 – 2/5 into – 2/11 is 1/5 + 4/55 would give us 15/55 which is 3/11

1/5 – 2/5 into 16/11; 12/5 – 32/55; 132 – 32 is 100

So 100/55 would give us 12/11

So $C_j – Z_j$ values will be these all = 0 and you end up getting (0, 0, 0, 0, – 1 and 0)

Now your first phase of the two phase method is over.

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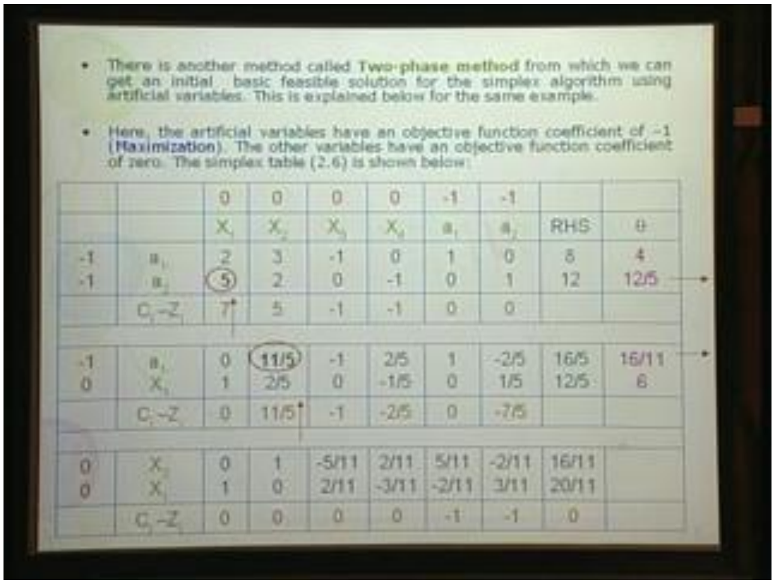

Let us move to the second phase. Now at the end of the first phase, we have a basic feasible solution with X_2 = 16/11 and X_1= 20/11.

We do not look at the value of the objective function because we have changed the objective function. So the purpose of the first phase is to give us a basic feasible solution without the artificial solution. Now from this solution we start the second phase and the second phase goes like this.

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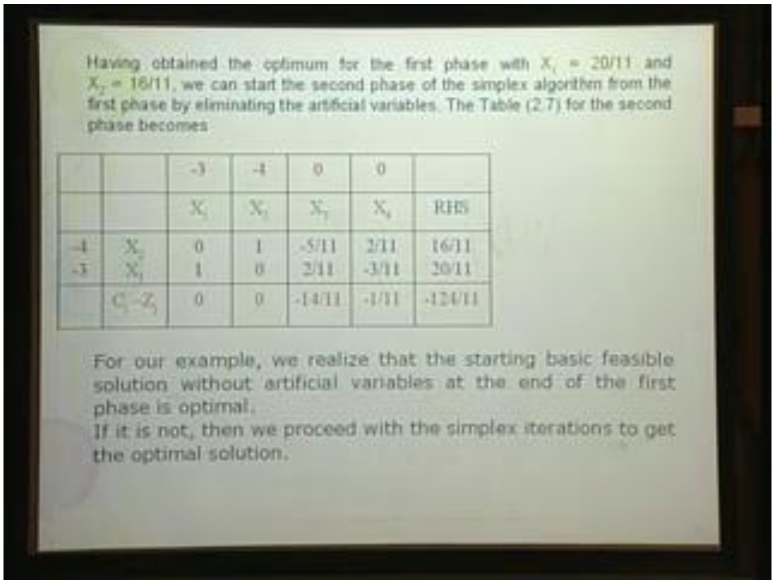

Now when we do the second phase, we leave out the artificial variables.

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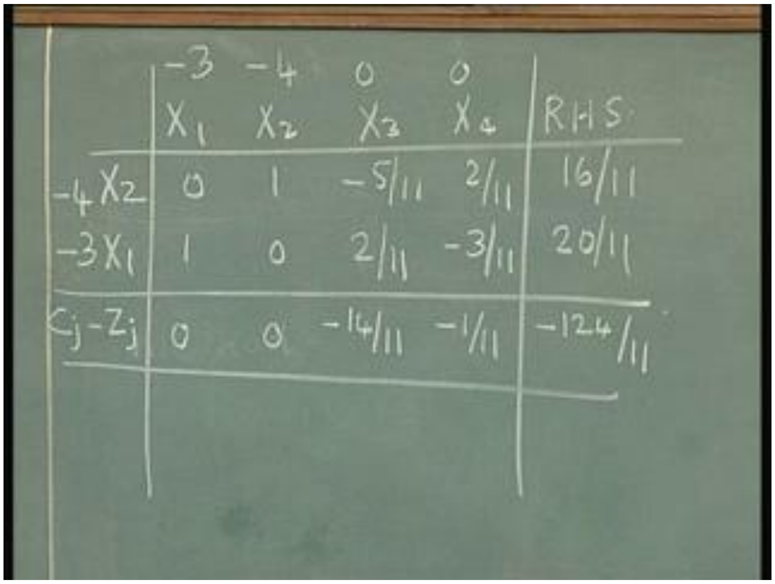

We have only X_1, X_2, X_3 and X_4 on the right hand side. For a maximization problem we would have (– $3X_1$, – $4X_2$, 0 and 0). We now have a solution which is this. X_2 with (0, 1, – 5/11, 2/11 16/11)

X_1 with (1, 0, 2/11, – 3/11 and 20/11)

The coefficients are – 4 and – 3 respectively.

Now we compute the values of $C_j – Z_j$. The values will be X_1 and X_2 being basic, would give us all 0s. We need to compute the $C_j – Z_j$ only for X_3 and X_4.

This is + 20/11 – 6/11 is + 14/11 i.e., (0 – 14/11) is – 14/11; – 8/11 + 9/11 is = + 1/11; 0 – 1/11 is – 1/11; – 64 – 60 is = – 124/11

Now it turns out that this is optimal because all the $C_j – Z_j$'s are negative. Thus within one iteration of the second phase we have got the optimal solution. If we go back and compare between the two phase method and the big M method you will realize you are actually trying to do the same thing without introducing the big M. The reason why two phase method is also important is when particularly when you start writing computer programs there is always a question of how to define this big M.

When you write a computer program you end up giving a large value of this big M and one needs to be a little careful about those computations that come in the big M method. In some sense the two phase does not have any big M appearing anywhere in the table and it is a very logical method of solving it because the first phase eliminates the artificial variables. It tries to give you a basic feasible solution without the artificial variables and then in the second phase you start with that solution. Now go back and bring the problem objective values. Then check whether remaining things are optimal.

So one could use either the big M method or the two phase methods to solve minimization problem and more importantly problems that involve these type of constraints. It is not the objective that matters because we know that if the problem is maximization then keep the objective function as it is. If the problem is minimization, we multiply the objective function with the – 1 therefore it does not matter to us. What is important or what is of concern is whenever we have this greater than or equal to constraint for a strictly positive right hand side value, the slack variable does not provide the starting basics. Therefore we need to introduce the artificial variable and the moment we introduce the artificial variable we need to use either the two phase method or the big M method to solve. Let us look at this again. We show the same to you, the second phase of the two phase method which gives the optimal solutions in this case.

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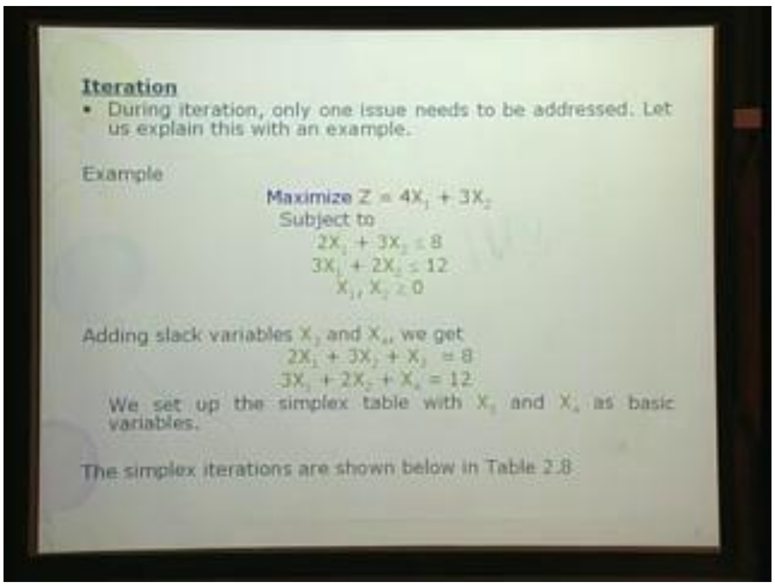

This is something to do with initialization. There are a few other issues that are with initialization. What are those issues? When do we introduce an artificial variable?

We said that if the problem has greater than or equal than to constraint and a positive value of the right hand side then it would result in a negative slack or a surplus variable. We also realize somewhere that this artificial variable is an additional burden that we are imposing into the problem. Artificial variables do not exist in the problem. We introduce them simply because we want an initial basic feasible solution to start this Simplex. Remember Simplex as an algorithm does three things. It always starts with a basic feasible solution. It iterates.

It does not evaluate any infeasible solution in between. It continues to evaluate only basic feasible solutions and it terminates. Because we are not able to get those basic feasible solutions here we introduce the artificial variables. We introduced the artificial variables. We added as many variables as the number of constraints in this example. We increase the number of problem variables and then we also introduced the big M and then we try to eliminate artificial variable and the big M. So obviously a good Simplex implementation should have minimum artificial variables. It is not that we end up introducing an artificial variable for every greater than or equal to constraint. There are situations where you do not really have to introduce an artificial variable.

We do not have to introduce it for the sake of introducing it. We actually need to minimize it. So these are the other aspects of the initialization that we will see in next lecture.