Fundamentals of Operations Research

Prof. G. Srinivasan

Indian Institute of Technology Madras

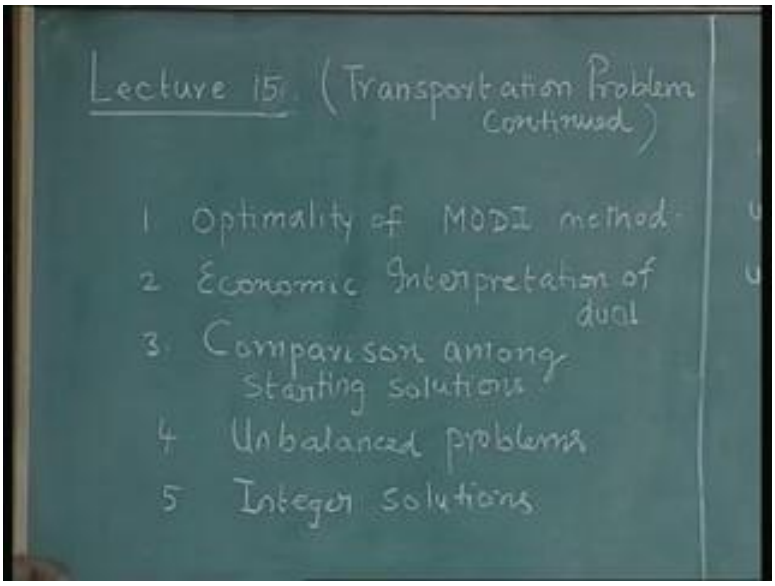
Lecture No. # 15

Transportation Problem - Other Issues

Assignment Problem - Introduction

In the last lecture we saw the 2 methods to solve the transportation problem, optimally which were the Stepping stone method and the Modified distribution method.

(Refer Slide Time: 01:23)

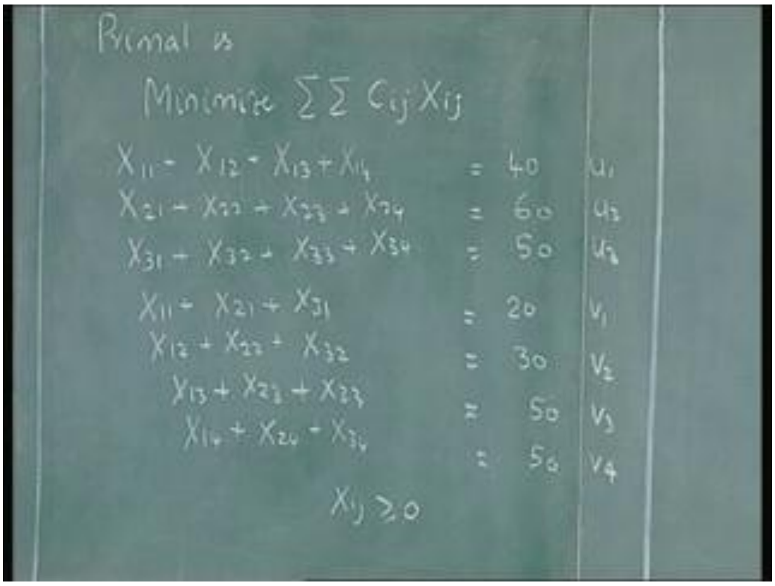


Now we look at all these aspects of the transportation problem which is

i. The optimality of the MODI method,
ii. Economic interpretation of the dual to the transportation problems,
iii. Comparison among the various starting solution solutions,
iv. How to solve unbalanced problems and
v. How to we ensure that we get integer solutions to the transportation problems.

Even if we solve it as a LP, we will look at all these dimensions 1 after another. First let us look at how the MODI method gives the optimal solution. To do that, we first write down the dual of the transportation problem.

(Refer Slide Time: 02:09)



Primal of the transportation problem is to minimize double sigma C_{ij} X_{ij}. Let us write the primal constraints with specific reference to the example that we are looking at. There are 3 supply points. When the first supply point is considered $X_{1\,1}$ + $X_{1\,2}$ + $X_{1\,3}$ + $X_{1\,4}$ less than or equal to a_1 is 40. As far as the second one is considered $X_{2\,1}$ + $X_{2\,2}$ + $X_{2\,3}$ + $X_{2\,4}$ less than or = a_2 and third one is $X_{3\,1}$ + $X_{3\,2}$ + $X_{3\,3}$+ $X_{3\,4}$ is less than or = 50. As far as the requirements are considered, first requirement point will be $X_{1\,1}$ + $X_{2\,1}$ + $X_{3\,1}$. This is the quantity received from supply points 1, 2 and 3 respectively. This should be greater than or equal to 20.

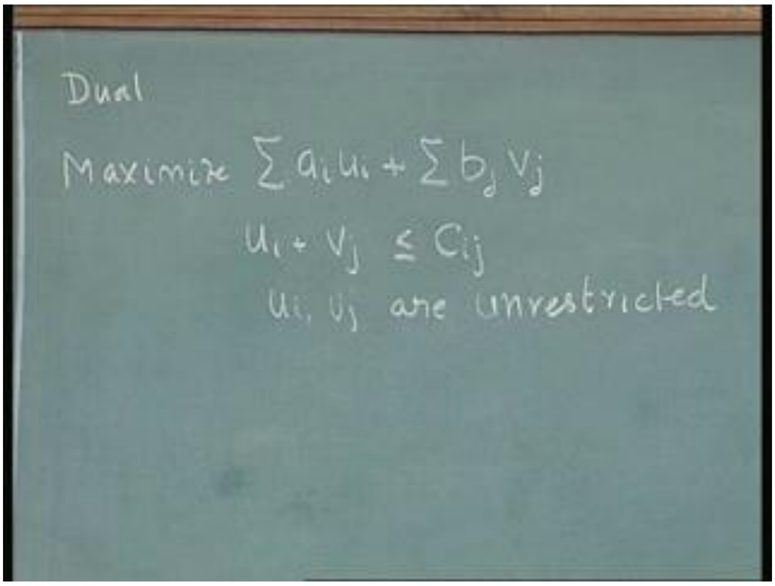
$X_{1\,2}$ + $X_{2\,2}$ + $X_{3\,2}$ greater than or equal to 30

$X_{1\,3}$+ $X_{2\,3}$ + $X_{3\,3}$ greater than or equal to 50

$X_{1\,4}$ + $X_{2\,4}$ + $X_{3\,4}$ greater than or equal to 50; we have X_{ij} greater than or equal to 0.

Now because we are solving a balanced transportation problem, all these inequalities now become equations. We have this entire equal to this and we have sigma a_i = sigma b_j.Now let us write the dual to this problem. Now we introduce dual variables u_1, u_2, u_3, corresponding to the 3 supply points and 4 dual variables v_1, v_2, v_3 and v_4 corresponds to the 4 demand points. So the dual will since the primal is a minimization problem.

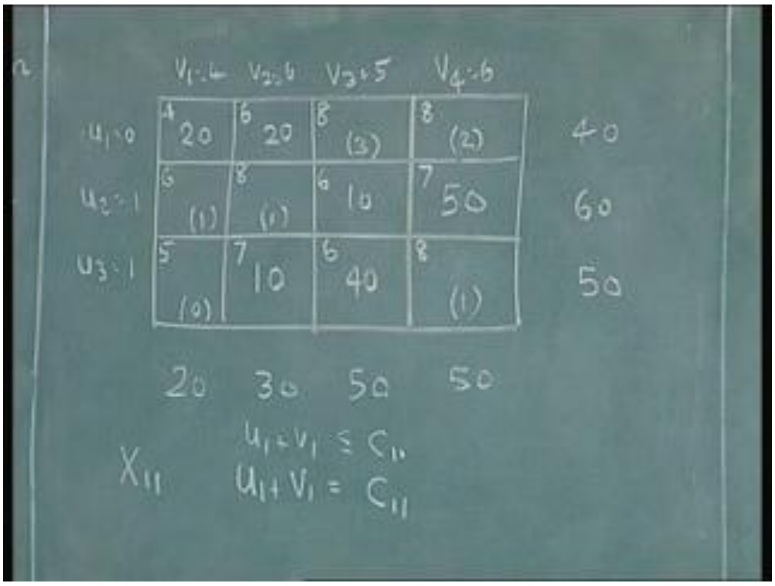
(Refer Slide Time: 04:46)



The dual is the maximization problem and dual is given by maximize sigma a_i u_i, 40 u_1 + 60y_2 + 50y_3 + sigma b_j v_j, 20 v_j + 30 v_2 v_j. Now we realize that every X_{ij} for example $X_{1\,1}$ is in 2 constraints, corresponding to u_1 and v_1. If we take X_2 to be 3, it is in 2 constraints corresponding to u_2 and v_3. So each X_{ij} appears exactly in 2 constraints so for a corresponding X_{ij}, we will have u_i + v_j is less than or equal to C_{ij} from the objective function.

For example we will have as many dual constraints as a number of primal variables we have 12 primal variables so will have 12 dual constraints which will be u_1 + v_1 less than or equal $C_{1\,1}$. u_1 + v_2 less than or equal to $C_{1\,2}$ and so on, we will have 12 constraints corresponding to the 12 primal decision variables and we will have as many dual variables as the number of primal constraints. We have 3 dual variables for 3 supply constraints and 4 dual variables for these four. Now this u_i's and v_j's are unrestricted because all the primal constraints are equations. We have already seen that corresponding to an equation; the dual variable will be unrestricted in sign. So u_i's and v_j's are unrestricted in sign. So this is the dual of the transportation problem. Now let us see how this dual helps us to understand the optimality of the MODI method.

(Refer Slide Time: 06:51)

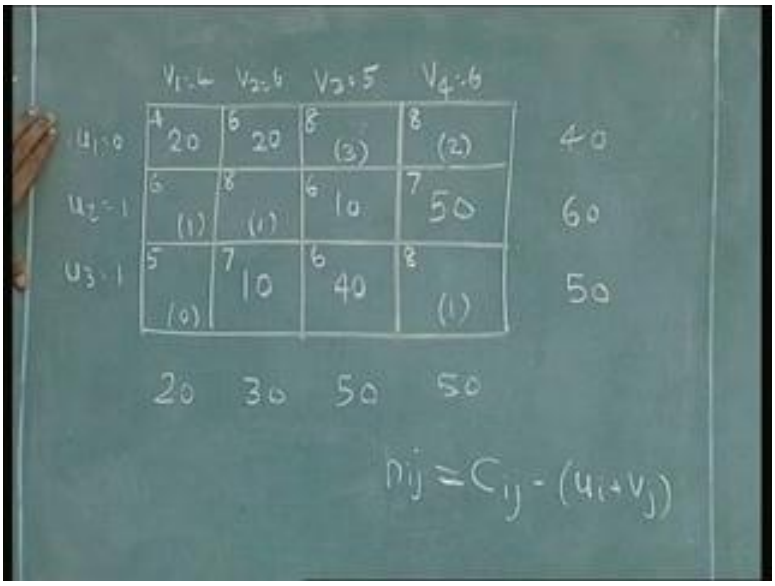


Now let us go back and look at the same problem that we saw in the earlier lecture. We represent the optimal solution here and the corresponding values of u_i and v_j as we computed in the last lecture. So if we take this situation now, we find corresponding to this allocation here, we fix $u_1 = 0$ to get $v_1 = 4$; $v_2 = 6$ would give us $u_3 = 1$ which would give us $u_3 = 5$, from which we get $u_2 = 1$ and we get $v_4 = 6$ and the u's and the v's are put here. What we have done by introducing these u's and v's is that, we are trying to evaluate the corresponding dual solution. Now these u's and v's represents the corresponding dual solution, corresponding to the primal solution given by these variables. The next thing we do here is to calculate $C_{ij} – u_i + v_j$ and 0 + 5 is = 5, so we get 3 here. We get 2 here. We get 1 here, we get 1 here, we get a 0 here and we get a 1 here. So we have calculated $C_{ij} – u_i + v_j$.

Now let us look at these. These represent the dual variables corresponding to the primal basic feasible solution which is here, so those are the dual variables u_i's and v_j's. Now how do we calculate the dual variables? Wherever the primal X_{ij} is basic, we define u_i v_j such that $u_i + v_j = C_{ij}$. Because $X_{1\,1}$ is basic, we make sure that $u_1 + v_1$ is = $C_{1\,1}$. If we go back to the dual constraint then $u_1 + v_1$ should to be less than or equal to $C_{1\,1}$. So now X_{ij} is basic. We find out u_i and v_j such that $u_i + v_j = C_{ij}$ indicating that the corresponding slack is 0 which means wherever a variable is basic we find u and v such that the corresponding slack is 0 which means we are satisfying complementary slackness condition.

Complementary slackness condition says that if a primal X is basic then the corresponding dual slack is 0. We compute the dual variables in such a way that complimentary slackness conditions are satisfied. Wherever X_{ij} is basic we have $u_i + v_j = C_{ij}$ indicating that the slack is 0 satisfying the complimentary slackness conditions. Wherever X_{ij} is non basic, we try to find out the value of $C_{ij} – u_i + v_j$. For example if we write the dual constraint $u_i + v_j + h_{ij} = 0$; wherever X_{ij} is basic we make sure that X_{ij} is 0 because $u_i + v_j = C_{ij}$.

(Refer Slide Time: 09:56)



Wherever X_{ij} is non basic, we try to find out the value of $C_{ij} – u_i + v_j$. For example we write the dual constraint as $u_i + v_j + h_{ij} = 0$. Now wherever X_{ij} is basic we make sure that h_{ij} is 0 because $u_i + v_j$ = to 0_j. Wherever X_{ij} is non basic we try to evaluate $C_{ij} – u_i + v_j$ which means we are trying to evaluate h_{ij} dual slack corresponding to this constraint or we are checking whether that constraint is feasible. Now all h_{ij}'s are greater than or equal to 0. It means the u_i's and v_j's are such that all the dual constraints are satisfied. If even one h_{ij} is negative it means that particular dual constraint is violated and the dual is infeasible. Dual infeasibility implies primal optimality. So you enter that variable and proceed towards the optimal solution. Now when all h_{ij}'s are greater than or equal to 0 which is the optimum, it means all dual constraints are satisfied. So your u_i's and v_j's are such that all the dual constraints are satisfied. Now u_i's and v_j's are unrestricted in sign so we need not worry about some of these taking even negative values. So far we did not encounter a situation where a u_i or a v_j took a negative value but you can have situations where these can take negative values and that is acceptable because your u_i's and v_j's are unrestricted inside. In fact that is precisely the reason why we solve a balanced transportation problem.

A balanced transportation problem has equations and therefore we have u_i's and v_j's unrestricted in sign. So we need not worry about the sign of the u_i's and v_j's as we compute that even if they are unrestricted in sign it does not matter to us. Therefore we are comfortable solving the balanced problems because we need not worry about u_i's and v_j's having to take strictly a non negative value and so on they can take any value.

Therefore the moment u_i's and v_j's are such that all the h_{ij}'s are greater than or equal to 0 which means all the dual constraints are satisfied then corresponding to this primal basic feasible solution, we have identified a dual solution which is also feasible and the complimentary slackness conditions are satisfied because wherever X_{ij} is basic we have $h_{ij} = 0$. So we have 3 things we starting with the basic feasible solution to the primal. We applied the complimentary slackness and found out the corresponding dual and we verified whether the dual is feasible and if the dual is feasible it has to be optimal because that is exactly the way the simplex algorithm works. We start with the basic feasible solution. You apply complimentary slackness and find out the corresponding dual. When the dual is feasible we have reached the optimal solution to both primal and dual therefore MODI method is optimal.

So based on this discussion the MODI method is optimal and we have already seen that the stepping stone method is exactly the same as the MODI method. So whatever we evaluate as a net gain or a net loss, by putting a, + 1 in the non basic position, completing the loop and getting the net increase is the same as finding $C_j - z_j$ or $C_{ij} - u_i + v_j$. So when $C_{ij} - u_i + v_j$ is negative it means it is profitable to enter into this basis. If we look at it from the MODI point of view, it is profitable to put something here and then adjust the allocation as we do in the stepping stone method. So stepping stone method is the same as MODI method as far as the results are concerned. Therefore both MODI method and the stepping stone methods are optimal. Now we move to the next part which is the economic interpretation of the dual variable. Now in the linear programming we gave an economic interpretation of the dual as the marginal value of the resource of the optimum. Now let us see what kind of an interpretation we have for these u_i's and v_j's. Now the X_{ij} represents this 20, which represents the quantity transported from the first supply point to the first destination point. Now what exactly are these 0's and 4's?

Let us look at the situation the primal of the transportation problem. If concerned about transporting the 40, 60 and 50 available in the 3 supply positions to meet the 20, 30, 50 and 50 which are the demands of the 4 demand points. Now one way of looking at it is if we solve the primal to find out exactly how much to transport from every i to every j, we get an optimal solution 20, 20, 10, 50, 40, 10 respectively. On the other hand let us look at the problem in a different way.

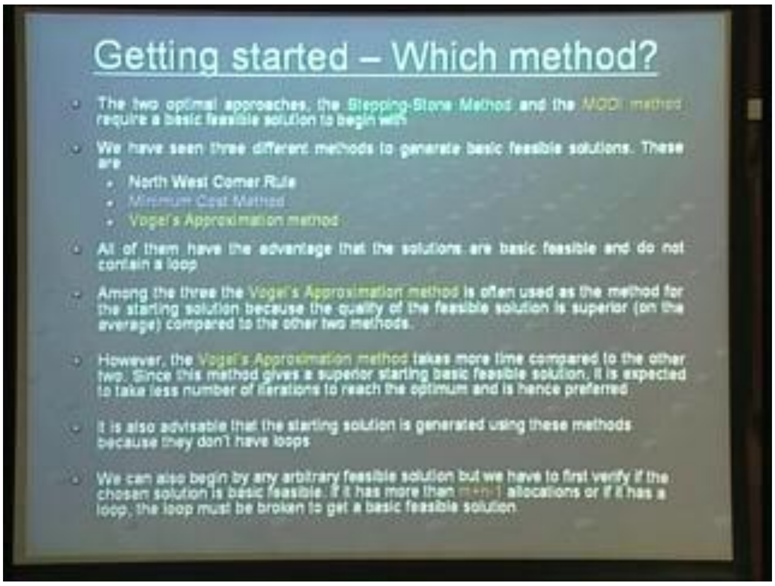
Now there are some supply points and there are people who have these 40, 50 and 60 as well as people who need these 20, 30, 50 and 50. Rather than directly transporting it, let us assume that a third party position is interested in buying all these things from the supply points from the suppliers and is interested in selling these to the demand points. So let us assume that there is a middle man or a third party person who is willing to buy this 40, 60 and 50 exactly at the supply points and then he is going to sell these 20, 30, 50, and 50 back to the people who are interested in buying at the demand points. In such a situation the u_i's will represent the value, the money that the person will be willing to buy and v_j's will represent the amount for which he will be willing to sell it at the destination point or for example if a middle person is going to buy one from here and sell, for example, to here, then the difference between buying price and the selling price should be equal to the cost of transporting which is 6. So the dual variables are interrupted as the marginal worth of the commodity at the supply points and the marginal worth of the commodities at the destination point when it comes to buying and selling respectively. Now in it is also obvious that we were actually not interested in finding the absolute values of the u's and v's but we are finding out the values of u's and v's by fixing $u_1 = 0$ always. For example if we had fixed u_1 to other number then the u's and the v's will get adjusted and that is acceptable to us because u_i's and v_j's are unrestricted in sign.

It is also important that we are not interested in the absolute values. We are only interested in the relative values. For example if this is going to be a basic variable at the optimum then no matter what price the person is going to pay and buy here, it is necessary for the person to sell it at 6, if this is rupees 6 more to this person. So that both are able to optimize otherwise if the third party person is willing to sell for something less than 6 then the person will lose and if the person is willing to sell for more than 6 then this person will not be ready to buy from this person. So the competitive market will ensure that this $u_i + v_j$ will be = C_{ij} where u_i and v_j are interpreted suitably as buying and selling respectively. So this is the economic interpretation of the dual variable. The dual variables can also be unrestricted in the sign. It does not matter. It is only the relative difference that matters and not the absolute value.

So they can be unrestricted in sign. So some of them can take positive values, some of them can take negative values as well.

Now let us also look at the third dimension we said there are 2 ways, 2 basic steps in solving the transportation problem. One is to identify the basic feasible solution and the second is to get the optimum from the basic feasible solution. So when we identify the basic feasible solution we looked at 3 methods the North West Corner rule, the minimum cost method and the Vogel's approximation method.

(Refer Slide Time: 19:08)



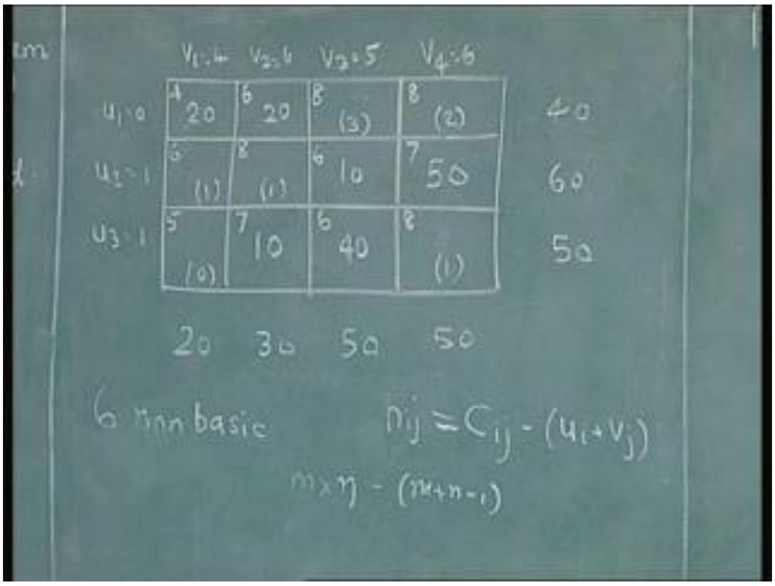
Let us look at some aspects of these. We could get started using the North West Corner rule or the minimum cost method or the Vogel's approximation method and we could get the optimal solution using the stepping stone method or the modified distribution method. When we look at the 3 North West Corner rule minimum cost and Vogel's, we realize that North West Corner rule is a simplest of that. All the 3 of them give basic feasible solutions. All the 3 of them will not give the solutions with the loop because every time when an allocation is made, they allocate the maximum possible which is the minimum between the supply and the demand. Now among the 3 we realize that the North West Corner rule is the fastest in terms of implementation it is quicker but it does not guarantee a solution because it does not use cost information while making the allocation. Then the minimum cost method is expected to give a value better than the North West Corner rule because it takes into account the cost while making the allocation and the Vogel is expected to perform better than the minimum cost because it goes to one more level of detail in computing the penalty associated with not allocating at the minimum cost time.

In terms of computational requirement, Vogel takes more time compared to the other two since this method gives the superior starting basic feasible solution it is expected to take less number of iterations to reach the optimum. It is also advisable that the starting solution is generated using these methods because they guarantee basic feasible solutions and they do not get into any loop. It is not absolutely necessary that we should use any 3. We could start with any arbitrary feasible solution. We need to check whether it is basic feasible.

We need to look at the number of allocations. If there is already a loop existing we need to break it and then we need to bring it into m + n – 1. If there is a loop, the loop must be broken to get a basic feasible solution.

In summary, Vogel's approximation is used normally even though it takes a little longer in terms of computations. It is expected to give a good quality basic feasible solution from which we can reach the optimum in fewer iterations. Between the MODI method and the stepping stone method let us look at it again.

(Refer Slide Time: 21:37)



Let us look at the same problem and let us compare what we did when we use the stepping stone and when we use the MODI. Now when we use the stepping stone, this problem has 6 non basic positions and in the stepping stone method, we calculated the net gain or loss when putting in each one of these 6 positions.

Each one is involved in computing a loop whereas in the MODI we did not do that. We calculated u_i's and v_j's using some properties as and having identified the entering variable we then computed 1 loop. So for small sized problems where the number of non basic variables are small, stepping stones could be used, but as the problem becomes larger when there are more non basic variables. We have actually m into n positions out of which m + n – 1 are basic. So mn – m + n – 1 non basic positions exist.

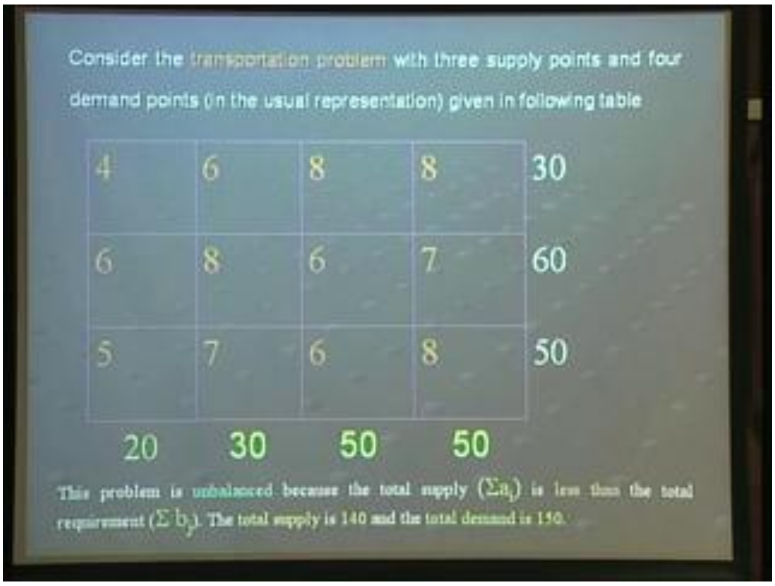
As m and n increase, we realize that this quantity will become large therefore it is advisable to use the MODI method or the uv method. Even though the MODI and the stepping stone are essentially the same, they give the same answer stepping stones computational requirement will be higher as the problem size increases. Therefore it is advisable to use the modified distribution method to solve. So in general it is advisable to start with the Vogel's approximation method or the penalty cost method to get a starting basic feasible solution and then approach the optimality using the uv or the MODI method.

Now let us look at one more dimension which is how to solve unbalanced transportation. So for when we started the formulation, we formulated all the supply constraints as less than or equal to constraints. All demand constraints as greater than or equal to constraints and we formulated an unbalanced problem and then we put or we added an additional restriction that sigma a_i = sigma v_j and we convert it into a balanced problem and solve the balanced problem. Much later when we looked at the dual we realized that the balanced problem is advantageous because the dual variables are unrestricted. The unbalanced problem would restrict the u_i's and the v_j's so we did not solve the unbalanced problem.

If the transportation problem is not balanced then how do we solve?

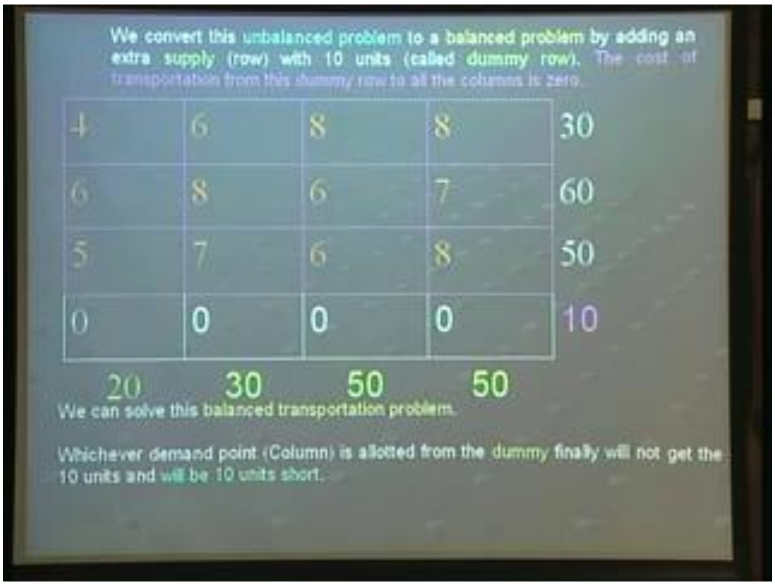
So we need to find out a way by which we are able to convert every unbalanced transportation problem into a balanced transportation problem. We see that through examples now.

(Refer Slide Time: 24:47)



Now let us look at a transportation problem given here. Now this problem has 3 supplied points and 4 destination points. The supplies are 30, 60 and 50 respectively. The requirements are 20, 30, 50 and 50. Now the supplied sum is 140 where as the requirement is 150. So we clearly have a transportation problem which is unbalanced. Sigma a_i is 140 sigma v_j is 150. So we have an unbalanced transportation problem so what do we do here?

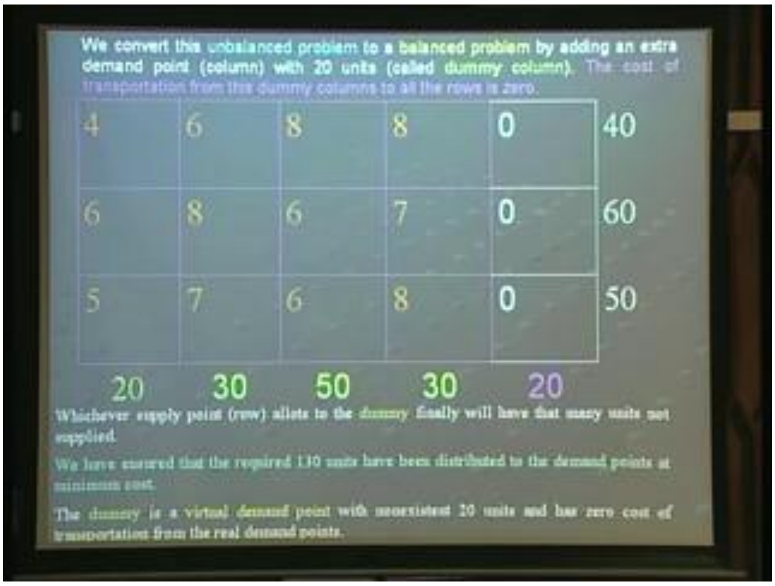
(Refer Slide Time: 25:40)



Now to make the problem balanced we add another supply or another row with the 10 here so that we are able to balance this to 150. Now we have added another row. We have created a 4th row with the balanced quantity of 10 and we have made it 150. Now the cost coefficients remain as they are, but this row that we have added is called a dummy row. This row does not exist in the problem so it is called a dummy row and this row is added with a very specific purpose of converting unbalanced problem into a balanced problem. The cost coefficient corresponding to the dummy is 0. Now we can solve the new, the present balanced transportation problem and get the optimum.

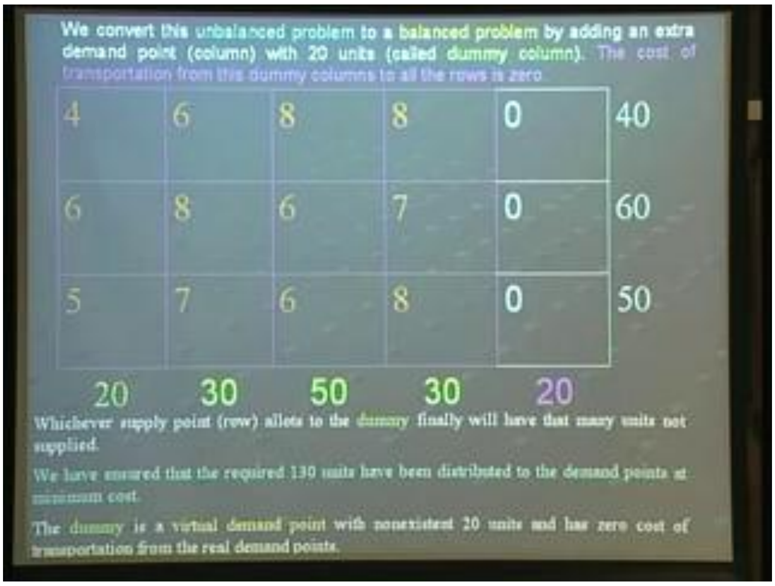
Whatever allocation comes out of these 10 however, will not be allocated at all because we actually have only 140 so somewhere 10 out of the 150 will not be met. Whichever of these 4 gets something from these 10; they effectively do not get anything because this 10 does not exist. In an unbalanced problem when the demand is greater than the supply after we solve a balanced problem, we realize that we cannot entirely meet all the demand a part of the demand will certainly be unfulfilled. Dummy will act as supply point within a nonexistent 10 and has cost of transportation from the real demand points.

(Refer Slide Time: 27:18)



We can look at another transportation problem which is given here. Again we have 3 supplied points and 4 destination points. The 3 supplies are 40, 60 and 50 respectively. The requirements are 20, 30, 50 and 30 respectively. The total supply is 150. Total demand is 130. So we have again another kind of an unbalanced transportation problem.

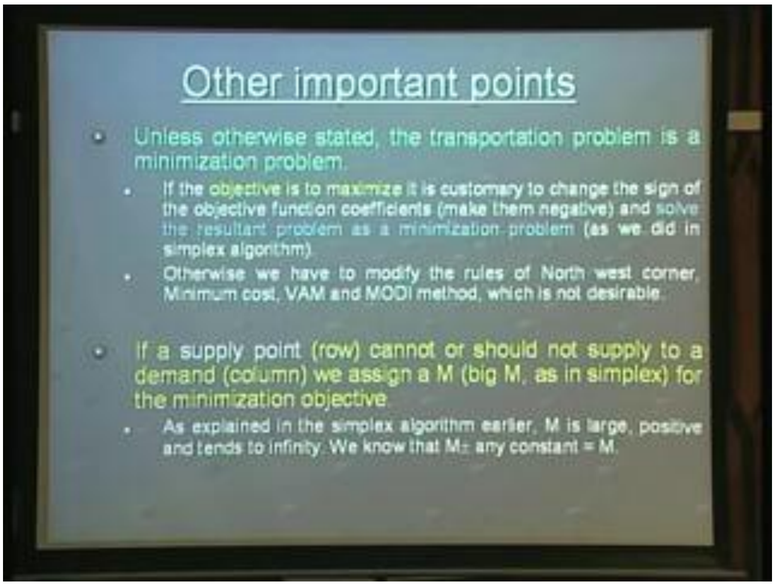
(Refer Slide Time: 27:49)



Here total demand is less than the total supply or the total supply is more than the total demand. We convert it to a balanced problem by adding a new column or a new demand point, a dummy, with value 20 such that the total demand now becomes 150. The total demand is = total supply, the dummy has cost = 0 and whichever supply point (we solved the balanced transportation problem) gives to this 20, one or more to that extent, that many supplies will not be taken from those supply points.

Now whichever supply points allots to the dummy, finally will have that many units not supplied at all and in this case since total supply exceeds total demand, we will be able to meet the entire demand from the existing supply points. Now by definition this dummy is a nonexistent or a virtual demand point which is created to balance this problem with a nonexistent 20 demand and a 0 cost of transportation from the real supply points.

(Refer Slide Time: 29:01)



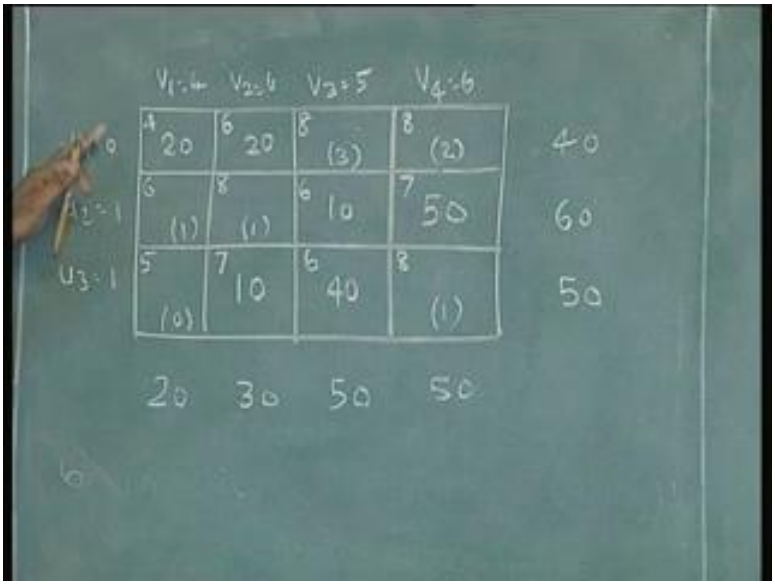
So this is how we convert every unbalanced transportation problem into a balanced transportation problem. Every unbalanced problem can be converted to a balanced problem by either adding a dummy row or a dummy column depending on whichever is necessary.

Always the dummies will have cost equal to 0. If we have a situation where the total supply is greater than the total demand then we have a dummy column and if the total demand is greater than the total supply, we have a dummy row. If the total demand is greater than the total supply then in the end we will have certain amount of demand unmet or unsolved.

However this way of converting every unbalanced problem to a balanced problem is easy for us and we can solve every unbalanced transportation problem very comfortably using this. We need to look at the last and some other aspects of the transportation problem. Now there are some aspects to it. First one is this, unless otherwise stated transportation problem is a minimization problem. For example we analyze otherwise stated we are always trying to minimize the cost of transportation. This 4 would mean the unit cost of transporting from this point to this point and objective of the function is to minimize the total cost of transportation problem. However there could be a situation where we can have a maximization transportation problem.

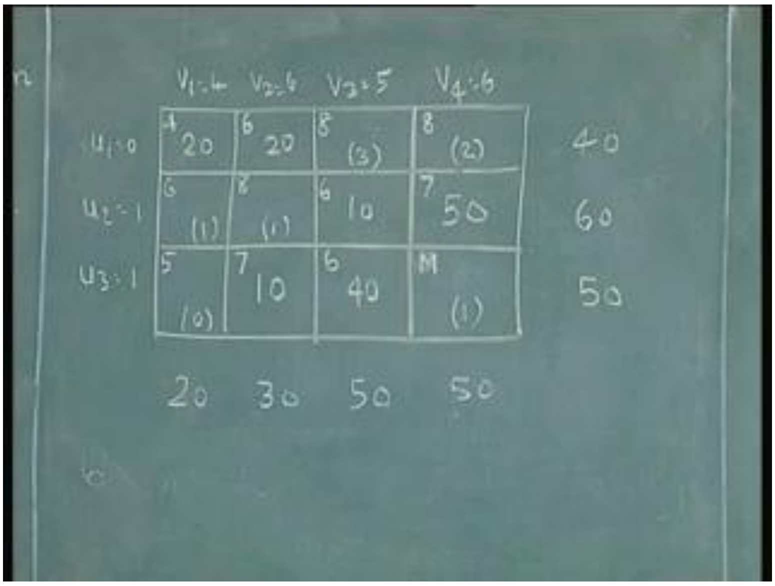
The common rule that is followed is to convert the maximization problem into a minimization problem by multiplying the objective function coefficient with the – 1. So if we were solving the maximization of objective then we would have all these negatives and then we can straight away use either the North West Corner or the Vogel's approximation to get the basic feasible solution. It is a common practice like a linear programming to convert the objective function in this case to a standard minimization by multiplying with the – 1. We do not try to modify the rules of minimum cost Vogel's or MODI which is not assigned. The easier thing to do is to change the objective function into maximization by multiplying with the – 1. For example we do not keep this as maximization and then go back and read and find the penalties by saying penalty will be the difference between the second largest and the largest and so on. We do not do that. We simply multiply the objective function with the – 1, convert it to a minimization problem and solve as we are using.

(Refer Slide Time: 31:35)



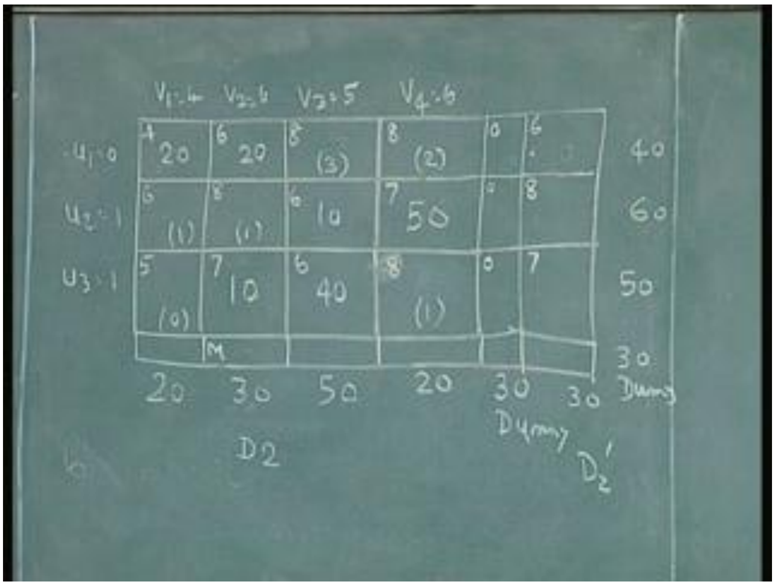
If a supply point cannot or should not supply to a demand column then we assign a big M considering a minimization problem. We assign a big M as a simplex for a minimization objective. Now once again the minimization is large positive and tends to infinity. M + or – any constant equals to M, so there will not be allocation.

(Refer Slide Time: 32:03)



For example if we say in the same problem that we cannot supply from supplier 3 to destination 3 then 8 will be replaced by a big M. Now the big M as in simplex is large and positive and because it is very large it will not attract of allocations. We are solving a minimization problem. We could also have different kinds of situations that have to be modeled.

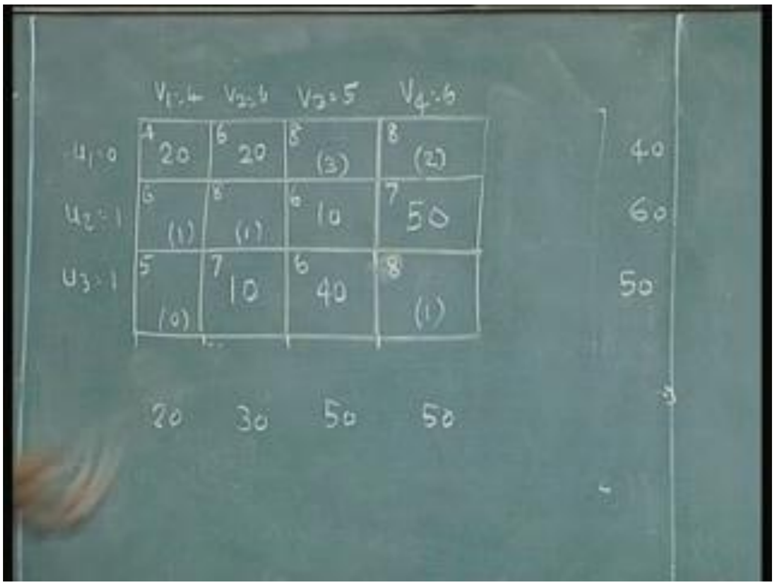
(Refer Slide Time: 32:38)



For example if we have an unbalanced problem, this problem is unbalanced because total supply is 150. Total requirement is 120. We call this as a demand point 2. This assumes that we need exactly 30 units in the demand point 2. Suppose we say that this person is ready to take any excess (that is also possible) then we can do another thing. For example we can convert this is unbalanced problem into a balanced problem by adding another dummy which has 30. This person is also ready to take the extra 30. It is not necessary that this person should be given 60 but this person is willing to take 30 or anything more. So what we do is we add another, i.e., a usual dummy that is added to convert it to a balanced problem.

We add another one called D_2 dash that also bids for the extra 30 and now we get 40, 60 and 50 and because this is D_2 dash we retain the same cost 6, 8 and 7. This is dummy which has 0, 0 and 0. Now the other thing we need to make sure is that now this once again becomes unbalanced because 50, 180 versus 150. So we add another row to balance it with another 30 dummy but we have to make sure that this person gets exactly 30 or more therefore this will be made as big M. So this person will get 30 out of the 150 that is available and whatever extra that comes into this person will also be given as additional from this 1. This way we can model other slightly more complex situations in the transportation problem.

(Refer Slide Time: 35:26)



In a normal transportation problem we will have something like this where these requirements are formed and these supplies are formed. But it is also possible to model situation saying that, for example demand D_2 in an unbalanced situation, this person can say I would like to take 30 or more or up to 50. If the person says I would like to take definitely 30 up to 50 then this 30 will be a column. The remaining will be here and there will be a dummy. Now to ensure that this person takes exactly 30 this dummy will have to put the big M. So that this 30 comes from the available supply and anything else that comes into the other dummy this person will take. So like this it is also possible to model other slightly different or difficult requirements as a transportation problem and solve it as a resultant balanced transportation problem.

The last aspect that we have to look at is the integer solutions to transportation. The transportation problem is the linear programming problem. We have seen the primal. The primal is to minimize C_{ij} X_{ij} greater or equal to 0. This (Refer Slide Time: 36:24) is the lineal programming problem. In all our problems that when we solved we had integer values for the supply and the demand points, we always had integer values solutions. We never had a situation where we had fractional values whereas in the linear programming problems, when you have integer values for your right hand side, one could get into situations where we could have the X_{ij}'s taking non integer or continuous values. In transportation problem we will not come across that situation.

If the a_i's and the b_j are all integer values then the X_{ij}'s will also be integers. This is also because of the way we have made the allocations. Essentially in all the 3 North West Corner minimum cost and Vogel, we allocated integer values, if the supplies and demands were integers and then we simply added and subtracted integers so we were getting only integer values. Now there is a slightly different way of looking at these integers. After all in any linear programming what we are trying to do is we are going to solve something like this. BX_b = small b if this entire thing is written as a vector then this is vector b. The right hand side is vector b. So we have BX_b = small b and you have X_B = B inverse B which is a familiar equation that we have. Now this is an integer value because we have seen that the supplies and demands are integers.

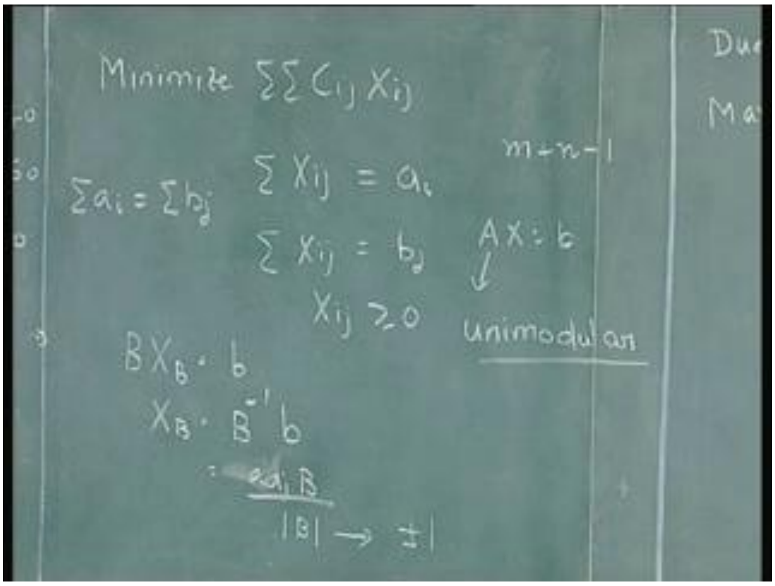
So this is an integer value. Now X_B will be integer, the B inverse is integer. Now B inverse is a cofactor of adjoint B by determinant B. Now if this B matrix is made up of integers then the adjoint B will be integers because every cofactor will be an integer adjoint is a transpose of the cofactor. So adjoint will also be an integer. Now if X_B has to be an integer then this determinant B should be + or – 1. If determinant B is + or – 1 then adjoint B by determinant B will still be an integer matrix. B inverse will be an integer. B inverse b will be also an integer because this is an integer. So we have to check whether this B is such that determinant B is + or – 1. Now if we go back to the transportation and write it in the form A_X is = b where A is the matrix and the b is the vector then this A is made up of 1's and 0's. If you look at these all 1's and 0s's, the coefficient is either 1 or 0.

Now from these coefficients, we can find out this. A matrix is such that every basis b can be drawn from here, after all this basis is pulled out from the coefficient matrix A. So every basic basis B that we pull out from A determinant b is + or – 1 or we can show that this A matrix is unimodular because this A matrix is unimodular. Every B will have determinant B is = + or – 1 and therefore X_B will be integer values. So X_B will be integer values. If a_i's and b_j's are integers then X_B will be integers. So if the supply and the demand points are integers then every solution here will be integers. Now going back to the transportation problem, the problem has m into n variables and m + n constraints like this and we also have in a balanced problem sigma a_i is = sigma b_j.

So if we take these constraints and simply add them vertically we will get sigma a_i is = b_j which is this. Therefore m + n constraints actually represent a linearly dependant system. Now the transportation problem therefore will be a degenerate linear programming problem because of this sigma a_i is = sigma b_j and because of the linear dependency, we realize that actually we have only m + n – 1 independent variables for the transportation problem and that is why in this transportation table we have exactly m + n – 1 basic variables 1, 2, 3, 4, 5 and 6. m + n – 1 is 4 + 3 or 3 + 4 – 1 which is 6. We should remember that even though the transportation is the linear programming problem that balanced transportation problem because of these equations and sigma a_i is = sigma b_j gives us a linearly dependant system and degenerate linear programming problem and we will have only m + n – 1 independent variables which will be there at the optimum. However we do not solve this by the direct application of the simplex.

We solve it by an algorithm which is very similar to the simplex starting with good basic feasible solutions and then moving towards optimal solution. We solve the balanced problem because of the fact that these equations give rise to unrestricted u_i's and v_j's which make it very easy for us to look at this problem through the MODI method and when we use the MODI method, we do not have to worry about some of these becoming negative simply because it is fine with us u_i's and v_j's are unrestricted in sign. So this is about the integer solutions of the transportation problem. So in general we have looked at the various aspects of the transportation problem and we have seen 3 important ways to solve the transportation problem to get the basic feasible solution using the North West Corner rule, minimum cost method or Vogel's approximation method and we said Vogel's approximation method is more desirable compared to the other 2. We saw 2 methods to get to the optimal solution the stepping stone method and the MODI method and we said the MODI is useful for solving large sized problem. We also saw the examples of how to solve a transportation problem which is degenerate by itself.

(Refer Slide Time: 43:53)



We later saw that the transportation problem itself is a degenerate and then we could get into the situation where we have degenerate transportation problem. The degenerate nature of the LP gives us $m + n – 1 = 6$; linearly independent variables but the degenerated transportation problem has one less and we had 5 allocations which was a degenerate transportation problem and then we looked at these various aspects of it, including how the MODI method is considered an optimal? We looked at the economic interpretation of the duals. We made a comparison among the 3 starting solutions and the 2 optimal seeking solutions. We also looked at the way to convert every unbalanced problem to a balanced problem and solve and then we also said that if the supply points and the destination points or the destination requirements are integers then the transportation problem will give integer valued solutions at the optimum. So we use the special algorithms to solve the transportation problem because of the very nature of the problem.

We could get good starting solutions, good basic feasible solutions and then move towards the optimum. In the simplex we normally do not worry about the goodness of the starting basic feasible solution. We were interested in getting a basic feasible solution and we move towards the optimum because of the special nature of the transportation problem where good basic feasible solutions can be obtained. We followed an algorithm which started with a good basic feasible solution with the Vogel's approximation method and then proceeded towards the optimal solution using the MODI or the stepping stone method.

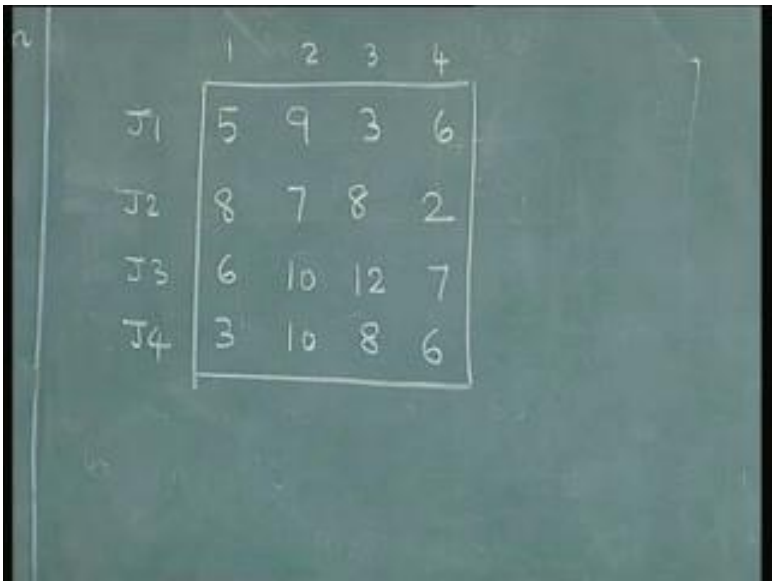
So transportation problems certainly use special algorithms compared to the simplex. However we should also one last observation we should look at is the transportation problem. The algorithm that we saw, particularly the MODI method does exactly what simplex would do. It starts with the basic feasible solution like what simplex would do and then the complementary slackness conditions are satisfied. It evaluates the corresponding dual and if that dual is feasible then it becomes optimum. So the transportation algorithm is also an example of primal algorithm. Remember that we saw some examples. We saw what a primal algorithm is and what is dual algorithm is.

A primal algorithm is one which will start with the basic feasible solution after they establish the basic complimentary slackness, evaluate the corresponding dual and if the dual is feasible then it will give optimal to both primal and dual which is exactly what this algorithm does. We evaluated the dual at every stage after satisfying the complimentary slackness and then when the dual is feasible, which means when all the h_{ij}'s or $C_{ij} - u_i + v_j$ is greater than or equal to 0 we got the optimum.

So like simplex, a transportation algorithm is also an example of a primal algorithm. Dual simplex is an example of dual algorithm. So with this we come to end of our treatment of the transportation problem and then we move into another important problem in linear programming which is called the assignment problem.

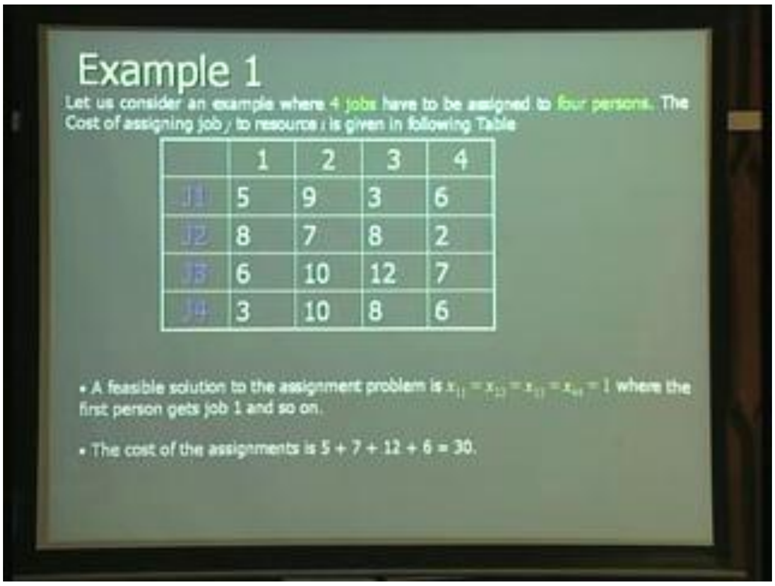
Now we look at the assignment problem. Let us consider the assignment problem using a small example. The example is as follows.

(Refer Slide Time: 47:53)



Let us consider a 4/4 problem. Let us assume that there are 4 jobs.

(Refer Slide Time: 48:34)



We call j_1, j_2, j_3 and j_4 that have to be performed and let us assume that there are 4 people whom we call 1, 2, 3, and 4 who can do these jobs. The assumption is every person here can do every one of these jobs. For example this person can do jobs 1, 2, 3, 4 and so on.

Now what is inside this matrix is the amount or the charge or the cost it takes if this person does this job. For example this person is going to charge 5 rupees to do this job, he is going charge 8 rupees to do this job. He is going charge 6 rupees to do j_3 and 3 to do j_4. Second person would charge 9 rupees to do j_1. He might charge 7 rupees for j_2 and so on. So basically whatever is inside the matrix is the cost that we or the money that we have to spend if this person is given this job. Now the assumptions are we want to give only 1 job to 1 person. 1 person gets 1 job and 1 job goes to 1 person. We do not want a situation where this person gets jobs j_1 and j_2 or j_1 could go to 2, 3 extra. We do not want to give 1 job to 1 person. So the problem is allocate or assign the jobs to people under the condition that 1 job goes to only 1 person and 1 person gets only 1 job and the total cost is minimized. Now such a problem is called the assignment problem. Now what are the decision variables? Each 1 job goes to 1 person and1 person gets only 1 job. So X_{ij} = 1. If job i goes to person j, now sigma a_j sigma X_{ij} = 1, for every i summed over j.

(Refer Slide Time: 50:51)

So j_1 for every i, i = 1, summed over j means this job goes to only 1 person. Now this for every job summed over people, now this person gets only 1 job when i sum it over I, this person gets only 1 job other would mean this job goes to only 1 person. I want to minimize double sigma C_{ij} X_{ij} which is the total cost of allocation and my X_{ij} is clearly a (0, 1) variable which means X_{ij} is 1. If job i goes to person j = 0 otherwise now, X_{ij} does not take any other vale other than 1 or 0. Now this problem is called the standard assignment problem and it is also a minimization problem and it is formulated this way. Now let us look at some aspects of the assignment problem. Now the first aspect of that this problem looks like a transportation problem. The objective function is similar and the constraints are similar except that I have different numbers here. I had a_i and a b_j for transportation problem.

Remember the assignment problem is normally a square matrix they have an equal number of jobs and equal number of people whereas in a transportation problem we had m supply points and n destination points. So the first thing is the assignment problem is a special case of a transportation problem where m is = n. You can always think of this as supplies and this as demands or this (Refer Slide Time: 52:00) as supplies and this as demands. So you could think of the assignment problem as a special case of a transportation problem with m = n and with all $a_i = b_j = 1$. It is like 1 unit is available in one of the supply points and that 1 unit is to be transported to each one of the demand points and what is the cheapest way to do it?

So the assignment problem is a special case of a transportation problem with m = n. Every one of the supplies $a_i = b_j = 1$, so X_{ij} greater than or equal to 0 will become X_{ij} = (0, 1).Now can we solve the assignment problem as a transportation problem because it is a special case or do we need other algorithms to solve it? The answer is what happens to the assignments? Let us take now any feasible solution, it would give, for example, if I give this to person 1, this to person 2, this to person 3, this to person 4, I get a feasible solution.

It may not be a best solution. It may not be the optimal solution but it is a feasible solution.

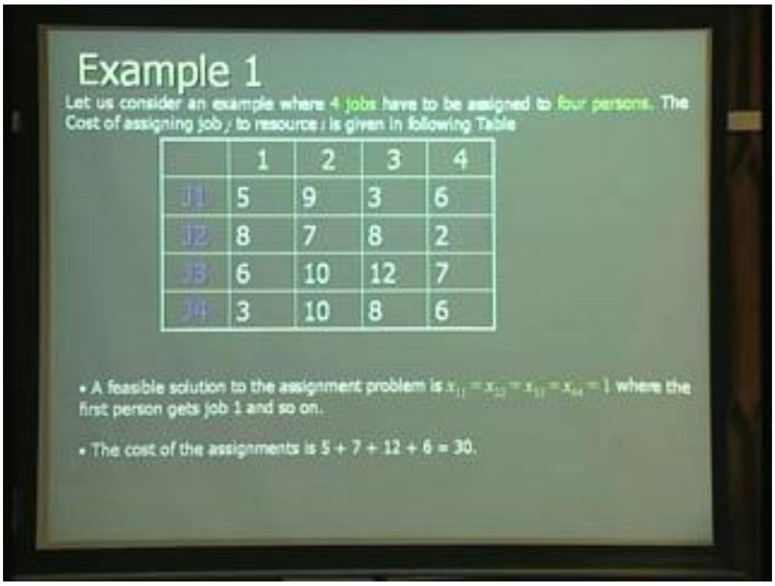
$X_{1\,1} = 1$; $X_{2\,2} = 1$; $X_{3\,3} = 1$; $X_{4\,4} = 1$ is feasible;

$X_{1\,2} = 1$; $X_{2\,4} = 1$; $X_{3\,3} = 1$ and $X_{4\,4} = 1$ is feasible

The point I am trying to convey is every feasible solution has exactly 4 allocations, 1 per row and 1 per column. So the assignment problem has if we are looking at n/n. n jobs are available and n people are available. We have n square variables or n square decision variables we have 2 n constraints n for this and n for this but the optimum solution or every feasible solution has only n allocations at the optimum or for every feasible solution which means if we have 2 n constraints then we need 2 n basic variables, out of which there are only n allocations which means out of the 2n basic variables, n basic variables take value 1 at the optimum and there are another n basic variables, which takes value 0 at the optimum.

So the assignment problem is a very degenerate transportation problem. It is very degenerate if this were a transportation problem then we would have 2n – 1 allocation because transportation problem has m + n – 1 allocation. Now out of these 2n – 1 allocation n allocations have value 1 and the remaining n – 1 allocation has value 0. So it is a very degenerate transportation problem and any degenerate transportation problem is not very comfortable to solve because it would have too many intermediate iterations without reducing the objective function. So we do not solve the assignment problem directly using the transportation algorithm. So we need special algorithms to solve the assignment problem. Let us also look at some aspects of solving the assignment problem.

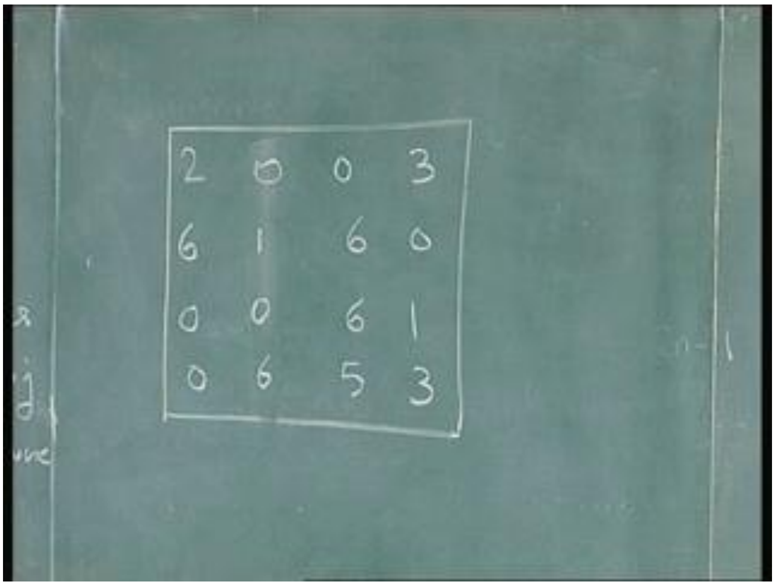
(Refer Slide Time: 56:25)



Let us look at the first condition. For this, the first condition would be the C_{ij}'s are greater than or equal to 0. It is a very reasonable assumption to have the C_{ij}'s greater than or equal to 0 because if some people are going to do some jobs, they are going to charge some money. We are not going to have the negative C_{ij}'s. Here when C_{ij}'s are greater than or equal to 0, a very simple result would mean that if I have a feasible solution with Z = 0, then it is optimal because we have C_{ij}'s greater than or equal to 0. If we have a feasible solution with Z = 0, that solution is an optimal has a very intuitive result because C_{ij}'s are greater than or equal to 0. Now we look at the first row of this. This job has to go to one of these people. If all of them increase their rate by say 1 rupee and they make it 6, 10, 4 and 7 respectively, the decision will not change because all of them have uniformly increased by 1 rupee.

The solution will not change but the objective function value alone will change. Similarly if all of them reduce their charge by 1 rupee, the solution does not change. Only the objective function will change. So what we try to do is we try to create as many 0’s in this matrix because we want a feasible solution with Z = 0. So we assume that each one can reduce by a certain quantity till the first one becomes 0. The solution does not change and only the objective function value changes. So subtract the row minimum from every row, so that the solution does not change and the objective function alone changes.

(Refer Slide Time: 58:24)



We get this row; subtract the row minimum which is 3 to get 2, 7, 2, 6, 0 and 3. Here the row minimum is 2. So you get 6, 5, 6, and 0. Row minimum is 6, 0, 4, 6, and 1. Row minimum is 3, 0, 7, 5 and 3. Now let us go back and do this for the column minimum. Now once again this person anyway should get a job. So if this person uniformly increases his rate by 1 rupee, for each one of these jobs, the solution again will not change and only the values of the objective function will change. Similarly if this person decreases it by 1 rupee, once again the solution will not change. So we can do column subtraction. Subtract the column minimum from every column and try to generate as many 0’s as we can. Now already these matrixes has been reduced to this, so we take this matrix and do the column minimum subtraction this is the only column which does not have 0.

Once again subtract the column minimum to get 2, 1. So we had 4 here. We had 5 here. We had subtracted the column minimum to get 0, 1, 0 and 6. You get 0, 1, 0 and 6. From this matrix, we have converted into a new matrix with every row and every column taking 0 values. Whether we solve this or whether we solve this, the solution will be the same. This has many 0. We try to get a feasible solution from this matrix with Z = 0 then it will be optimal. We will see how we get the feasible solution. To get to a feasible solution with 0 costs from this matrix, will be dealt with later. So right now we have just identified the assignment problem and we have understood that the feasible solution with Z = 0 is optimal. We have converted this to a matrix with 0’s. We will get the optimal solution for this in the next lecture when we meet.