Fundamentals of Operations Research

Prof. G. Srinivasan

Department of Management Studies

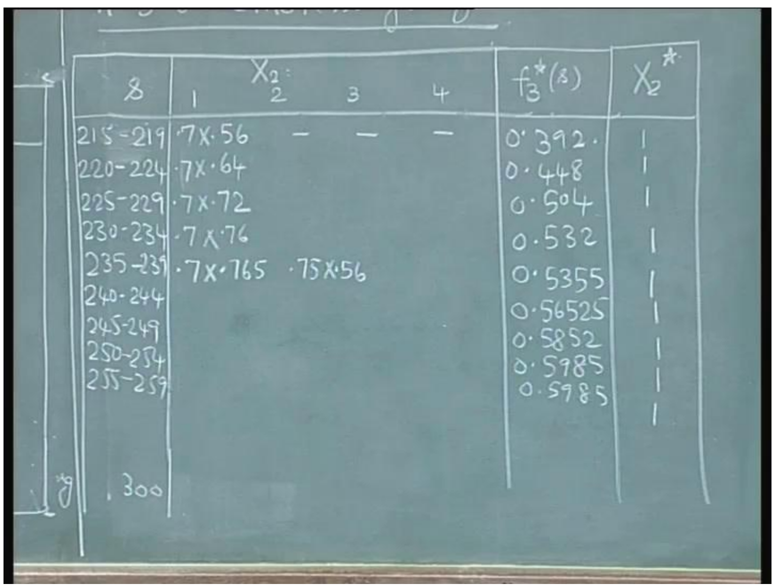
Indian Institute of Technology, Madras

Lecture No. # 18

Dynamic Programming - Examples Involving Discrete Variables

In this lecture we continue our discussion on the reliability problems. We had seen 2 stages in the previous lecture. So we continue and move to the third stage where we have n = 3; 3 more stages to go.

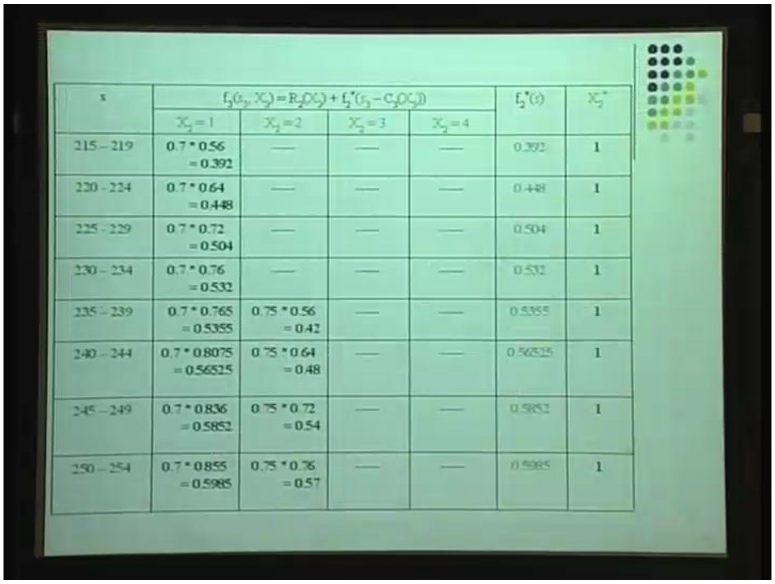
(Refer Slide Time: 01:32)



So here we make decision for item or component B and we have to decide how many we can buy of this component B, so the state variable will be s i.e., the amount of money that is available. So decision will be X_2 = 1, 2, 3 and 4. Now what are the values that the state variable can take? The minimum value that state variable s can take should be the minimum money required to buy at least one of B, one of C and one of D, which means the minimum money that we should have is 80 + 75 = 155 + 60 is 215 that is the minimum amount of money. The maximum that we would have if we had bought only one of A and spent 100, so the maximum would be 300. So if we start with state variable = 215 and we decide to buy one of B so we spend 80 rupees on B to get a reliability of 0.7. So we would get 0.7 and out of 215, we have spent 80, so we have remaining 135 which we carry to the previous stage. When we have 135 then the maximum we could get is 0.56 so this will be 0.7 into 0.56 which will be 0.392. Now we cannot be evaluating X_2 = 2, 3 or 4 because we will not have money to buy the other C and D. So the bare minimum is 215. There is only one alternative that is possible, so we do this. Very similar to the previous table we have to now look at the value of the state variable from 215 to 300 which means we have to look at about 86 values starting from 215 to 300. We do not individually consider all the 86 values. We look at the range in which the maximum value will remain intact and wherever there is a scope to bring in another alternative or wherever there is a scope for one of the numbers to change we evaluate that separately. This will be from 215 to 219. We can have only one option which is this option. So we say from 215 to 219, we have only one option the moment we have 220 now we see that something else can happen. When we have 220, we can buy one of B and spend 80 and with the balance 140, we can go here and the best one becomes 0. 64, so we get 0.8 into 0.64 and $X_2 = 2$ would give us 0.7 into 0.64. We cannot consider $X_2 = 2$ because we would have spent 100 on B which means we would be carrying 120 and we do not have anything. So we need at least 135 here. There is only one option that is possible here and that would give us a reliability of 0.448 and the only option is the best option. This would hold from 220 to 224. Now the moment we have 225, we find some other options possible. We buy one and spend 80, so we have the remaining 145 and so we could get 0.7 into 0.72.

Even here if we want to buy 2 of B and spend 100, we would still be having only 125, so only when the state variable becomes 235, there is a possibility of the next option coming in. So 225 to 229 would give us only one option which would mean 0.504. So similarly we look at 230 to 234, the only option is we spend 80 here and for the balance we go back to 150 so for the 80 rupees spent, we get 0.7. Now for 150 that is available in the previous table, we find that we could get a reliability of 0.76 and therefore this number becomes 0.532 and once again 1. The moment we have 235 we realize that 2 possibilities exist. We could buy one spend 80 and carry the 135. So spend one, 0.7, buy one, and spend 80. We get 155 and for 155 we get 0.765. Now we could spend 100, buy 2 of B, get reliability of 0.75 and with the balance 135, we go back to this table and get reliability of 0.56. So 2 possibilities exist now and between the 2, we have to choose that one which is higher. We get 0.5355 and go back to 1. So like this we have to complete this table. This is a fairly lengthy table. This would be valid for the range 235 to 239 and we would have another range coming from 240 to 244 and for this range we would have 0, 56525 and 1, for the range 245 to 249 we would have 0.5852 should be 1, 250 to 254, 0.5985 and 1. 255 to 259, we have the same 0.5985, 1 and so on. Now all these are shown in these 2 tables. We first go to the previous table.

(Refer Slide Time: 09:19)



s	$f_2(s_2, X_2) = R_2(X_2) + f_1^*(s_2 - C_2(X_2))$				$f_2^*(s)$	X_2^*
	$X_2 = 1$	$X_2 = 2$	$X_2 = 3$	$X_2 = 4$		
215 – 219	0.7 * 0.56 = 0.392	—	—	—	0.392	1
220 - 224	0.7 * 0.64 = 0.448	—	—	—	0.448	1
225 - 229	0.7 * 0.72 = 0.504	—	—	—	0.504	1
230 – 234	0.7 * 0.76 = 0.532	—	—	—	0.532	1
235 – 239	0.7 * 0.765 = 0.5355	0.75 * 0.56 = 0.42	—	—	0.5355	1
240 – 244	0.7 * 0.8075 = 0.56525	0.75 * 0.64 = 0.48	—	—	0.56525	1
245 – 249	0.7 * 0.836 = 0.5852	0.75 * 0.72 = 0.54	—	—	0.5852	1
250 – 254	0.7 * 0.855 = 0.5985	0.75 * 0.76 = 0.57	—	—	0.5985	1

Now 215 to 219, we have only one alternative 0.392 that is shown here in the beginning of this table and that is shown here as well. The last one that is shown here is 250 to 254 which is this. For 250 to 254, we have 2 possibilities, so we quickly explain both these possibilities. One is we decide to buy one of B, $X_2 = 1$ which means you spend 80. So spend 80, with the reliability of 0.7. We see 0.7 here in the first part here, we see a 0. 7 here and with the remaining 170 we go back to the previous table and we could get 0.855. So we have 0.7 into 0.855 which is 0.5985. Now the other possibility of buying 2 of B is $X_2 = 2$ which you can see there in the third column of the table. $X_2 = 2$ in green and there we realize that we to buy 2, we spend 100, get a reliability of 0.75 which is shown here. We spend 100, so for the balance 150, we could go back to this table and get a reliability of 0.76, so you have 0.75 into 0.76 which is 0.57 which is shown here. Between the two, 0.5985 is higher, so the best value is 0.5985 with the best decision at 1.

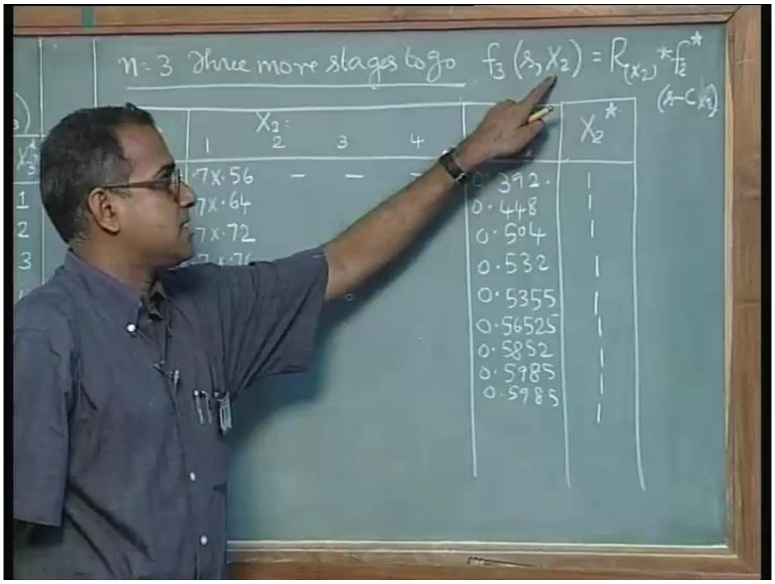
(Refer Slide Time: 10:53)

255 – 259	0.7 * 0.855 = 0.5985	0.75 * 0.765 = 0.57375	0.8 * 0.56 = 448	—	0.5985	1
260 – 264	0.7 * 0.855 = 0.5985	0.75* .8075 = 0.605625	0.8 * 0.64 = 0.512	—	0.605625	2
265 – 269	0.7 * 0.855 = 0.5985	0.75* 0.836 = 0.627	0.8 * 0.72 = 0.576	—	0.627	2
270-274	0.7 * 0.855 = 0.5985	0.75* 0.855 = 0.64125	0.8 * 0.76 = 0.608	—	0.64125	2
275-279	0.7 * 0.855 = 0.5985	0.75* 0.855 = 0.64125	0.8 * 0.765 = 0.6120	0.85 * 0.56 = 0.476	0.64125	2
280-284	0.7 * 0.855 = 0.5985	0.75* 0.855 = 0.64125	.8 * 0.8075 = 0.646	0.85 * 0.64 = 0.544	0.646	3
285 – 289	0.7 * 0.855 = 0.5985	0.75* 0.855 = 0.64125	.8 * 0.836 = 0.6688	0.85 * 0.72 = 0.612	0.6688	3
290 – 294	0.7 * 0.855 = 0.5985	0.75* 0.855 = 0.64125	.8 * 0.855 = 0.684	0.85 * 0.76 = 0.646	0.684	3
295 – 299	0.7 * 0.855 = 0.5985	0.75* 0.855 = 0.64125	.8 * 0.855 = 0.684	0.85 * 0.765 = 0.65025	0.684	3
300	0.7 * 0.855 = 0.5985	0.75* 0.855 = 0.64125	.8 * 0.855 = 0.684	.85* .8075 = 0.686375	0.686375	4

In the next table we continue this from 255 to the maximum 300 and we now have several alternatives coming in. We will just go through a couple of instances of calculation. For example let us look at the case when the state variable is 270. Let us say we are here at 270. Now there are 3 possibilities. For example when you could decide to buy $X_2 = 1$ and get a reliability of 0.7 and so, among 270 we have spent 80 so there is a balance of 190. So 190 would give us another 0.855, so you would get 427, 0.7 into 0.855 which is 0.5985. Now we could also consider $X_1 = 2$, spend 100 get a reliability of 0.75. Now out of 270 we have spent 100, so we have 170 and for 170, you go back to the previous table you get 0.855. So this number is 0.75 got from here into 0.855 got from here (Refer Slide Time: 12:12) which is shown as 0. 64125. Now we could also consider $X_2 = 3$, so spend 120, get a reliability of 0.8. There is a balance of 150 which is carried out, that 150 would give us 0.76 so we get 0.8 into 0.76 which is shown here which is 0.608. We cannot consider $X_2 = 4$ because $X_2 = 4$ would mean you spent 140, so you have a balance of 130 that is carried to buy C and D which is not enough.

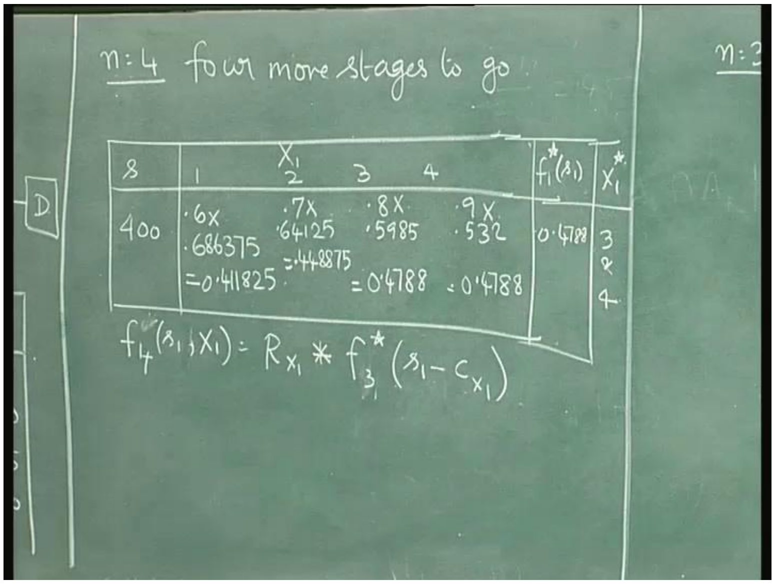
We need a minimum of 135, so we do not look at the 4th one. So there are 3 alternatives possible. Now among the 3, the best one is 0.64125 with the best decision at $X_1 = 2$. The moment we come to 275, we realize that all 4 options are possible. So when we have 275, the 4th option is possible. So you could buy 4, spend 140, get 0.85. 0.85 is shown there and spend 140 out of the 275. To carry 135 and for 135 you get 0.56 so you have 0.85 into 0.56 = 0.476. But meanwhile the rest of the reliabilities have changed. So the best value is again 64125 among the 4 values that we have computed. So like this, we complete this table and go up to 300 and we also realize that for 300, the best one would be to buy 4 of this. So we have completed the third table. We have only shown a part of it here. The entire table is shown here as 2 tables in this slide. So the recursive equation here would be f_3, 3 more stages to go.

(Refer Slide Time: 13:46)



I have s state variable, s rupees available with me. I decide to buy X_2 units of this. So I get a reliability of R of X_2 into f_2 star of (s – C) of X_2. I start with s. In order to buy X_2, I spend C of X_2 so s – C of X_2, I carry to the previous table, f_2 star of this. Now the best value f_3 star is to minimize this. So f_3 star would be to minimize f_3 of sX_2 subject to the condition that C of X_2 is less than or equal to s. I cannot exceed the amount that I have plus more of course so accepted. This is how the third table is completed. So at the end of the third table, we had the state variable taking a value of 300 which is shown here and the best value is 0.686375 with X = 4. Now we go to the last stage which is the stage here. We have 4 more stages to go, so we go back and look at n = 4.

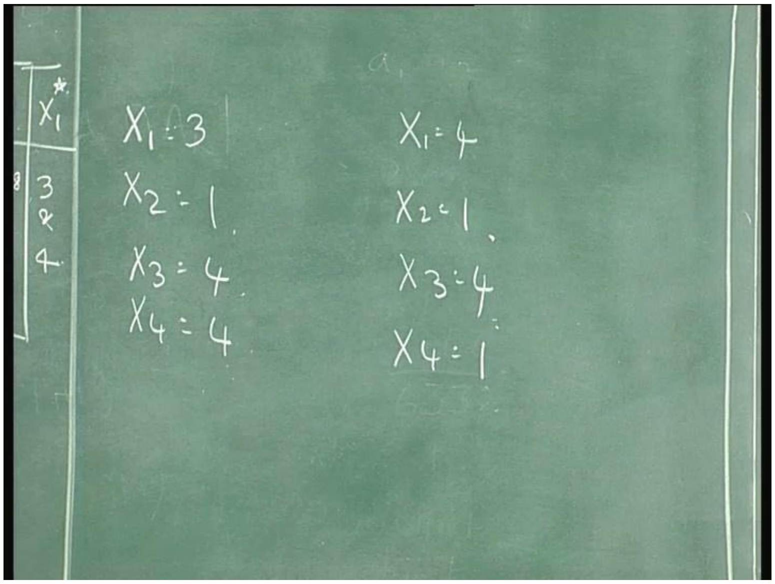
(Refer Slide Time: 15:14)



We now move to n = 4; 4 more stages to go so we are at the beginning here and we have 400 available with us. s is the state variable which is the amount of money available. X is the decision variable or X_1 in this case is the decision variable, where we like to choose the number of parallel units of A that we are going to have. X_1 can take any one of these 4 values. You have X_1 = 1 2 3 and 4 and we have 400 that is available with us. We evaluate when $X_1 = 1$, you have a reliability of 0.6 that comes out of it. You spend 100. So there is a balance of 300 that is available. So we go to the previous stage n = 3 and find out that for 300 the best value is 0.686375, so this multiplication would give us a value 0.411825. Now we can consider $X_1 = 2$ and when we have $X_1 = 2$ we get a reliability of 0.7. Then out of the 400 we spend 130, so we have a balance of 270 and for 270 from the previous table, we would get 0. 64125. When we multiply these two 0.7 into 0.64125, we would get 0.448875. When $X_1 = 3$, we get a reliability of 0.8 and we spend 150 out of the 400 available.

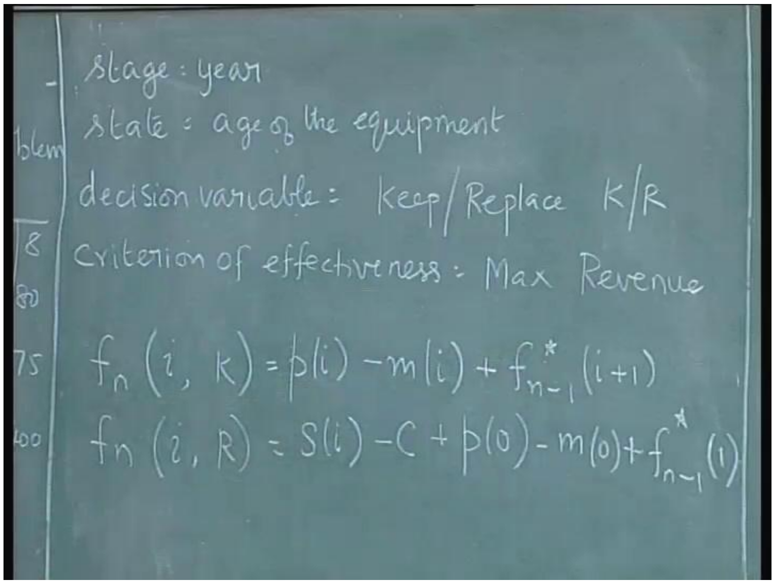
We have a balance of 2150, so we go to the previous table to find out for 250 the value is 0.5985. This is 0.5985 and this multiplication 0.8 into 0.5985 gives 0.4788 and for $X_1 = 4$, you could get a reliability of 0.9. We spend 170 so we have a balance of 230 would which would give us 0.532. We spend 170 there is a balance of 230. So you get 0.532, this on multiplication also gives us 0.4788. Now the best value which is f_1 star of s_1 happens to be 0.4788 and X_1 star, we have 2 values which are 3 and 4. We also have to write the recursive relationship. So for this, the recursive relationship will become f_1 star of s_1, f_1 of s_1X_1 would give us R of X_1. The reliability that you get multiplied by f_2 star or f_4 star of f_1X_1 would give us f_3 star of s_1 – C of X_1 i.e., we have s_1 = 400 here, out of which we can take any one of the 4 values. There is an associated cost. So 400 minus that cost goes into the next stage from which we took these 4 values so this is the recursive relationship. Now as far as the solution is concerned we have 2 possibilities $X_1 = 3$ and $X_1 = 4$. So we have some kind of an alternate optimum.

(Refer Slide Time: 19:59



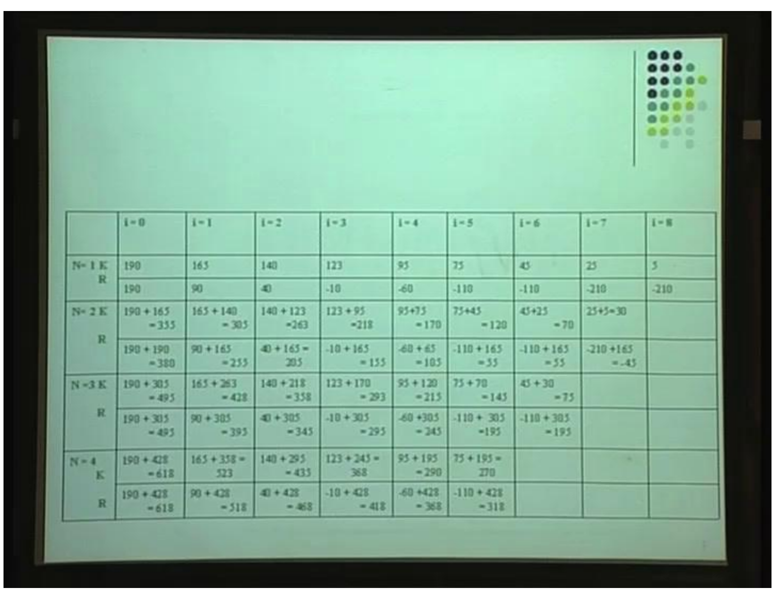
So we look at both these cases. We look at the case $X_1 = 3$ and $X_1 = 4$. $X_1 = 3$ would mean you spend 150 and there is a balance of 250. Going back to the 250 you would get $X_2 = 1$ for 250 here. so this gives us $X_2 = 1$, so from 250, $X_2 = 1$ means you have spent another 80, so you get 170 and now you go to 170 here you get $X_3 = 4$ from here, 90. There is an 80 that is remaining so you would have $X_4 = 4$. Alternately when you have $X_1 = 4$ you have spent 170, there is a 230 that is remaining. So from this you get again $X_2 = 1$, so from the 230, you have spent 80. So 150 remains and for 150 you would get $X_3 = 4$ which means you spend 90, you have another 60 so that would give us $X_4 = 1$. So there are 2 solutions that we have here. One is 3144 and the other is 4141. 3144 would mean 3144 and the other would mean 4141. Both have the same 0.4788 reliability.

(Refer Slide Time: 21:35)



We look at one more example where we consider decision variables taking discrete value I get s of i which is a salvage value. I have to buy a new one so – C or s of 0, they mean the same. Now I have a new machine with age 0, so I have + p of 0. I have the profit associated with a new machine. I have a maintenance cost associated with the new machine. Then at the end of the year I go back. I have to make a decision f n – 1 star but the age of the equipment is one at the end of the year. So I have f n – 1 star of 1. So these are the recursive relationships with which we will work and then we will make a table and proceed with our calculations.

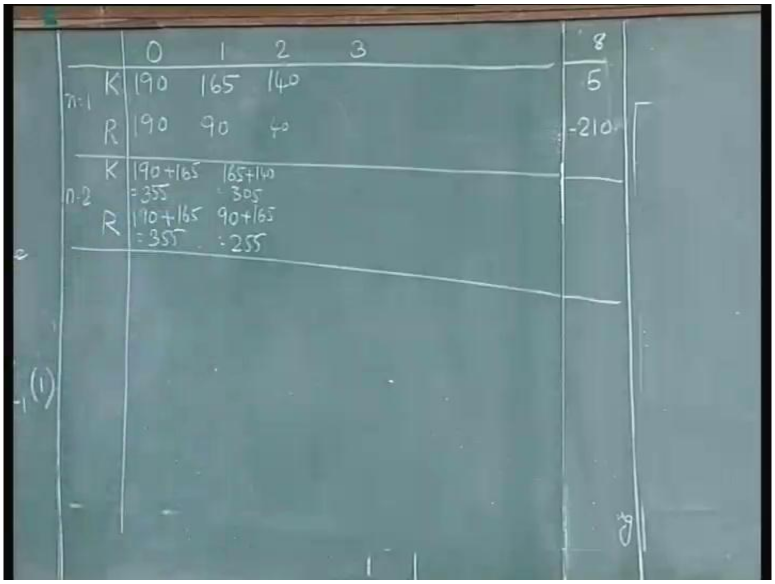
(Refer Slide Time: 22:54)



	i = 0	i = 1	i = 2	i = 3	i = 4	i = 5	i = 6	i = 7	i = 8
N = 1 K	190	165	140	123	95	75	45	25	5
R	190	90	40	-10	-60	-110	-110	-210	-210
N = 2 K	190 + 165 = 355	165 + 140 = 305	140 + 123 = 263	123 + 95 = 218	95+75 = 170	75+45 = 120	45+25 = 70	25+5=30	
R	190 + 190 = 380	90 + 165 = 255	40 + 165 = 205	-10 + 165 = 155	-60 + 65 = 105	-110 + 165 = 55	-110 + 165 = 55	-210 +165 = -45	
N = 3 K	190 + 305 = 495	165 + 263 = 428	140 + 218 = 358	123 + 170 = 293	95 + 120 = 215	75 + 70 = 145	45 + 30 = 75		
R	190 + 305 = 495	90 + 305 = 395	40 + 305 = 345	-10 + 305 = 295	-60 +305 = 245	-110 + 305 = 195	-110 + 305 = 195		
N = 4 K	190 + 428 = 618	165 + 358 = 523	140 + 295 = 435	123 + 245 = 368	95 + 195 = 290	75 + 195 = 270			
R	190 + 428 = 618	90 + 428 = 518	40 + 428 = 468	-10 + 428 = 418	-60 +428 = 368	-110 + 428 = 318			

This is n = 1, one more stage to go.

(Refer Slide Time: 22:49)

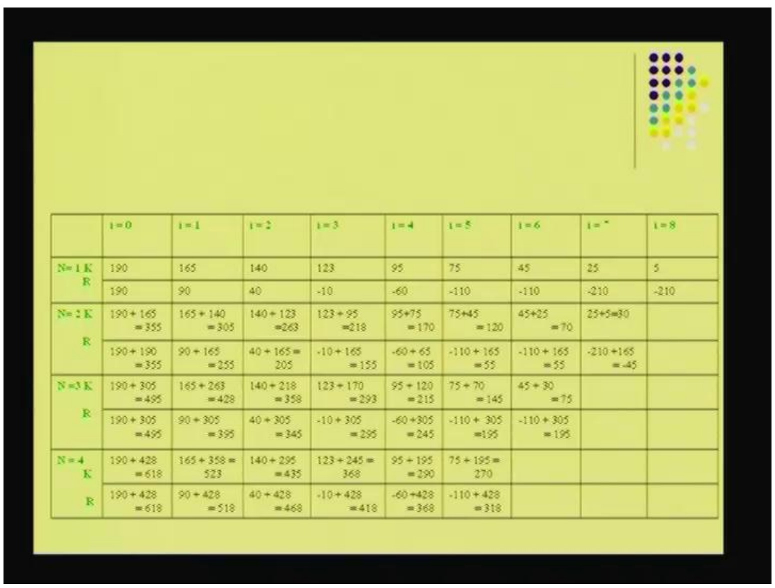


Decision to keep (K) or replace (R), the age is age is given by 0, 1, 2, and 3 and so on. So we get into the situation where the age is 0 and I have one more stage to go. I keep the machine. When I keep the machine, my decision would be here. So when I keep it as p of i – m of i, 200 – 10 which is 190, I do not have anything else to do. So I have 190 which come up. When I have one more year to go and I have a brand new machine, I already have a brand new machine so even if I decide to replace it, we assume that when you sell a brand new machine you get 800. You incur another 800 to buy. So they cancel out and then you have a new machine. You have p of i and m of i so you will get 200 – 10 which is again 190. When I have a machine that is one year old, I have p of i – m of i, 180 – 15 which is 165. When I decide to replace, I sell this. So I get 700, I have to buy a new one. I spend 800. So net is – 100 and then I made 200 – 10 = 190. So 190 – 100 is = 90. So you have 90. So we can complete the rest of this table and we will show it again. We do not have this f n – 1star coming in because we have only one more stage to go. So when the age is i, we decide to keep the value will be p of i – m of i and when we decide to replace, it will be s of i – C + p of 0 – m of 0. So that is shown as a first table here. For example the 190, 190, 165 and 90 can be seen here. When i = 2, we get 140 which is this 160 – 20 and to replace we would get a 40 which comes like this.

650 + (– 800) is = – 150 and then you get a new machine + 190, so 190 – 150 is = 40 and that 40 is shown. Like this we complete this table and go up to i = 8 to write 5 and – 210. Now when we have n = 2; 2 more stages to go and I decide to keep. Now when the age of the machine is 0, even though I have 2 more stages to go, I decide to keep. So the expenditure, if the age is 0 would simply be p of 0 – m of 0. This means whatever be the situation if I decide to keep for that age; this is going to be 190. When I decide to keep a 0 1at the end of the year, the age will be 1. So I go back to the previous table and when the age is 1, the best decision here is 165. So I get 190 + 165 which is 355 here and that 355 is shown there. When I decide to replace, as long as the age is 0 for that year, the value is 190. At the end of the year, the age is 1, so go back to the previous table. The best decision is again 165 so you get 355. When I have a 1 year old machine that I decide to keep the money will be 165 for that year and then at the end of the year the age

would become 2, so go back to the previous table and see what the values of 2 are. That is shown here as 140 and 40. So now coming back when I have n = 2, no matter what this n is, if I decide to keep and if the age is 1 the profit is 165 which is shown here which comes here. The age being 1, I decide to keep at the end of the year, the age will become 2, so go back to the previous table and see for age = 2. The best decision is to further keep and add a 140. So this will be 305. When n = 2 and I decide to replace. The moment I decide to replace, no matter what n is if the age is 1, then I get 90 for that year. When I replace at the end of the year, the age of the equipment is 1, so go back to the previous table and see what is good here so 90 + 165 = 255, so that is shown here as 305 and 255 respectively. Similarly you can work it out. But you can work it only up to i = 7 here, because we cannot work out i = 8 because you need i = 9 for this. Wherever you keep, you realize that you are picking something from the previous table, but the one to the right because age goes up by one. Since we have data only up to i = 8, we cannot do i = 8 here because that would require numbers for i = 9 of the previous table. So you do only up to i = 7 and write this. Similarly we can continue with this and all these are shown in this table now.

(Refer Slide Time: 29:25)



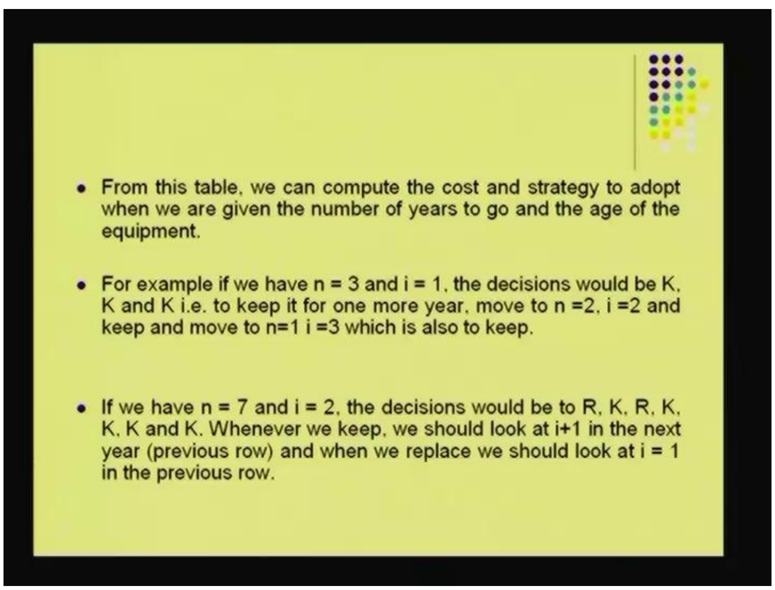
	i = 0	i = 1	i = 2	i = 3	i = 4	i = 5	i = 6	i = 7	i = 8
N= 1 K	190	165	140	123	95	75	45	25	5
R	190	90	40	-10	-60	-110	-110	-210	-210
N= 2 K	190 + 165 = 355	165 + 140 = 305	140 + 123 =263	123 + 95 =218	95+75 = 170	75+45 = 120	45+25 = 70	25+5=30	
R	190 + 190 = 355	90 + 165 = 255	40 + 165 = 205	-10 + 165 = 155	-60 + 65 = 105	-110 + 165 = 55	-110 + 165 = 55	-210 +165 = -45	
N =3 K	190 + 305 = 495	165 + 263 = 428	140 + 218 = 358	123 + 170 = 293	95 + 120 = 215	75 + 70 = 145	45 + 30 = 75		
R	190 + 305 = 495	90 + 305 = 395	40 + 305 = 345	-10 + 305 = 295	-60 +305 = 245	-110 + 305 =195	-110 + 305 = 195		
N = 4 K	190 + 428 = 618	165 + 358 = 523	140 + 295 = 435	123 + 245 = 368	95 + 195 = 290	75 + 195 = 270			
R	190 + 428 = 618	90 + 428 = 518	40 + 428 = 468	-10 + 428 = 418	-60 +428 = 368	-110 + 428 = 318			

For example when you are at n = 3 and you decide to keep it and i = 0 and decide to keep that 190 comes because no matter what n is if you decide to keep a 0-year-old equipment you get 190 which is from the first table which is this 190. When you decide to keep at the end, the age is one so go back to the previous table. The best is to choose between 305 and 255. To choose between 305 and 255, the best value is 305, so add 305 to 190 to get 495. Similarly when you decide to replace for that year, it is 190. The 190 comes from i = 0. Replace this 190. The moment the age is 0, no matter what n is, you get this 190. Now you have already replaced it. At the end of the year the age is one go back to the previous table and realize that 3 not 5 is better so 190 + 305 = 495. You will also realize that this column will have the same number coming in for all of them. It makes more sense to start looking at it from the second column where the numbers will be different. Now when you are at n = 3, you can proceed only up to i = 6, you cannot go beyond because you cannot do i = 7 because you do not have i = 8 for n = 2

(Refer Slide Time: 30:58)

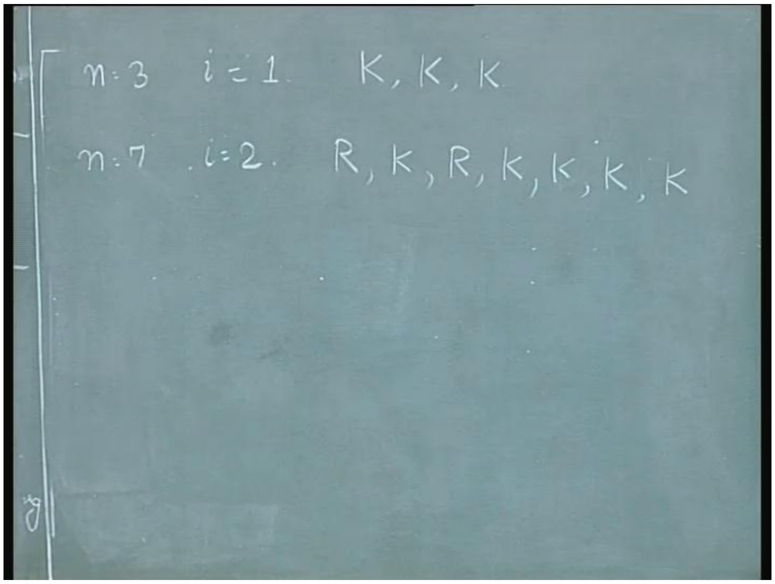
Like this we proceed and do it till n is = 8 because we have data here for i = 8 up to 8 years now, and we can go back and complete this table up to n = 8 and you have only one entry there. Once this table is completed we can start evaluating the decisions depending on where we are.

(Refer Slide Time: 31:25)



For example, from this table we can compute the cost and the strategy when you are given number of years to go and age of the equipment.

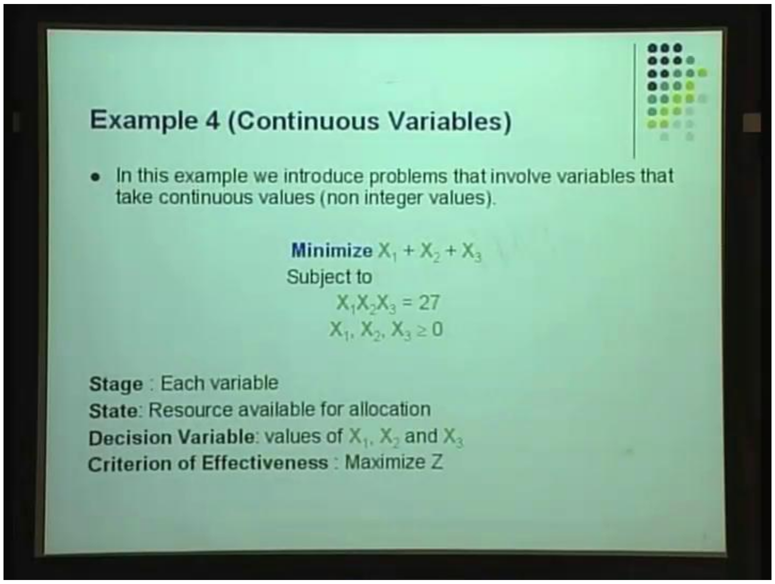
(Refer Slide Time: 31:33)



Fo example if we are looking at n = 3, 3 more years to go and the age of the equipment is 1. Let us go back and see what the decisions are. Go back to the previous table when n = 3, I have i = 1, now for n = 3; i = 1 when we keep the value as 428. When we replace, the value is 395, higher one being 428. We decide to keep for the first year, so we decide to keep. Now once you decide to keep at the end of that year, the age goes up by 1, so go to the previous table n = 2 and look at the age also = 2. So n = 2, age = 2, to keep it is 263, to replace it is 205, so decision again is to keep. Now the decision is to keep at n = 2, i = 2, so age becomes 3 at the end of that year. So go back to n = 1 and i = 3. So decision would be once again to keep, so in this case when n = 3 and i = 1, the decision would be to keep. When you have n = 7 and i = 2 let us see what we do. Go back to the table now at n = 7 and i = 2, is this next one. We have 763 and 768, so the better decision would be to replace for the first year to replace. When you replace for n = 6, you have to look at i = 1, because you have brought a new machine so for n = 6 look at i = 1, so decision would be to keep. Now when you decide to keep at i = 1, then the age goes up by one more, so I have to look between 558 and 563, so the decision again is to replace at n = 5, decision is to replace.

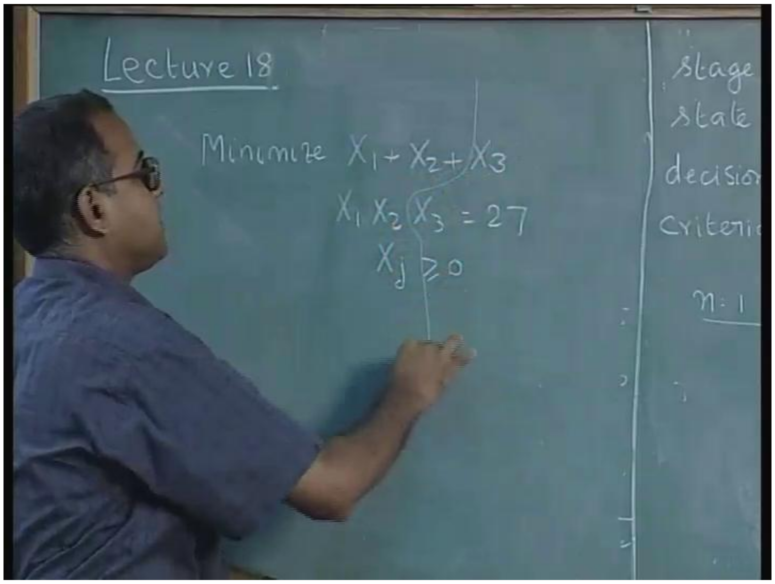
Now go back to n = 4 and look at age = 1, because you have replaced it so n = 4, age = 1 is between 523 and 518, 523 being higher decision is to keep. Age becomes 2, go back to n = 3 and i = 2, the relevant values are 358 and 345 so once again the decision is to keep. Now go back to n = 2; i = 3, so 218 and 155, once again the decision is to keep. Now at n = 1 between 95 and – 60 the decision again would be to keep so these would be the decisions. For this equipment replacement problem, now once the table is drawn, now depending on the given values of n and I, we could go back and look at the table and try to find out what the corresponding decisions are. The only drawback in some sense is that when we begin the problem with 8 years data, then with every stage we are able to evaluate one less. So we can go only up to n = 8 in this problem and we cannot go beyond that and we also realize that the alternatives that we have are also becoming lesser and lesser. Alternately if we have more data here and the other thing that will happen is that the table will become too cumbersome. So depending on the data that is given depending on the values of p, i, m, i and s, i_1 can construct this table, part of this table is shown in the board while the complete table is shown in the power 0 slide. Then depending on the value of n and the value of i, depending on how many stages we have, how many years to go and the age of the equipment, we go back into the table and then we would complete this. Now coming back to this for n = 7 and i = 2, the relations are shown. Whenever we decide to keep, we go back to the previous stage, go back to the previous table and shifted i + 1, move 1 to the right and so on. When you replace, you should look at i = 1 because you bought a new machine at the end of the year. The age of the machine will be 1.

(Refer Slide Time: 36:26)



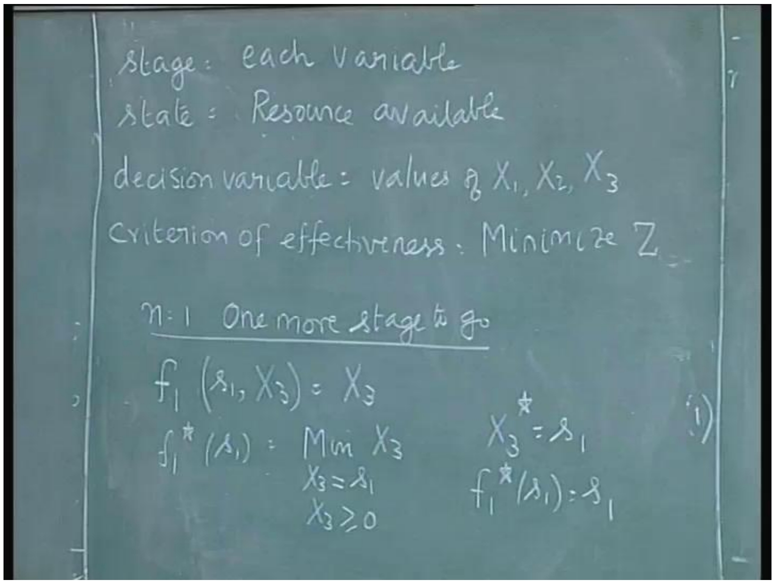
Now from after this example we now start looking at problems where we look at variables that have continuous values. So far we have seen 3 examples and in all the 3 examples the decision variables had discrete integer values and now we look at examples where the decision variables would take continuous values. Let us look at some simple problems that we are familiar with. We could even know the solution to these problems but then we will try to show how we apply dynamic programming to solve these kinds of nonlinear problems.

(Refer Slide Time: 37:10)



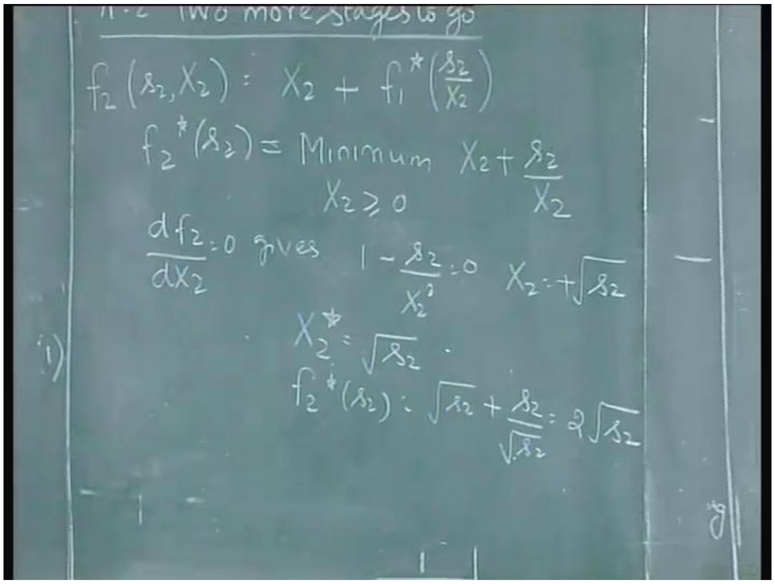
The problem that we choose is to minimize $X_1 + X_2 + X_3$ subject to X_1, X_2, $X_3 = 27$, X_j greater than or equal to 0. There is no integer restrictions on the $X_{j,}$. X_j's can take any continuous values. It is a nonlinear problem. Because of the constraint which exhibits nonlinearity there is a product form in the constraint. Let us first define the stage, state, decision variable and the criterion of effectiveness for this problem before we proceed. Here there are 3 variables and we will be solving them one at a time. So here in all these kind of problems, the stage is each variable is a stage and we will be solving for one variable at a time the state always represents the resource that is available. There is a single constraint. We do not know what this constraint physically represents. In all the 3 earlier examples, we knew there is some physical entity associated with the problem like age of the machine or the money that is available or the city from which the person moves. We do not know what this constraint is so we simply say that the state variable is a resource available, decision variable are the values of X_1, X_2, X_3 and the criterion of effectiveness is to minimize Z. We now start the problem with n = 1 more stage to go.

(Refer Slide Time: 39:05)



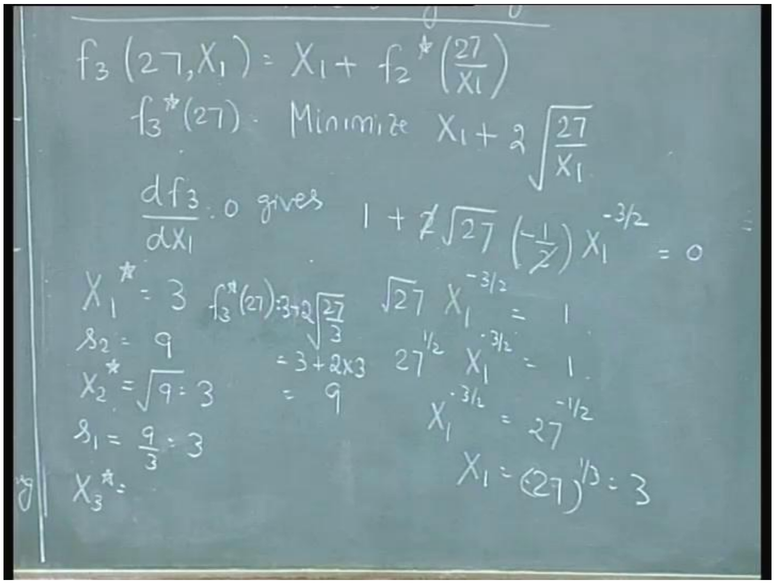
When we have one more stage to go we are solving for the first variable. Basically we are at this part of the problem. So we try to minimize an X_3, subject to X_3 = something, X_3 greater than or equal to 0. So $f_{1,}$ one more stage to go, I have s_1. Let us say, I am trying to solve for X_3. This will be = X_3 because, this is the function that we are trying to minimize. f_1 star of s_1 is a minimum value of X_3 subject to $X_3 = s_1$, X_3 greater than or equal to 0. What we are trying to solve here is minimize X_3 subject to $X_3 = s_1$ is available for allocation to X_3 which means something has been allocated to the product X_1 X_2 and available for X_3 And because it is an equation we have to allocate everything that is available to this variable X_3. If this was an inequality, then we have a choice of whether we allocate everything to X_3 or something less and so on. Now because we have an equation here, all of s_1 has to go to X_3 and X_3 has to be greater than or equal to 0. Now we are going to make an assumption that since X_1, X_j is greater than or equal to 0, we assume that the state variables cannot be negative. So state variables themselves are greater than or equal to 0. So straightaway for this case you have X_3 star = s_1, because you have to allocate everything. f_1 star of s_1 will be s_1 because we allocate everything. So the minimum value that it will take is s_1, it cannot take anything lower than that.

(Refer Slide Time: 41:11)



Now when we have 2 more stages to go n = 2, 2 more stages to go, f_2 of s_2X_2. Here we are somewhere in the middle trying to solve for this, so I have s_2 resources available which I have to allocate to X_2 in a certain form. Now this will be X_2 plus because the objective function is X_2 plus if I have s_2 and I allocate X_2 here then s_2/X_2 is carried as s_1 because of the product form of this. Whatever is carried as s_1 is actually s_2/X_2, so this will become f_1 star of s_2 /X_2. Once again we assume when we have 2 more stages to go, that there is some s_2 available here which is obtained after something is allocated to X_1. Please remember it is not subtraction, it is a product form. So something is allocated to X_1 and there is an s_2 that is available for allocation to X_2 and X_3. So we allocate something to X_2. So s_2/X_2 will go to X_3 or we are looking at a constraint which is like X_2, $X_3 = f_2$. So if this X_2 goes then the balance will be s_2 /X_2 will go as X_3. So you have f_1 star of s_2 /X_2. For f_2 star, the best value would be that value of X_2 which minimizes for minimum of X_2 plus. We know that f_1 star of s_1 is s_1 itself, so f_1 star of s_2 /X_2 will be s_2/X_2 subject to the condition, of course X_2 greater than or equal to 0, less than or equal to, need not be because it is a product form. So it need not be less than or equal to f_2. Because of the product form if it were an addition, then you could go back and say X_2 should be less than or equal to f_2 here. This we need to minimize and the minimum is obtained by equating the first derivative to 0, 2 = 0 gives, 1 – s_2/X_2 square = 0 from which X_2 = + root of s_2 + root of s_2 because X_2 is greater than or equal to 0. So X_2 star is = root of s_2 and f_2 star of s_2 is X_2 which is root of X_2 + s_2/root of s_2 which is 2 root s_2.

(Refer Slide Time: 44:25)



Now we have n = 3, 3 more stages to go which means we are making a decision for variable X_1. So we are trying to solve $X_1 + X_2 + X_3$ subject X_1 into $X_2 = 27$, so something out of the 27 goes to X_1 and because of the product form whatever is carried to X_2, X_3 is $27/X_1$ so we have f_3. We start with 27 and we decide to allocate X_1. This will be $X_1 + f_2$ star of $27/X_1$ because from this 27 we have given an X_1 here, so whatever gets carried as s_2 is $27/X_1$. So now f_3 star of 27. So the best decision will be that value of X_1 which minimizes $X_1 + f_2$ star of s_2 is 2 times root of s_2. So f_2 star of $27/X_1$ is 2 times root of $27/X_1$. Now this the minimum value of X_1 which is got by differentiating with respect to X_1 and setting the first derivative to 0. So $df_3/dX_1 = 0$, gives 1 + (2 root 27), this X_1 to the power – 1/2, so – $1/2X_1$ to the power – 3/2 = 0. Now this 2 and this 2 gets cancelled so we have root of 27 into X_1 to the power – 3/2 is = 1. Now this is 27 to the power 1/2. So 27 to the power half into X_1 to the power – 3/2 = 1. X_1 to the power – 3/2 is = 27 to the power – 1/2 from which we would have X_1 = cube root of 27, 27 to the power 1/3 which is 3. The best value of X_1 which is X_1 star is given by X_1 star = 3, now f_3 star of 27 will be 3 + root of 27/3 which is 3 plus. So 2 root 27/3, 3 + 2 times root of 27/3, so 3 + (2 into 3), this is = 9. X_1 star = 3, which means we have given 3 to variable X_1. The product whatever goes as s_2 is 9, so s_2 is 9. Now X_2 star is = root of s_2 which is root of 9, which is 3 + 3 and because of this, s_1 will be s_2/X_2. So we have 9/3 which is 3 and X_1, X_3 star, X_3 star is = s_1 which is also = 3. So we have $X_1 = 3$ $X_2 = 3$; $X_3 = 3$. The minimum value will be 3 + 3 + 3 which is = 9. So we have actually solved this problem to get the solution even though the problem to the solution is well known even by looking at the problem.

Now there are a couple of other things that we need to address. One is this. When we made this differentiation and said, we set the first derivative to 0 we got the value of $X_1 = 3$. We have to make sure that $X_1 = 3$ actually minimizes. So that is verified by looking at the second derivative and from this, one would realize that the second derivative would be positive. Because there is a minus term here and differentiating further would give us another – 3/2 X_1 to the power – 5/2 which would make the term positive for X_1 positive. When X_1 is positive, X_1 to the power negative term also is also a positive number, so second derivative is positive indicating that it is a minimum. Here again the first derivative = 0, it would give us this $1 - s_2/X_2$ square = 0. If we do a second derivative we would again have a + term coming in and X_2 to the power something. X_2 to the power something is again a positive term, so here again this value + X_2 minimizes this. We have to ensure that not only do we set the first derivative to 0 and get the minimum or the value minimum or the maximum, we also have to look at the sufficient condition and look at the second derivative to 0 and check that it actually minimizes in both these cases. Setting second derivative = 0, gives a positive coefficient which means these corresponding values minimize the objective function. The next thing is, in these situations, these are problems where the decision variable takes continuous values. They do not take integer values like the 3 previous examples. They take continuous values. For example it is incidental that the values were nice round integers but on the other hand if these were not 27, but these were something else then we would get continuous values here. If this was 30, then each would be cube root of 30 to get 3 times cube root cube root of 30. So the decision variable could take continuous values here.

Now in all these problems we need not separately write the recursive relationships unlike in the discrete case where wherever we use the tabular form to solve. It is necessary for us to write the recursive relationships separately. In all these cases when we work with continuous variables, the equations themselves represent the recursive. After all the recursive relationships are written and from which they are solved. So one need not explicitly write the recursive relationship nor is it necessary for one to write a general recursive form. We have done it for each and every stage as it is. The other thing that we need to look at here is there is a single constraint. Most of the problems that we will be addressing would be single constraint problems and in this case it is an equation. Because the single constraint is an equation, when we have one more stage to go and we have one variable to allocate, we would always get into a situation of $X_3 = s_1$.

Whenever we have an equation in the first stage at n = 1, we do not optimize the resource that is available for this variable. If the constraint is an equation and when we have one more stage to go, we do not optimize. We simply allocate as we did here because $X_3 = s_1$, whether you minimize or maximize or whatever be the thing, X_3 is forced to take the value s_1, so there is no optimization in this case. Whereas in the subsequent stages n = 2 and n = 3, there is scope for optimization. So here we evaluate the function and then we realize we have to minimize something. So you set the first derivative to 0 and then you optimize here and then show that actually at root of s_2 it minimizes by looking at the second derivative and showing that the second derivative is positive. So in all the problems involving equations, we do not optimize in the first stage or stage n = 1, we optimize only at subsequent stages. So whenever we have an inequality coming in as a constraint, then there is a scope to optimize right in the first stage. We need to look at those aspects as well in the subsequent examples that we will be looking at. We would also be trying to address problems which have maximization instead of minimization. Of course the methodology would more or less remain the same. It will not change except that when we address maximization problems and when we optimize, we have to go back and say that the second derivative is negative at the value. So in that case we need to again look at the second derivative, set it to 0 or evaluate the second derivative at the optimum 0. Show that the second derivative is negative, indicating maximum. So in the next lecture, we will be looking at some more examples which involve maximization, which would involve inequalities and which would also involve linear functions.