Fundamentals of Operations Research

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Lecture No. # 17

Assignment Problem – Other Issues

Introduction to Dynamic Programming

We begin this lecture continuing our discussion on the assignment problem.

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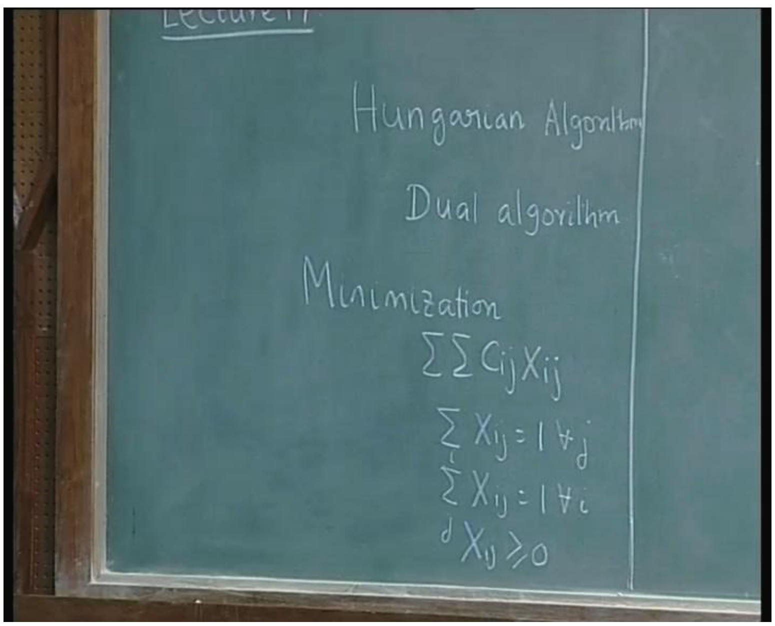

We have already formulated and solved the problem using the Hungarian algorithm. We have also seen how Hungarian algorithm works and how it (Refer Slide Time: 01:42) provides the optimal solution to the problem. We have also shown that Hungarian algorithm is an example of a dual algorithm like dual simplex. It finds a feasible solution to the dual of the problem and once the primal becomes feasible the optimal solution is obtained. There are still a couple of things to be seen with respect to the assignment problem.

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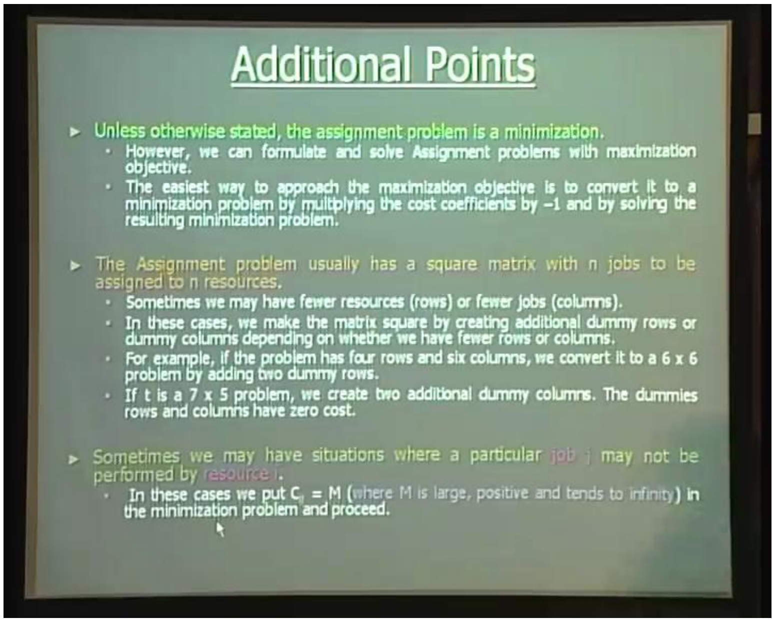

The first thing is (00:02:07) unless otherwise stated; the assignment problem is a minimization problem. Remember that the assignment problem has a minimization objective of minimizing double sigma C_{ij} X_{ij}. There could be situations where we have to solve a maximization assignment problem and what we do there is we convert it into a minimization problem by simply multiplying the objective function coefficient with – 1.

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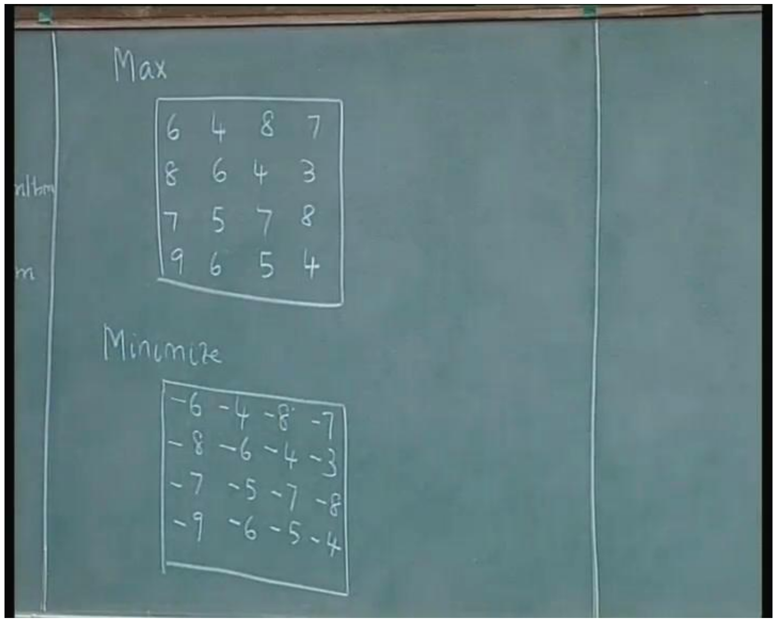

For example if we had a maximization assignment problem, say with numbers such as this, we convert it into a minimization problem by multiplying the objective function coefficients with –1 and then we can do row column subtraction from this matrix and proceed as it is. Normally it is not recommended to do the row column directly. It is always better to convert it to a minimization problem. Treat this as row minimum and subtract from every element and so on.

The assignment problem usually is a square matrix, we have we have seen. This is a 4/4 example where 4 jobs have to be assigned for 4 people unless otherwise stated. The assignment matrix is a square matrix but we could have situations where we could have say for examples 5 jobs and they might have to be allocated only to 4 people.

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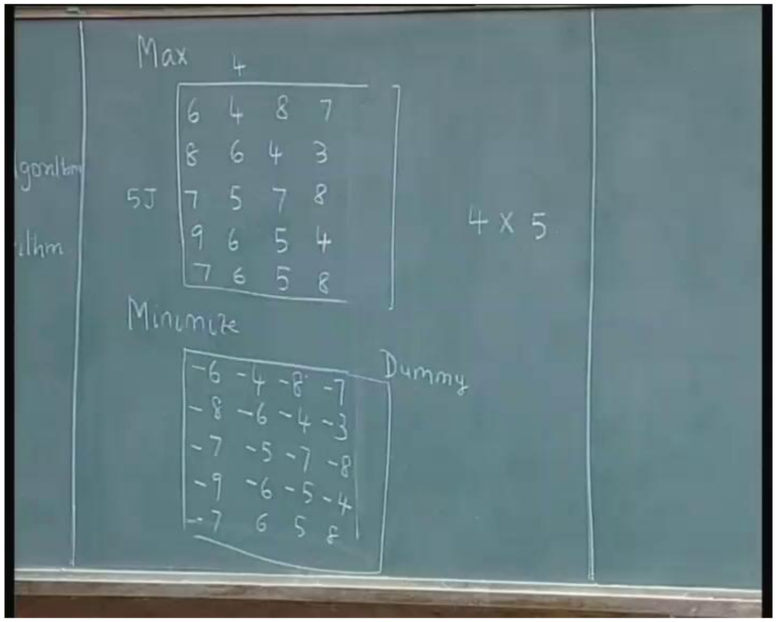

For example we could have a 5th job which might have something like 7, 6, 5 and 8. Now in situations such as this we need to convert this into a square matrix by adding a dummy as we did in the transportation. Dummies will have 0. We then proceed after converting this into a square matrix we proceed as we normally do. This problem being a maximization problem, we go back and we need to write this. We do not put a dummy here. This will become – 7, 6, 5 and 8 and then we could also put a dummy here and converted it to a 5/5 problem.

Sometimes we could have an assignment problem such as 4 rows and 5 columns in which case we add an additional row which will be a dummy row, make it into a 5/5 problem and then solve. The only thing that will happen is if we have an unbalanced assignment where the number of rows and number of columns are different then either all jobs cannot be assigned to all people or all people will not get a job. For example if I have 5 jobs and 4 people and then we make it into a 5/5, now whichever job is allocated to dummy is effectively not allocated at all. Similarly if there are 5 people and 4 jobs then in the final solution 1 person does not get any assignment. But when we solve an assignment problem, we have to convert it into a square matrix and then solve. There could be situations where a particular job may not be performed by a particular resource.

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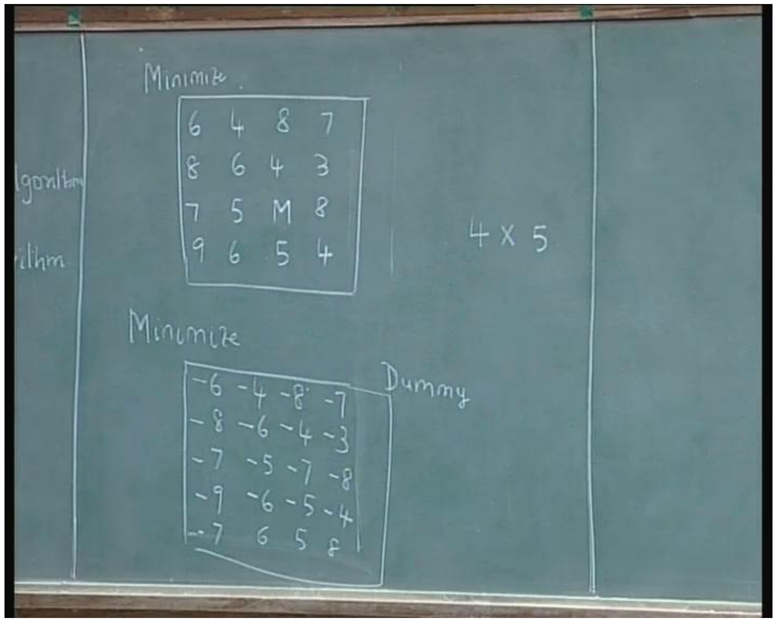

For example if this is a minimization problem, and say it is a 4/4 minimization problem and suppose we say that this job cannot be performed by this person, and then we put a big M here. Big M as we defined M in the simplex once again, M is large and positive when tending to infinity for a minimization problem. So that when we make the allocations we will not have an allocation at all here. This M + constant = M, this M – any constant is = M. So this will never get 0. Therefore this can never be a place for allocation. So this way we handle situations where a particular job cannot be handled by a particular person.

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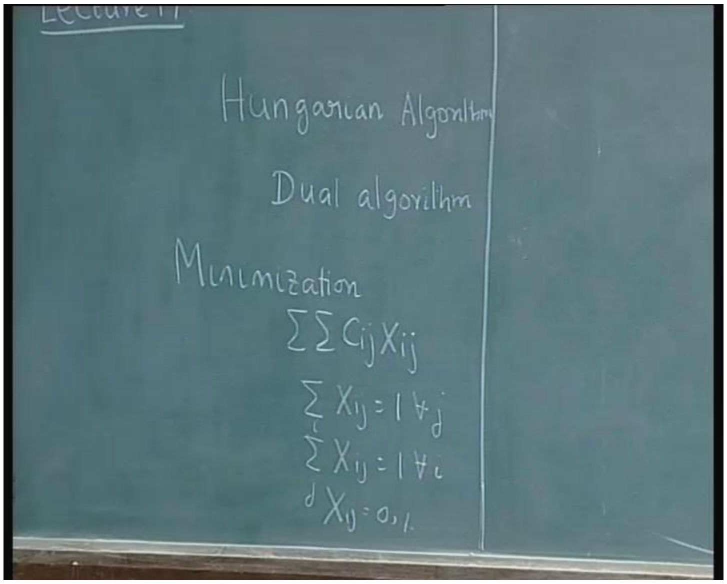

The last thing perhaps we need to look at is this. When we formulated the assignment problem, this is the objective function. We have constraints sigma $X_{ij} = 1$; sigma $X_{ij} = 1$, we defined it as $X_{ij} = 0, 1$. So summed over i for every j and summed over j for every i. This is the formulation of the assignment problem that minimizes the total cost of assignment. Now these 2 constraints indicate that every job goes to exactly 1 person and 1 person gets exactly 1 job. It is a square matrix with n square decision variables.

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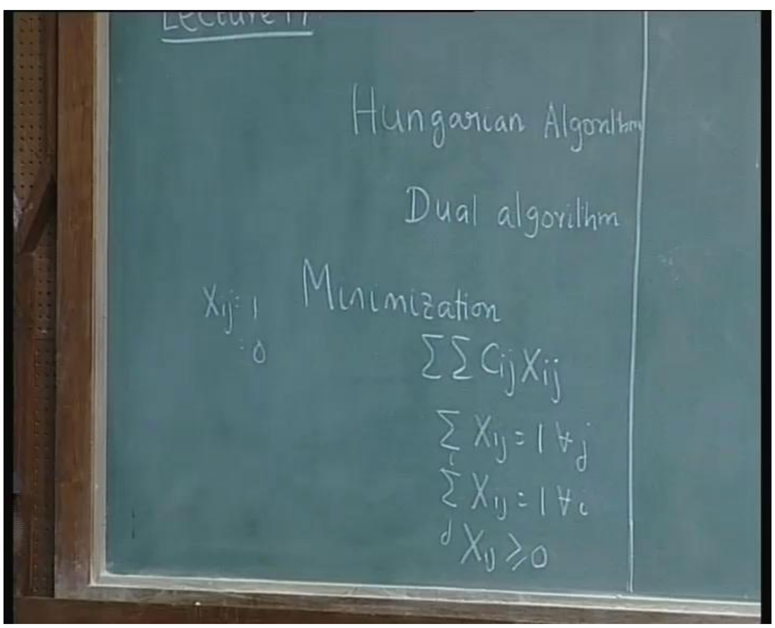

Now one more thing is 0, 1 is now relaxed and this problem is written as greater than or equal to 0. It becomes a liner programming problem and we solve this linear programming problem by simplex or any other liner programming solver. Then we would get the solutions where these X_{ij}'s actually take either 1 or 0. We will not have a situation where these X_{ij}'s will take fractional values. X_{ij} will continue to take 0 or 1, even if we relax the 0, 1 restriction and solve the resultant l 2. This is very similar to what we saw in transportation problem. We also saw in the transportation that we had a_i's and b_j's and we said that as long as the a_i's and b_j's are integers we said this solution to the transportation problem would be an integer. Now here we have 1 which is an integer, so the solution to the assignment problem even if we solve it as a LP would still be integer value which is 0 or a 1. However we do not solve this using a liner programming because we have a much degenerated LP. There are n square decision variables out of which only n variables are basic at the optimum. So we use different method such as the Hungarian algorithm and solve the assignment problem. So with this we come to the end of our treatment on the assignment problem and we look at the next topic which is called dynamic programming.

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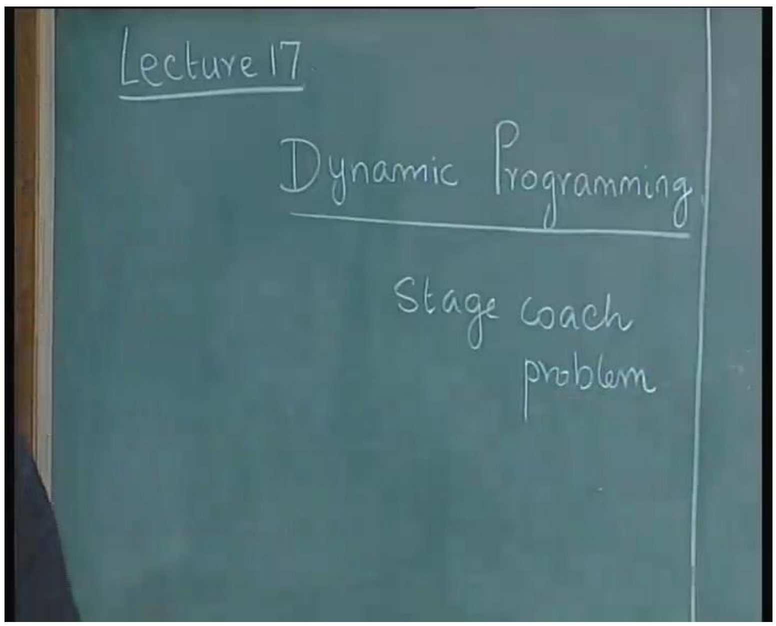

Now under dynamic programming we are going to try and solve different examples to show to you the various terminologies that are associated with DP. In dynamic programming, we divide the problem into several stages and solve the problem for each stage.

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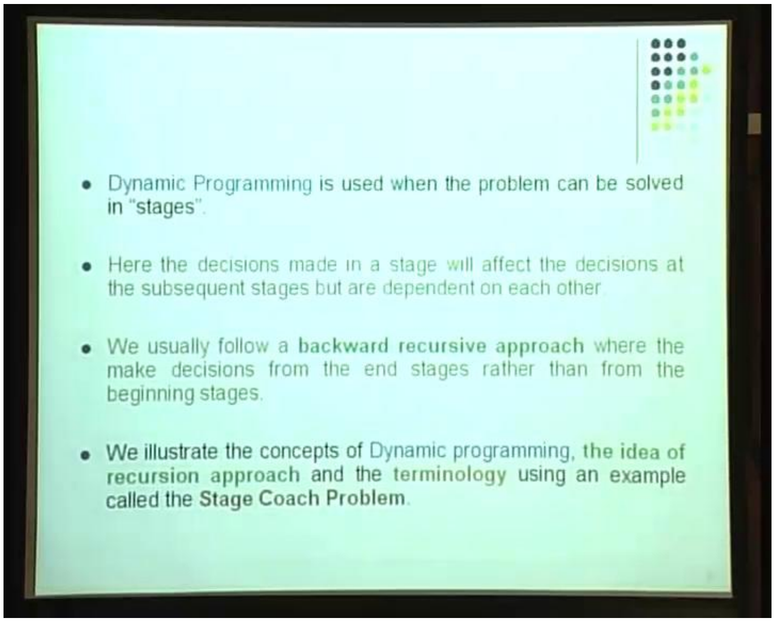

Now we go back and see the dynamic programming is used when the given problem can be solved in stages. There the decisions are made in a particular stage which will affect the decisions at the subsequent stages but they are dependent on 1 another. We also called something called backward recursive approach in dynamic programming. What we are going to do is we will illustrate the concepts of DP idea of the recursive approach and the terminologies that we are using including the notation by using an example which is called as Stage coach problem. We will solve the stage coach problem first.

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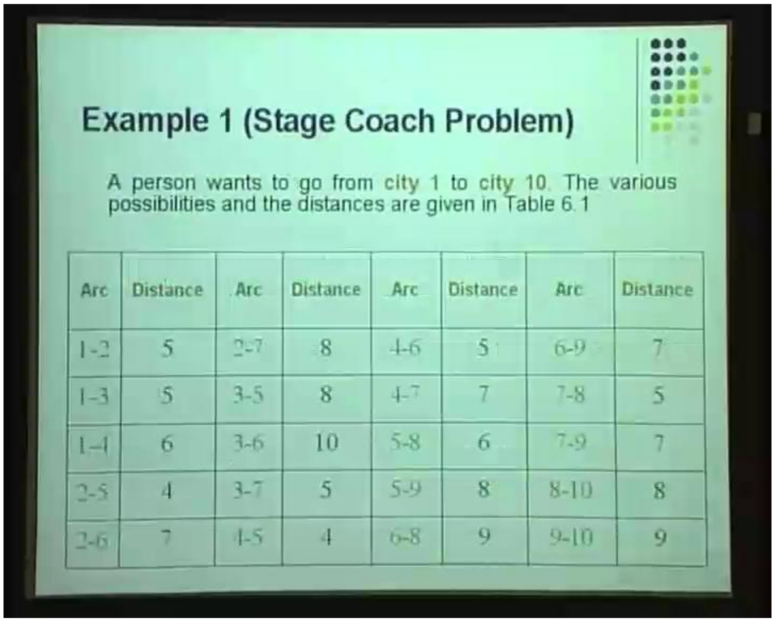

We explain the terminologies as well as the general framework to solve all DP problems. Now stage coach problem is essentially the problem of finding the shortest distance from a given source to a given destination which is nothing but the shortest path problem so we take a stage coach problem like this.

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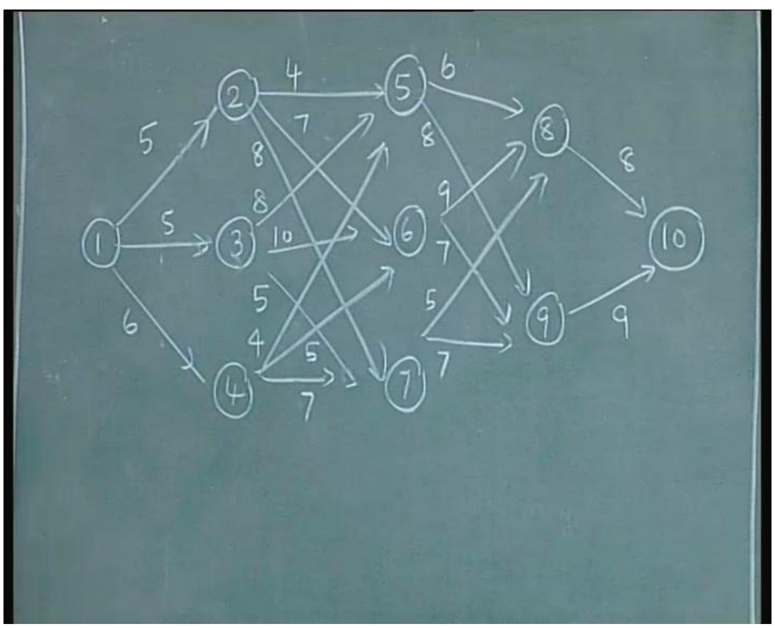

Now we have 10 nodes or 10 cities that a person has. The person is right now at the start node or city node and wishes to go to the destination node or city 10. This is the network that is connecting the nodes. This is a problem where you can easily decompose the decision making into different stages and you we will also realize that here you do not actually have connectivity between say, 1 and 5 or 3 and 8 and so on. You can easily decompose this into several stages and you find connectivity from the nodes of$_1$ stage to the next stage. It is a simple shortest path problem but we are going to use in this problem to illustrate the concepts behind dynamic programming and the terminologies that we are going to follow. Now the distances are also given here. So we have 5, 5 and 6.

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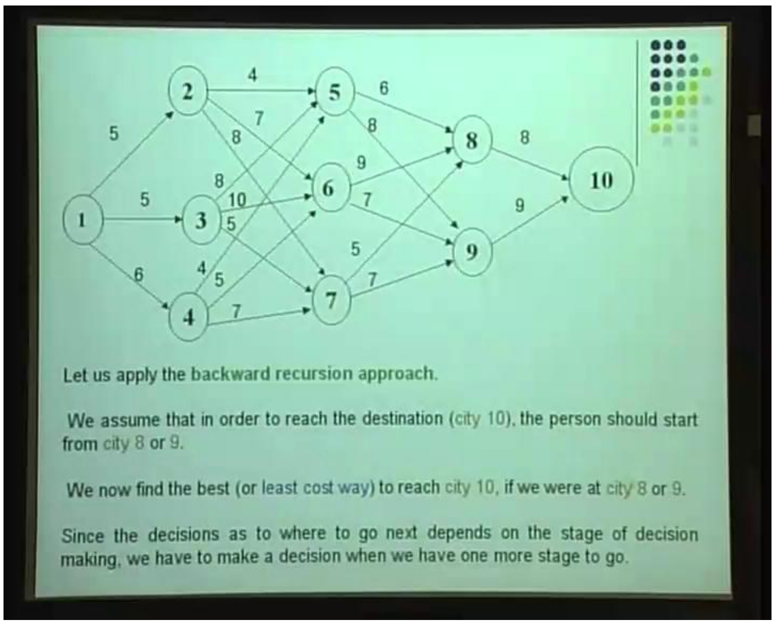

With 5 is 4, 7 and 8, 8, 10 and 5, 4, 5 and 7, 6, 8, 9, 7, 5, 7, 8 and 9. We follow something called the backward recursive approach. In order to reach 10, now this person has to either start from 8 or 9 and then the person has to go to 10. So we are right now assuming that the person is here and we call this 1 more stage to go and the person is either at 8 or 9. Now the node or the place where the person is is called the state of the system. So the state of the system is either at 8 or 9 and then given the state of the system, we try to find out the best decision that the person has to make at that stage to go from 1 stage to the next stage and since we are following a backward recursion we are going to say when we are here, we have 1 more stage to go. When we are here we have 2 more stages to go and so on. So the first table would look like this.

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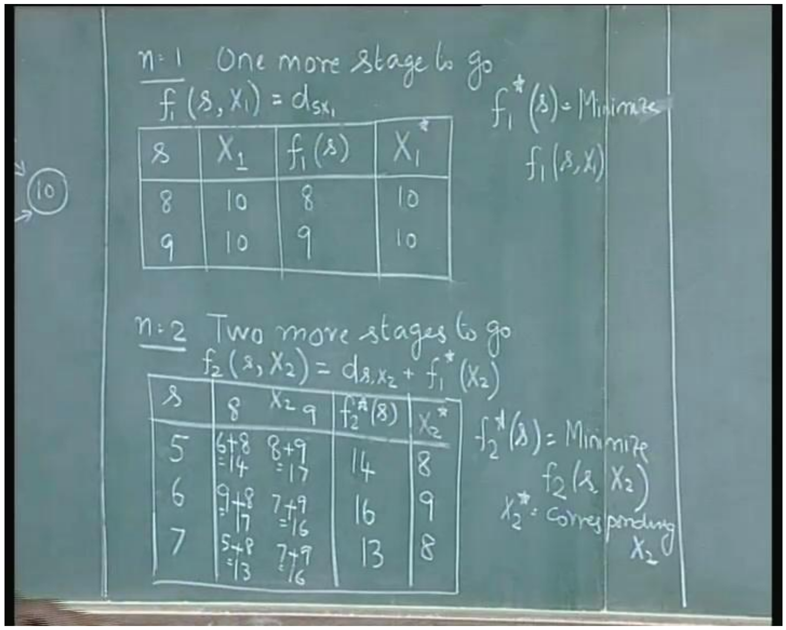

We will have n = 1, one more stage to go. Now s is called the state variable and then we are here with 1 more stage. To go the state variable it is either at 8 or at 9. The person can be either at node 8 or at node 9. Then the person has to make decision which we call X_1, X_1 as the decision that is made when you have 1 more stage to go. Now whether the person is at 8 or the person is at 9 the decision is very clear that is to go to 10 because there is no other destination. So in this case the $X_{1,}$ the decision is to go to 10. Also the decision is to go to 10 and then the value of the objective or the distance is the distance minimization problem. So the distance that the person will cover is given by f_1. f_1 is the distance function. Now f_1 is far from s. In this case from 8 to 10 the distance is 8 and from 9 to 10 the distance is 9 and we have X_1 star which is called the best decision which will be to go to 10 whether you are going from 8 or from 9.

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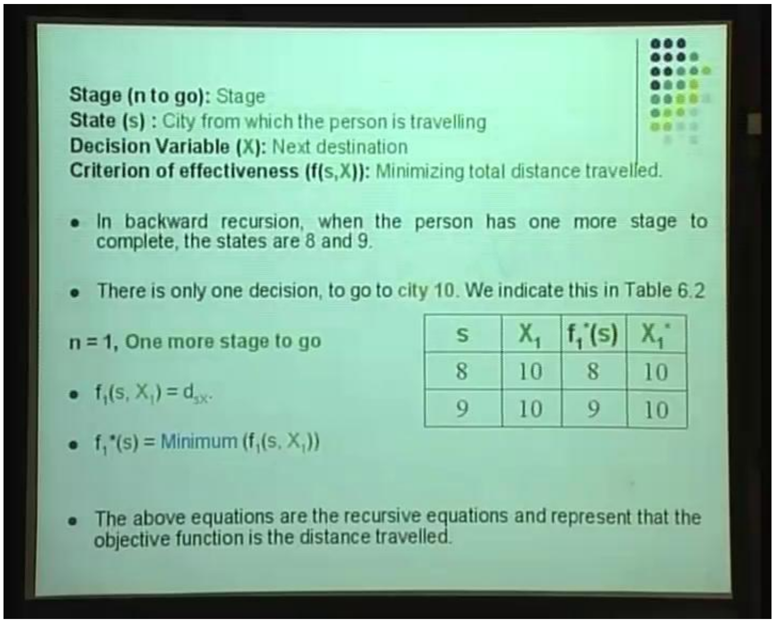

Since we have only 1 alternative for X_1, it automatically becomes X_1 star which is the best decision. So if the person is at 8, there is only 1 choice which is going to 10. So that is best choice so the person would travel a distance of 8 and go to 10 which is the best choice. If the person is at 9 then there is still only 1 choice to go, 10 and the person will spend 9 or will travel 9 units and go to X_1 star which is 10. Now when we have n = 2 or 2 more stages to go which means, we are say, somewhere here. Now this is 1 stage. This is another stage, so there are 2 more stages to go. Now you will realize that there are more alternatives than 1. When the person is here and he has 2 more stages to go, then he could either be in 5 or 6 or 7. So the state variable s takes 3 values which is 5 or 6 or 7. Now when the person is in 5, the person has to make a decision here so the decision is where to go next which are either 8 or 9. So now the X_2 which is the decision that have to be made can take 2 values which is 8 and 9.

When the person is at 5 and decides to go to 8, you incur a distance of 6 plus, already the best way from 8 to reach the destination 10 is already written in the previous table, so you add another 8 to this to repeat again. The state could be 5, 6 or 7. If the person is at 5 and decides to go to 8, now to travel from 5 to 8 you travel a distance of 6 and the best way to reach from this to the destination is already seen in the previous table. So if the person is at 8 then the best decision is to go to 10 to incur another 8. 6 + 8 = 14. Now if the person is at 5 and decides to go to 9 there is a distance of 8 and the best way to go from 9 to destination is already there in the previous table. So that will be 9 to destination, and an additional distance of 9. So this becomes 8 + 9 = 17. Now this is the state variable which is at 5, the person is at 5 and in order to go from 5 to the destination, the person either goes through 8 or 9 and the total distance travelled from 5 to destination is already seen here. So the best thing the person would do is to travel a distance of 14 and go via 8 because this is the minimum distance. If the person is at 6 and decides to go to 8, there is 9 here and 8 to destination is already seen, which is the same 8 which is here. So this becomes 9 + 8 = 17. If the person is at 6 and decides to go to 9, it is 7 and 9 to destination is already an additional 9 which is the same 9 here. So 7 + 9 = 16 and the minimum of them is 16, so the total value is 16 and the best decision now is to go via 9 and not via 8. Similarly if the person is at 7 then you have 5 to go to 8 and from 8 to destination is another 8 which is 13 and 7 + another 9 which is 16. The smaller 1 is 13 and the decision is to go to 8. We have now completed 2 of these stages. We have 2 more stages to go. The next the person would be here to make a decision and then the person would come back to make a decision but then since we have written a lot of numbers, let us define what we have actually written. What we have here is this. When I have 1 more stage to go I define something called f_1. I am in a state s or s_1, you could define it suitably. We call it as s and the person decides to go to X_1. Now the value of the function that we are going to have is dsX_1. So in this table the person is at 8 decides to go 10, so the value of the function is dsX_1 which is distance from 8 to 10 which is 8. Similarly from 9 to 10 which is 9. The best decision here would be f_1 star which is the best decision.

For a given state s it is going to be the best one so this will be the 1 that minimizes f_1(s X_1). In this case it will become much more obvious and clear. Now we have 2 stages to go. So f_2 I am in a state s, I decide to go to city X_2. Now that is dsX_2 for example, if I am in 5 and decide to go to 8 it is d58 plus the best way to reach from 8, which is f_1 star of 8. f_1 start of 8 is another 8 which is added here, so this will become + f_1 star of X_2. From s I go to X_2 so from s to X_2, from 5 to 8, I incur a distance of 6 and then once I reach 8, the best way to reach the destination from 8 is given by f_1 star of 8. f star 1 of 8 is this 8. So we add 6 + 8 = 14. Now when I am in state s = 5, I have 2 choices. For X_2 = 8 and 9, so I have evaluated this function as 14 and 17, so the best way to go from 5 to destination would be f_2 star of s will be minimizes f_2 of (sX_2) and the corresponding X value is written as X_2 star. When I am in 5, now the best decision f_2 star of 5, the best way to go from 5 to destination f_2 star of 5 is given by 14 and the corresponding X_2 star is 8. So X_2 star is the corresponding value. So if I am at 5 the minimum distance to travel is 14 with the corresponding value of 8 from 5 to 8, 6, 16 and next destination is 9 and if I am at 7 the best way to reach is to travel to 13 and go to 8.

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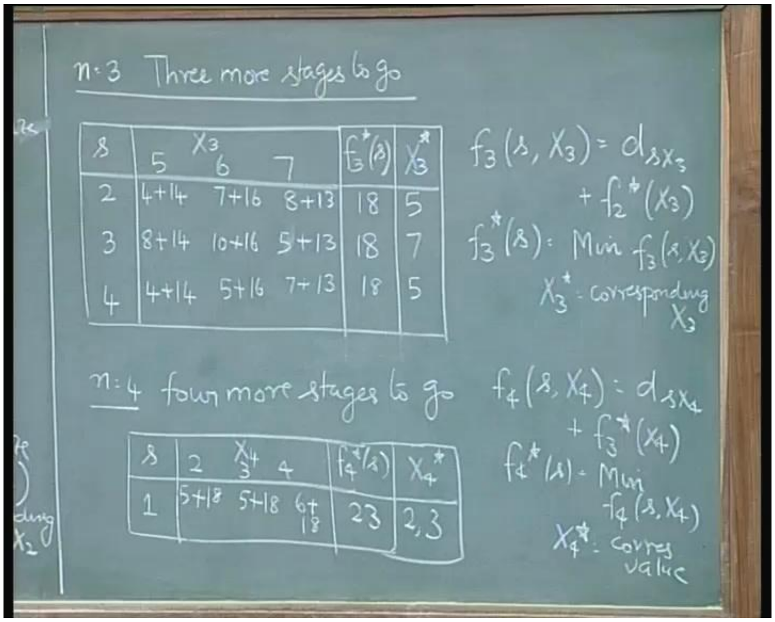

Let us continue with the next 3 more stages. I am here so I could be either in 2 or 3 or 4 now my state variable s can be in 2 or 3 or 4 my decision variable X_3 can be to 5 or 6 or 7.
So if am in 2 and I decide to go 5 from 2 to 5 the distance is 4 + 5 to destination. I already found i which I can get from the previous table, so 5 to destination f_2 star of 5 is 14. If I am in 2 and decide to go to 6 I have to travel 7 + 6, to destination which is already here, minimum value of 16 and if I am in 2 and decide to go to 7 it is 8 + from 7 to destination, it is another 13 so this is 4 + 14 = 18; 23 and 21, so the best value is 18 and X_3 star is 5.

If I am in 3 and I decide to go to 5, it is 8. 5 to destination is this 14 which is written here. So you can directly copy this 14, 3 to 6 is 10, 6 to destination is 16 which is already written here as 16, so 10 + 16 and 2 to 7 is another 8. 3 to 7 is 5. So I have a 5 here and 7 to destination is 13 so this is 22, 26 and 18 and minimum is 18 and the best decision is to go with 7. If I am at 4 then I incur 4 to 5 which is 4, 5 to destination is already here as 14. The same 14 is here. 4 to 6 is 5, 6 to destination is 16 which is here as well as here and 4 to 7 is 7, 7 to destination which is 13. The same 13 is here as well. So the values are 18, 16 + 5 = 21 and 20 so it is again 18 and the best decision is to go to 5. We also have written down the equation. So f_3, 3 more stages to go and at state s. I decide to go to X_3. This will be dsX_3 plus, we go back to the previous stage and find out the best value. So f_2 star of X_3 and f_3 star for a given s will be to minimize f_3 of X, X_3 and X_3 star is equal to corresponding X_3.

We go to the last part n = 4, 4 more stages to go which means we are here. The person is at the beginning which is in city 1, so state variable has only 1 value. We are now here. s is 1, now from 1, the person can go to either 2 or 3 or 4, so you would have X_4 which can take values 2 or 3 or 4. So 1 to 2 is 5, 2 to destination is already here, which is given as f_3 star of 2 which is another 18. 1 to 3 is another 5 and f_3 star of 3 is again 18, so 3 to destination is another 18. Now 1 to 4 is 6 and 4 to destination is also 18 so f_4 star of s is 23 and f_4 star is 2 and 3. We have to write down the equations so we write f_4 of s, X_4 is = dsX_4 + f_3 star of X_4 indicating the define s and decides to go to X_4. I travel a distance from s to X_4 + the shortest distance from the X_4 to destination is given in the previous table, so f_3 star of X_4. Now f_4 star of s is to minimize f_4 of + X_4 and X_4 star = corresponding value. Now we have solved the problem. Now the shortest distance is 23 from 1 to 10 and there are 2 paths. So we start from 1 and you could go to 2 from this or from 1 you could go to 3, you could go to 3.

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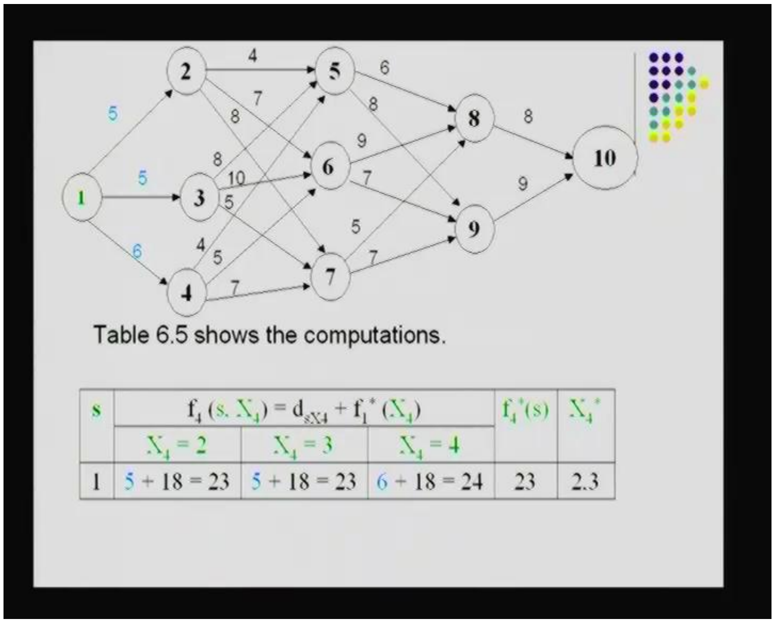

Now go back to this table and see if I am in 2. The next destination is 5. If I am in 3, the next destination is 7, so 1 to 2, 2 to 5, 1 to 3, and 3 to 7; this table will tell you if you are in 5. The next destination is 8. If you are in 7, the next destination is 8 and this table will tell you if you are in 8. The next destination is 10, so there are 2 paths here which are the shortest paths that have a total distance of 23. We also need to introduce some terminologies which are consistent with all DP problems which we will do. We have to define 4 terms which are called as stage, state, decision variable and criterion of effectiveness.

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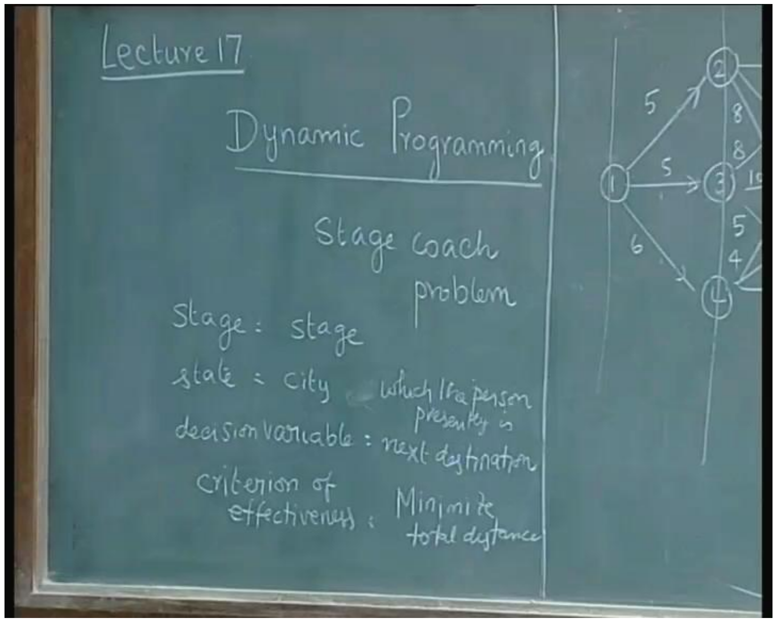

These 4 parameters are common for every DP problem and have to be defined for each problem. Now this problem stage is actually a stage in the sense that we have defined the problem in such a way that the person goes from 1 stage to another. Each stage is characterized by a set of alternatives or cities that are available. So here each 1 is a stage. So stage is actually the state of decision making, where at every table you realize that the state is the place where the person is already from where he or she has to make a decision. So state is city at which the person presently is or you can say city in which the person is located. Decision variable is the next destination where we have to go next and the criterion of effectiveness is the minimization problem. So we need to minimize the total distance travelled. So now we have completed this problem or solved this problem using DP. Now dynamic program involves certain things. First thing is we solve the problem, stage wise so even before we do all these things, the first thing we need to do is to write down the stage state decision variable and criterion of effectiveness.

May be for this example and perhaps the next example we will try to write this in the end but then as we proceed with more examples it is absolutely necessary that we define this first. In fact once the state variables and decision variables are defined the problem becomes easy to solve here, since we are introducing the terminologies we solve the problem and then later said that these 4 terms has to be introduced for every problem. In fact first thing we need to do is to introduce these 4 terms and define them to this specific problem. Then we follow what is called a backward recursive approach. We assumed that we have reached the destination and then we go backwards and see that in order to reach 10 we need to go 1 stage back and you can start from 8 to 9 which is the 1st, 2nd, 3rd and 4th respectively. At each stage the calculations are shown in the form of table here because the state variable and the decision variables take discrete values. They are not continuous variables so here the state could be either 8 or 9 and decision is to go to 10. Here state is discrete values 5, 6 and 7 and decision is 2, two values 8 or 9 and no other continuous variable. So we form our tabular approach. For every table we have to define what it is in the table, we have to define the relationship. Now this relationship is called recursive relationship. You could define a recursive relationship for every table as we have done or you could write a general recursive relationship for this problem which would be like this, consistent with our terminologies, fn, f is the function, n more stages to go, I am at state s and have decided to go to $X_{n.}$ X_n is the decision variable, s is the state variable then I travel a distance of dsX_n from s to X_n and then I go to the previous table f(n – 1) star which is the best value of X_n. For example if I am in 5 and decide to go to 9 then somewhere we would have this. Here I am in 5; I decide I go 10 here so I have 2 more stages to go so s X_2 is dsX_2 + f_1 star of. So the best value is taken from the previous value so I have n more stages to go, I go to the previous table which is n – 1 star of X_n and the best value would be fn star n more stages to go given state s the best value is the 1 that minimizes fn of sX_n and the value of X_n star is the corresponding X_n for which this value is minimum. This kind completes the first example along with that they will use few terminologies. We will now take few more examples, at least a couple of more examples for the discrete case and then we would also look at some more examples where the decision variables take continuous values. So let us look at the second example here.

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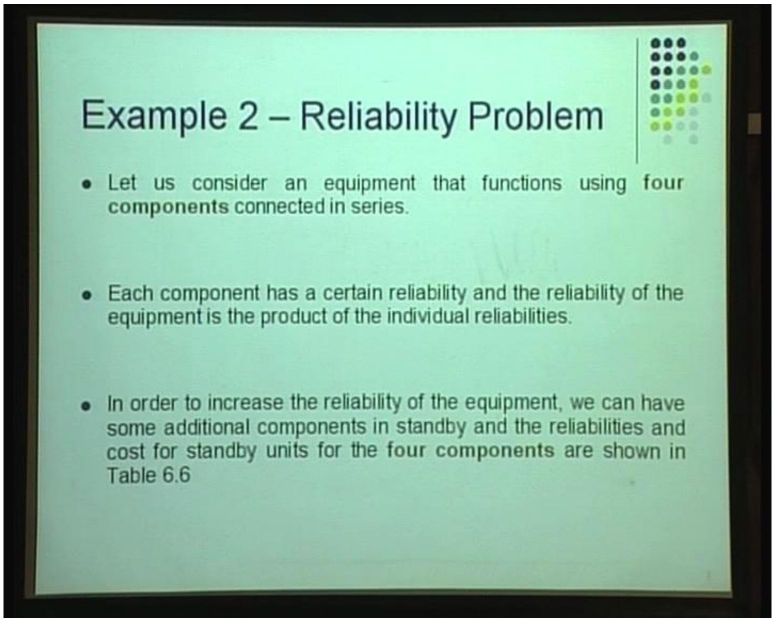

The second example is what we call as the reliability problem.

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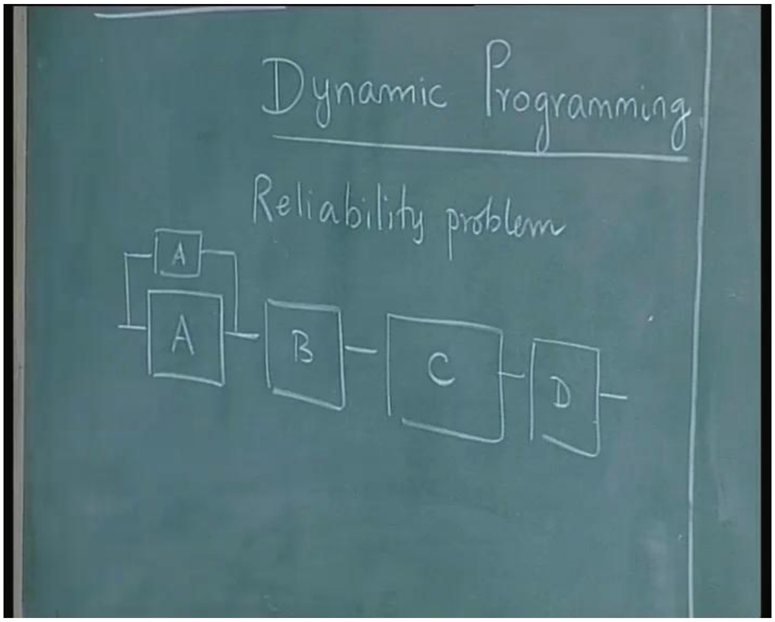

Here we have assumed that we have equipment which is made up of say 4 components. We call them A, B, C and D which are connected in series in order for the equipment to function. Now each component has certain reliability in the series system. The total reliability of the system is the product of the individual reliabilities. In order to improve the reliability, we can also add another component in parallel or keep them as a stand-by and so on. For example we could add another A here or use it in such a way that the reliability of A together is increased and therefore the overall reliability is increased. Now we have the data that tells us, for this problem that the data is like this.

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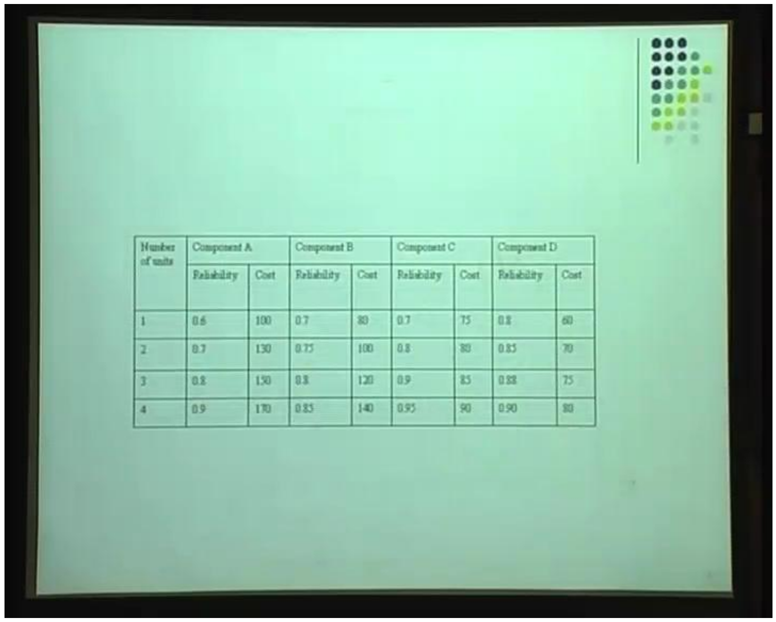

Number of units	Component A		Component B		Component C		Component D	
	Reliability	Cost	Reliability	Cost	Reliability	Cost	Reliability	Cost
1	0.6	100	0.7	80	0.7	75	0.8	60
2	0.7	130	0.75	100	0.8	80	0.85	70
3	0.8	150	0.8	120	0.9	85	0.88	75
4	0.9	170	0.85	140	0.95	90	0.90	80

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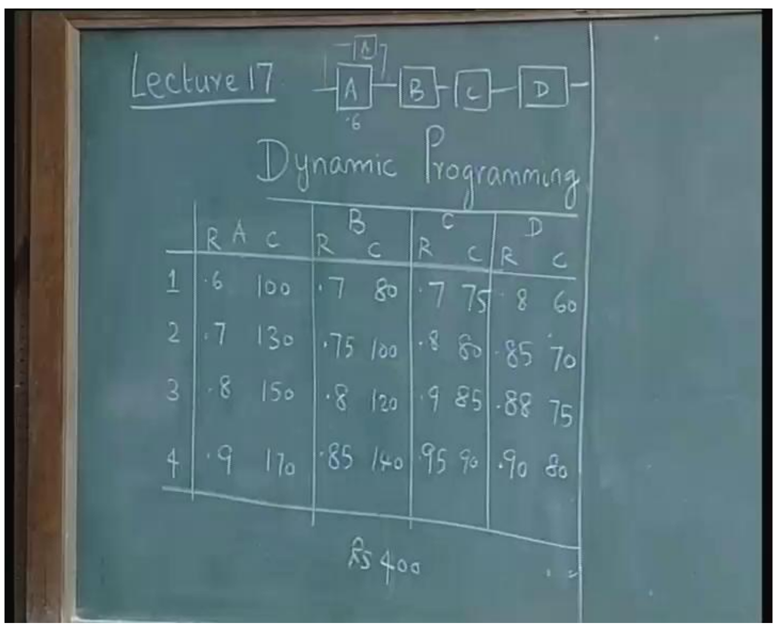

Now for these 4 components the number of units that we could have could be 1, 2, 3 and 4. Now this is component A, so we have 2 terms reliability and cost reliability and cost C and D reliability and cost. 0.6, 100, 0.7, 130, 0.8, 150, 0.9, 170, 0.7, 80, 0.75, 100, 0.8, 120, 0.85, 140, 0.7, 0.75, 0.8, 0.80, 0.9, 0.85, for example if we have these 4 components here, I have only one of A then I get the reliability of 0.6. I have to spend 100 Rs. If I decide to add another one of A here then I would have to spend 130 and the total reliability of this A alone will be 0.7 and so on. Similarly if I decide to put say, 2 of C, then I get a reliability of 0.8 and the cost is going to be

80, obviously there is a restriction on the amount that we have that we can spend. That is given by Rs 300. We assume that Rs 400 is available as amount and we try to solve this problem. The problem now is how many units will have A B C D such that we maximize the total reliability of this system and at the same time we have a budget restriction or the money restriction Rs 400 with us. So as we start, we realize that we first have to define the 4 important terms Stage, state, decision variable and criterion of effectiveness.

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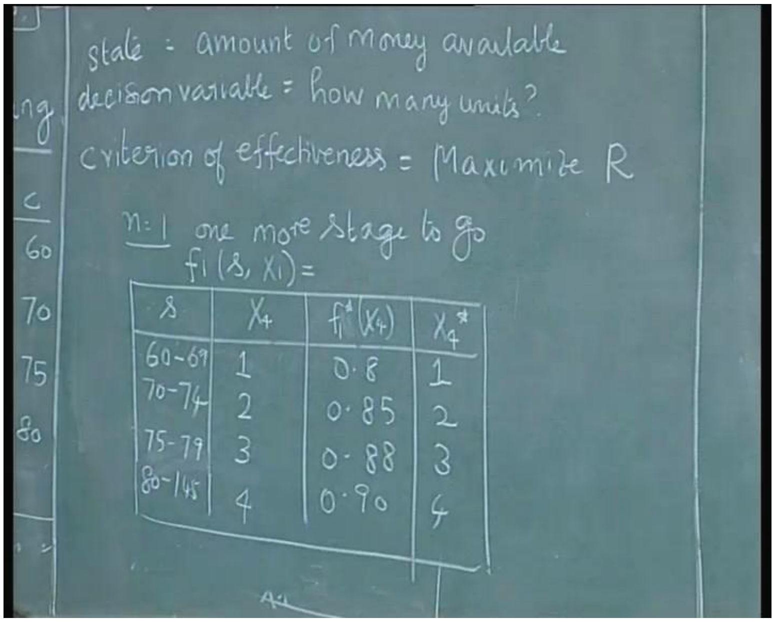

Now as far as this problem is concerned, if we follow a backward recursive approach then when we have one more stage to go, we would be somewhere here. If we have 2 more stages to go then will be here, 3 more will here, 4 more will be here. So as far as this problem is concerned, stage is each component because we make a decision for every component as we follow a backward recursive. So stage is component. Decision variable is easier to define in this than the state variable. So we define this. We define this when we come back to this. Now decision variables are how many units are we going to have for each one of these components, so decision variable would be how many units? Criterion of effectiveness is the objective function with which we begin.

The objective is to maximize the reliability so criterion effectiveness is to maximize the reliability. What is state variable? State variable in this example is the amount of money available for allocation, because the decision is going to be based on the resource that we have and the resource in this case is the money. If we have unlimited money then the decisions are obvious. Try to put the maximum now because there is money restriction depending on how much money that is available to make a decision in every stage. The decision variable changes, so state in this example is the amount of money available or in general where ever there is a constraint the resource availability represents the state of the system. The state can be directly linked to resource availability. In this case, state variable is the amount of money which is available for allocation to that particular component. So having defined these 4, let us go back and try to solve this problem. Now when we have n is = 1, we have one more stage to go. We call it as X_1 or we call X_4 as the number of units that we are going to have of D, X_3 for C, X_2 for B and X_1 for A. So X_4 is the variable which says how many units of D and we have f_4 of X_4 or f_1 star of X_4 and then we have the corresponding X_4 star. Now what are the possible values that the state variable can take if I have 1 more thing to go? Now one easy thing say is we can say the money that is available for allocation of D could be anything from 0 to 4 hundred but we do not follow that we try to see what is the bare minimum that we need and what is the maximum that we can have? now as far as this is concerned that bare minimum money that we need is enough money to buy at least 1 of D which is at least 60, so we need to start with at least 60 otherwise we will not able to buy even 1 of D and the equipment will not function.

It does not make sense to look at state variables of 0 to 59 for this because we cannot buy anything of D and the reliability will be 0. So it makes sense start with saying that we have at least 1 each of A B C D there which will I have to buy and over and above that whatever money I have. I try to spend on buying more of A B C D. So the minimum money that I need to have here is 60. The maximum money that I can have for D is actually if I brought only 1 of A B and C, which means if I have spent exactly 100 on A, 80 on B, 75 on C, which is 255. If I had spent then the balance 145 is the maximum that I could have for D. So D, the maximum that state variable will have here is 145 because if I have anything more than 145 allocated for D which are available for allocation to D which it means that I will not able to buy 1 of A, 1 of B, 1 of C and therefore my reliability will be 0. So I make an assumption that I have at least one of A B C and D therefore the resource that is available for D can vary between anything from 60 to 145. Now if it is from 60 to say 70 or 69 then I can buy only one of this. So X_4 is 1 and the reliability is 0.8 and X_4 star is 1.

If I have between 60 and 69 Rs I can buy only 1 of D therefore my reliability is .8 for D and I buy exactly 1. So when I have 70 to 74, I can buy 2 of D that is what I will do. I will not buy 1 of D and keep the remaining money un-utilized. The reason is by buying 2 of D, I can increase the reliability of D and the overall reliability of the system. If I have more than 70, I will not consider the possibility of buying only 1 of D and keeping the remaining money unspent. So we try to buy as much as D as possible because the objective is to maximize reliability and not to minimize the money spent. So the decisions are always made keeping in mind, the criterion of effectiveness which is the objective function. So when state is from 70 to 74, X_4 will be 2 and I can have a reliability of .85 and X_4 star may be 2. Even though there are 2 alternatives here which is $X_4 = 1$ and 2, the best value will be 2. Now similarly when I have 75 to 79, i.e., enough money to buy 3 of D, X_4 is 3, reliability is .88 and X_4 star is 3 and when I have anything greater than 80, 80 to 145, I will have 4 of this to get .90, I have 4 of this. Now what is the recursive equation here? Recursive equation is f_1 of s, X_1. I have s Rs available. I have s_1 Rs available or s you could use it suitably. If you use s_1 here, you will have s_1 here that is all and this will be f_1 star of s or s_1. So we use s here. You could consistently use s, so I have 1 more stage to go, 1 decision to make. That is D if I am left with s Rs and I decide to buy X_4 units of D then this will be reliability associated with X_4. For example if I have 70 to 74, I can buy a maximum of 2, so I go back and see that my reliability is 0.8.

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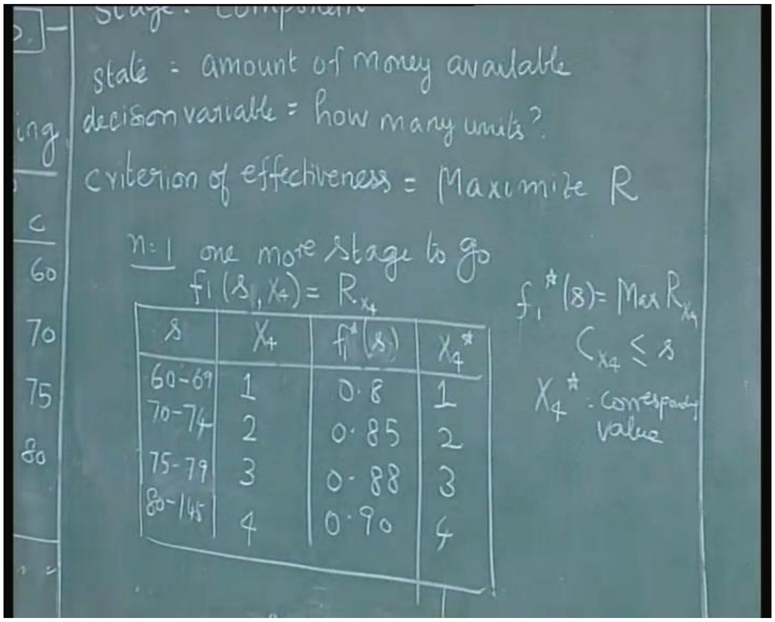

Now what is the best value given? f_1 star of s will be maximized R of X_4 subject to the condition C of X_4 should be less than or equal to s. For example when I have state variable, suppose I have 72 Rs available here, I can buy 2 s, as long as the money that I have for 2 which is 70 is less than the state variable. So this constraint has to be satisfied by the best value and the best value X_4 star will be the corresponding value of X_4. Now when we go back and try to solve for this C and D put together which means, we have 2 more stages to go. So we are at n = 2, 2 more stages to go.

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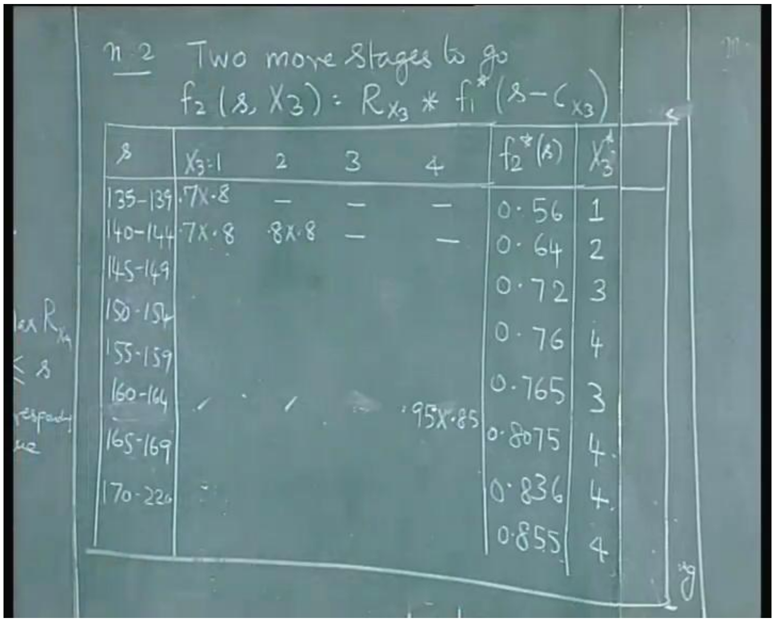

Now we have the state variable s and you could have decision variables this is X_3 = 1, 2, 3 and 4. There are 4 alternatives that are possible then we have f_2 star of X and we have X_3 star. Now what is the minimum value that s can take, when I am here? When I have 2 more stages to go, the minimum amount of money that I definitely need should be sufficient to buy at least 1 of C and 1 of D. So this would begin with 75 + 60 which is 135 and the maximum would be, if I had bought 1 of A and 1 of B, so if I had bought 1 of A and 1 B and spent 180 then we had 400 so the maximum I could have would be 220. Now let us complete this table.

When I have exactly 135 here and I buy X_3 = 1, so X_3 = 1 gives me a reliability of 0.7 and I spend 75 on this, so the balance money that I have out of the 135 is 60, 135 – 75 which is 60 which I carry to the next item. So when I have 60, I get another reliability of .8 which I have. Now when I have 135, I decide to put by 2 of C which means I spent 80, I have the balance 55 available for this. I cannot buy anything. So I do not look at this alternative. So when I have 135 I have only 1 alternative which gives me a reliability of .56 and X_3 star is 1. When I have 136 now, what we need to do is we need to look at all the state variable values from 135 to 220 which is roughly about 66 + 20 = 86 different values that we need to look at. So we do not tabulate all 86 here. We try to bring them into the range like what we did here. So now we look at 136 when we are at 136, the same thing will happen. For example I can buy only 1 of C and spend 75. I would have balance of 61 from which I go to this range and I could have only 1 of this.

So this will happen from 135 to139. I will have only 1 possible that exists. I will be able to buy only 1 of C and get this. Now when my state variable moves to 140, what will happen is I could have 2 alternatives. I could buy 1 of C, spend this 75, and get 0.7. The balance would be spending 75 out of 140, so the balance would be 65. So 65 would give me again .8, but I can do another thing. I can spend 80 and buy 2 of C to get .8. I have already spent 80 so I carry the remaining 60 to the next one, so at 60 I get another .8. So this becomes .64. Now these two are not possible. Now the best decision here between these two is .64 and X = 2. Now like this we have to build this table now. Wherever we find another alternate coming in, we move to the next definition of the stage now. We show the entire table here.

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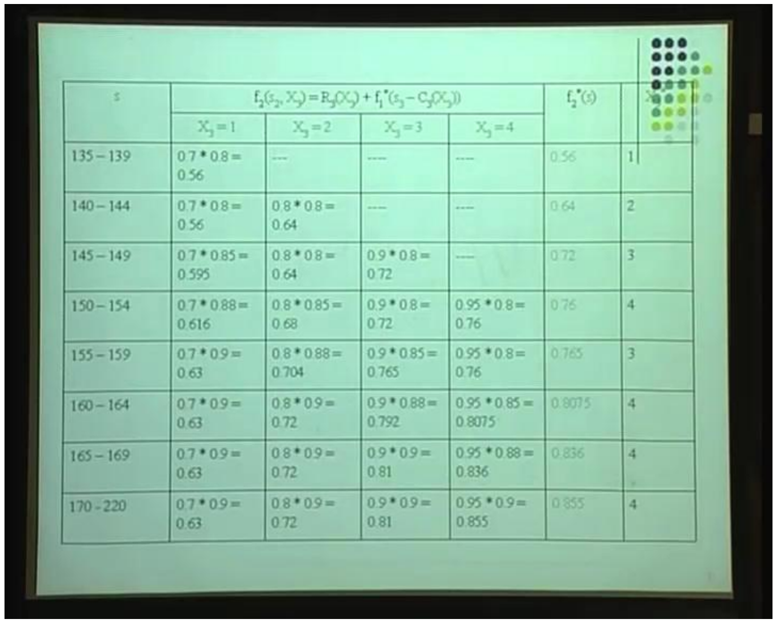

s	$f_2(s_2, X_3) = R_3(X_3) + f_1^*(s_3 - C_3(X_3))$				$f_2^*(s)$	X
	$X_3 = 1$	$X_3 = 2$	$X_3 = 3$	$X_3 = 4$		
135 – 139	0.7 * 0.8 = 0.56	---	----	----	0.56	1
140 – 144	0.7 * 0.8 = 0.56	0.8 * 0.8 = 0.64	----	----	0.64	2
145 – 149	0.7 * 0.85 = 0.595	0.8 * 0.8 = 0.64	0.9 * 0.8 = 0.72	----	0.72	3
150 – 154	0.7 * 0.88 = 0.616	0.8 * 0.85 = 0.68	0.9 * 0.8 = 0.72	0.95 * 0.8 = 0.76	0.76	4
155 – 159	0.7 * 0.9 = 0.63	0.8 * 0.88 = 0.704	0.9 * 0.85 = 0.765	0.95 * 0.8 = 0.76	0.765	3
160 – 164	0.7 * 0.9 = 0.63	0.8 * 0.9 = 0.72	0.9 * 0.88 = 0.792	0.95 * 0.85 = 0.8075	0.8075	4
165 – 169	0.7 * 0.9 = 0.63	0.8 * 0.9 = 0.72	0.9 * 0.9 = 0.81	0.95 * 0.88 = 0.836	0.836	4
170 - 220	0.7 * 0.9 = 0.63	0.8 * 0.9 = 0.72	0.9 * 0.9 = 0.81	0.95 * 0.9 = 0.855	0.855	4

Now when I have from 140 to 144, my best decision is .64. Now from 145 to 149 that is the next range and there I realize that I could get into a third alternative which is here. Now this alternative is trying to buy 3 of C and spending 85 and with the remaining 60 buying 1. So from 145 to 149, we have 3 alternatives and the best value is .72 with X_3 star = 3. Similarly I move again from 150 to 154. I have 1 more alternative that comes in now when I am at 150 I could think of X_3 = 4 which means I could think of buying 4 of C spending 90 getting a reliability of .95 and with the remaining 60, I buy 1 of this. Now that is given as the 4^{th} option here which gives me .95 from here and the remaining 60 would give me 0.8. So 0.95 into 0.8 gives me a 0.76. This gives me 0.76 and the best alternative is still with .76 and 4.

When I am in the range 155 to 159, all the 4 possible exists for buying C but then one of the values would change. One could see that when you are in 151 – 159, if you decide to buy 1 here and spend 75 you get a balance 80. So you are moving into this range to get a reliability of .9. So you get .7 into .9 which is .63. This would be .8 into .88. If I buy 2 and spend 80 and then I have a 75. So I move into this range. So I get .8 into .88 with is = .704. So all the 4 alternatives exist but they have different reliabilities. So the best value would be .765 and this happens when X_3 is star = 3 which is shown here. Now again at 160 to 164, we need to calculate. So at 160 to 164, we realize that the change actually happens here. 0.9 Into 0.88 and .95 into .85. So if I have 160 and I decide to buy 3, I have 85. 85 + 75, so I move into the next range so I decide to buy 4. I have 90 giving me .95 into .88 which gives me .8075. So when I am at 160 to 164 I have .8075 which comes as the maximum reliability and my decision would be to buy all 4. Similarly when I am at 165 to 169, I again have all 4 possibilities and the best value would be .836 with X_3 star = 4 and I am from 170 to 220, I realize that the 0.855 X star = 4. Now we have not shown the individual calculations in this table but we have shown the individual calculations in the other table and one need to go back and see that final particular range and all the 4 decisions are possible in this case. Now depending on the decision you have to see the money that you have spent here and then you carry the balance money to the previous table and find out the best result. We will just right down the recursive relation for these 2 for the second one right now. So I have 2 more stages to go. So if I have s Rs which is my state variable and decide to buy X_3 units of this, I will have R of X_3 + or multiplied by f_1 star of s – C of X_3. For example just to calculate say, some something like this if I have 160 with me and say I decide to buy 4 here, then I have 160 for 4. I spend 90 and get a .95 here. I spend 90. I have a balance of 70. This 70 would give me another .85 which is from .8075. So like this we actually complete this table. The most important thing here is that we have s – C of X. I start with s, I spent some C which is the cost. The balance I carry to the previous stage and put f_1 star of s – C X_3. So we have right now completed 2 stages. There are still 2 more stages in this problem. We will look at the other 2 stages in the next lecture.