Fundamentals of Operations Research
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Lecture No. # 1
Linear Programming Formulations

This course is titled in Fundamentals of Operations Research. In this course, you learn various tools of operation research such as Linear Programming, Transportation, Assignment problems and so on. Before you begin let us see what operations research is.

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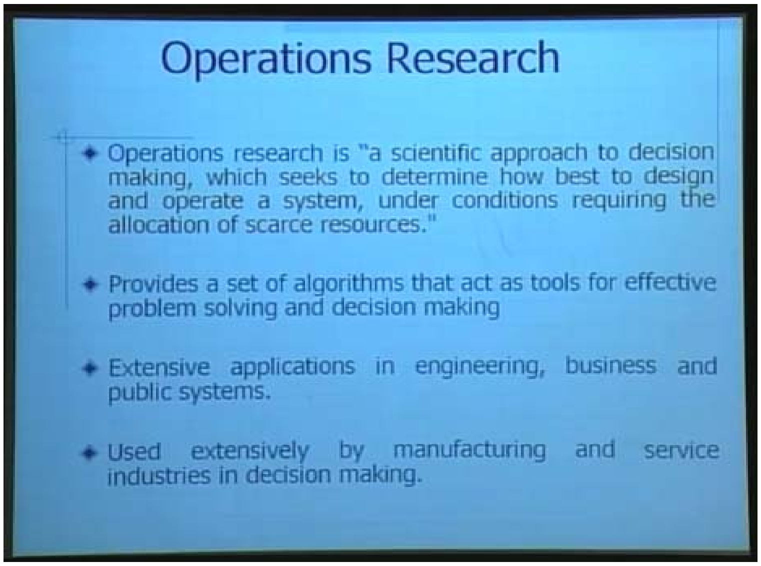

Operation Research is also called OR for short and it is a scientific approach to decision making which seeks to determine how best to design and operate a system under conditions requiring allocation of scarce resources. Operations research as a field, primarily has a set or collection of algorithms which act as tools for problems solving in chosen application areas. OR has extensive applications in engineering business and public systems and is also used by manufacturing and service industries to solve their day to day problems.

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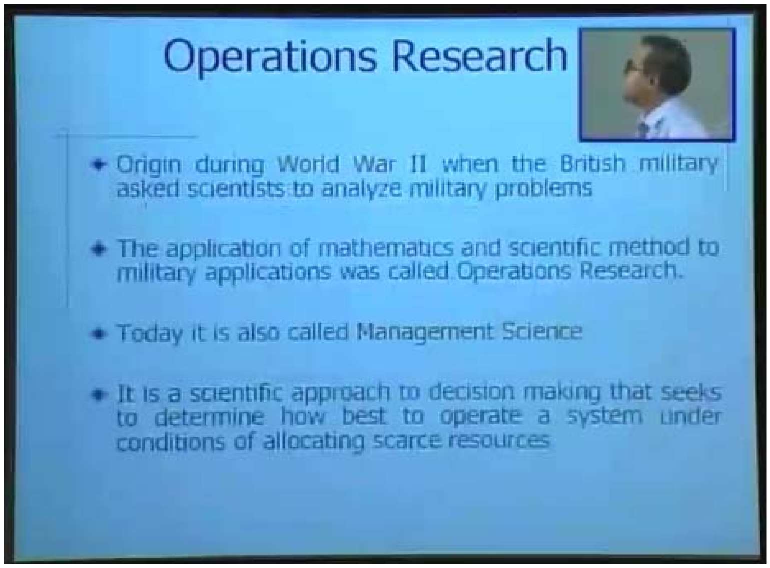

The history of the OR as a field as goes up to the Second World War. In fact this field operations research started during the Second World War when the British military asked scientists to analyze military problems. In fact Second World War was perhaps the first time when people realized that resources were scarce and had to be used effectively and allocated efficiently. The application of mathematics and scientific methods to military applications was called operations research to begin with. But today it has a different definition it is also called management science. A general term Management Science also includes Operation Research. In fact these two terms are used interchangeably. A set of tools that are used to solve problems is also called Management Science. Today OR is defined as a scientific approach to decision making that seeks to determine how best to operate a system under conditions or allocating scarce resources. In fact the most important thing in operations research is the fact that resources are scarce and these scarce resources had to be used efficiently.

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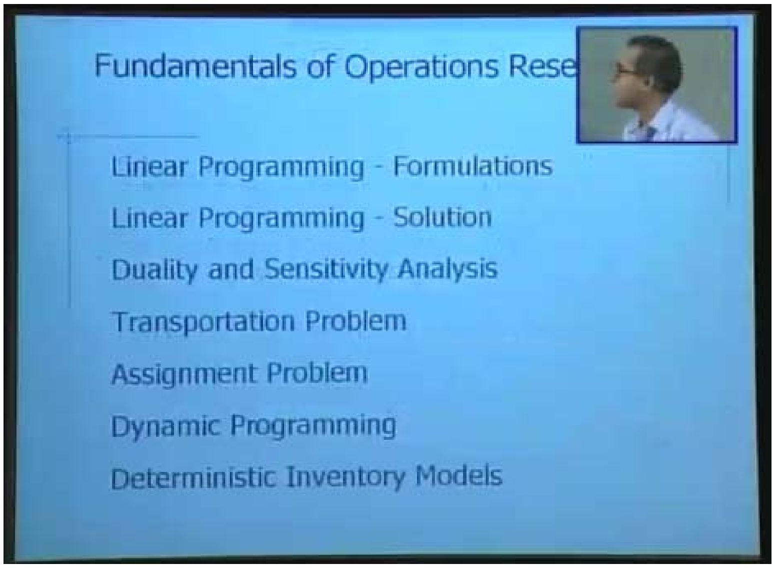

In this course we will primarily handle the above topics. We started with Linear Programming. We introduced the formulations or problems to formulate it in Linear Programming. We will do four examples there.

We will also spend lot of time on solving linear programming problems and understand the various issues associated with the solution. We then get into a specific title called duality and sensitivity analysis where we explore this linear programming topic further. Then we move on to two important problems called the Transportation problem and the Assignment problem and when we do Dynamic Programming and then we also spend some time on Deterministic Inventory Models. So these will be the topics that will be covered in this course.

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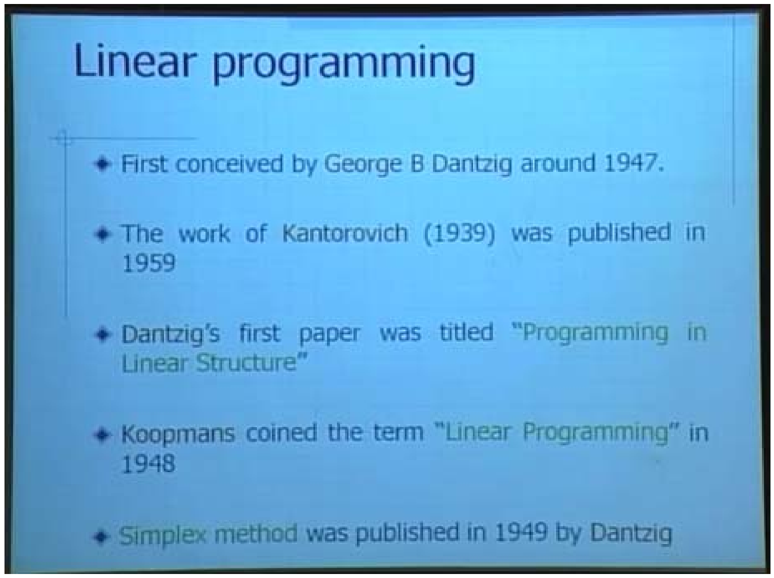

Linear programming was first conceived by Dantzig, around 1947 at the end of the Second World War. Very historically, the work of a Russian mathematician first had taken place in 1939 but since it was published in 1959, Dantzig was still credited with starting linear programming. In fact Dantzig did not use the term linear programming. His first paper was titled 'Programming in Linear Structure'. Much later, the term 'Linear Programming' was coined by Koopmans. The Simplex method which is the most popular and powerful tool to solve linear programming problems, was published by Dantzig in 1949. So this is the brief history of this field called Linear Programming.

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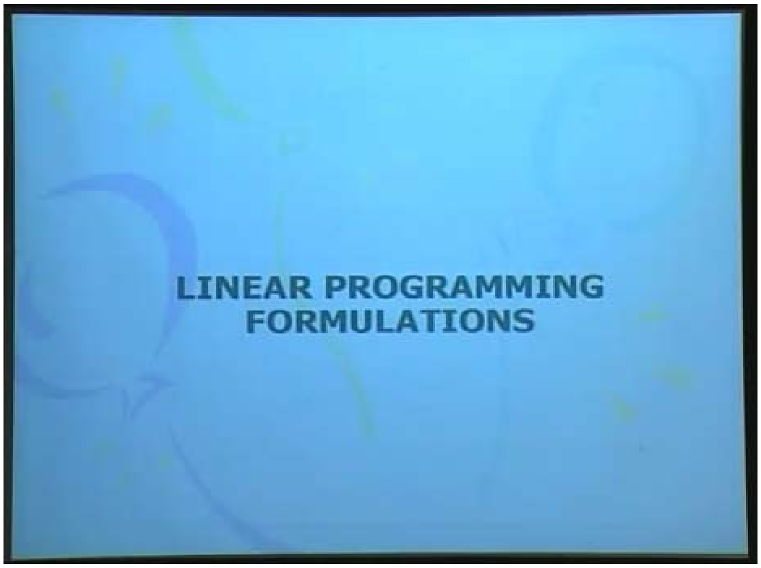

The first title is called linear programming formulations where we try to introduce to you how to formulate real life problems as linear programming problems and to understand the various terminologies that is used in formulating linear programming problems. We begin linear programming with an example.

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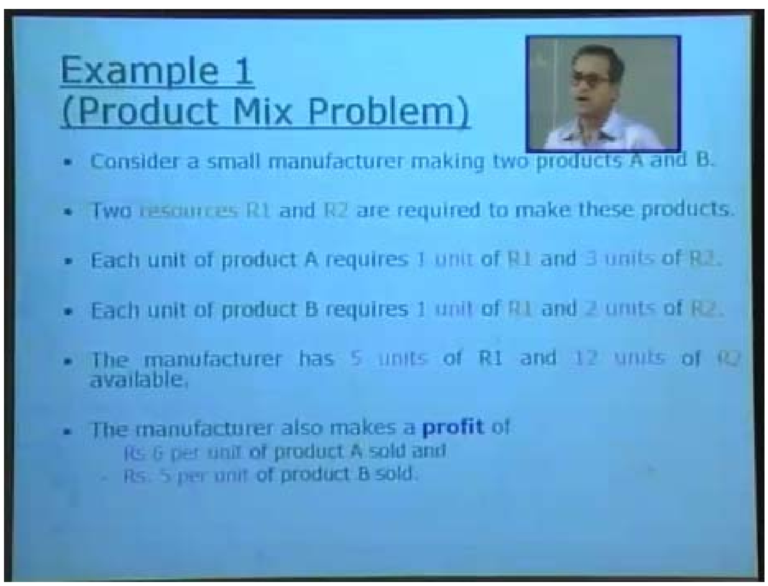

So let us consider a small manufacturer making two products called A and B. Two resources are required which we call as R1 and R2 to make these products. Now each unit of product A requires one unit of R1 and 3 units of R2.

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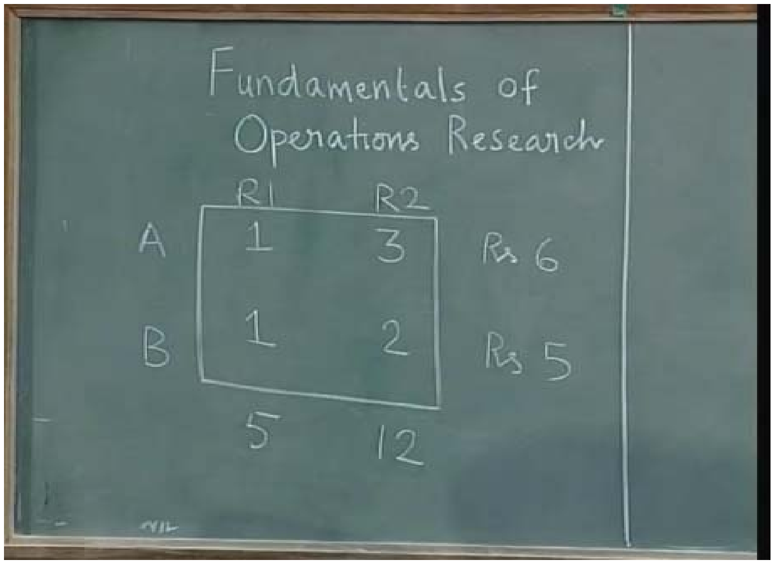

For example we have these two products A and B. Two resources are need R one and R2. A requires one unit of R1 and three units of R two. B requires one unit of R one and two units of R two. Manufacturer has 5 units of R one available and 12 units of R two available. The manufacturer also makes a profit of rupees 6 per unit of A sold and rupees 5 per unit of B of sold. So this is the problem setting that we are looking at. Now what are the issues that this manufacturer faces?

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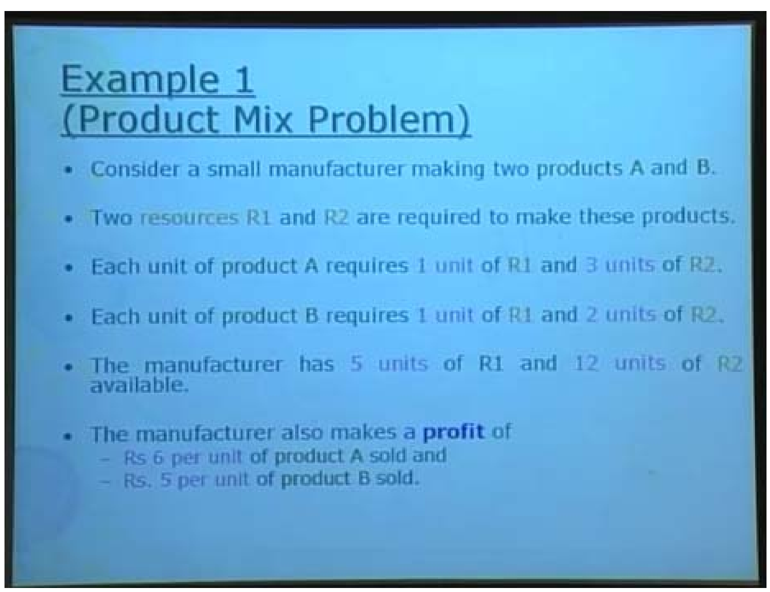

One, the manufacturer would like to produce in such a way that the profit is maximized. How does the manufacturer go around formulating his problem?
The first thing is that the manufacturer has to decide on is the number of units of A and B that is to be produced.

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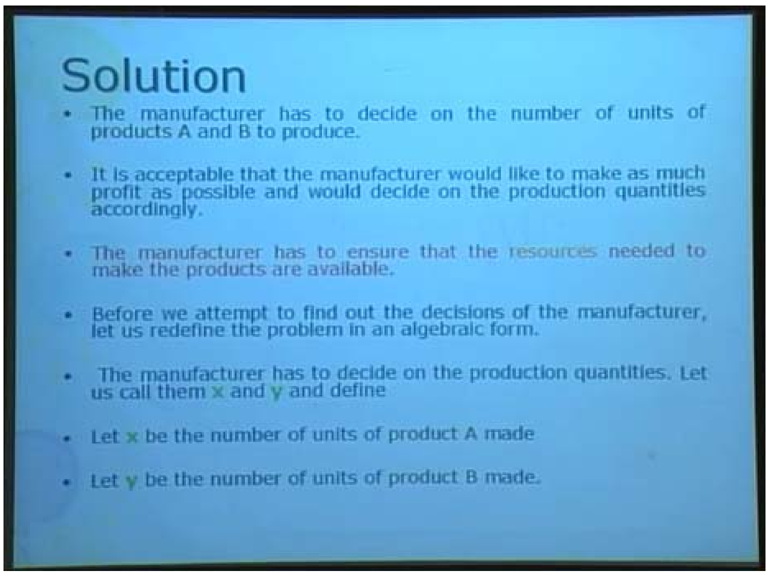

It is also reasonable to assume that the manufacturer produces this A and B in such a way to maximize the profit.

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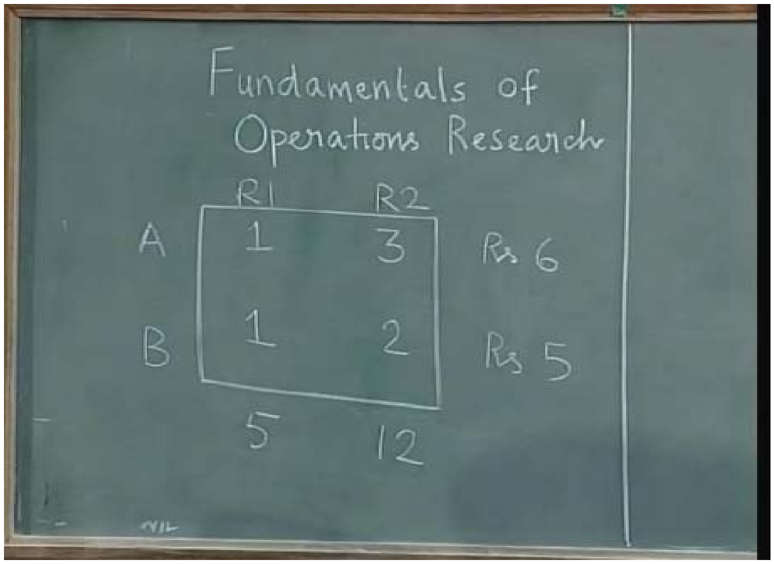

So first thing that the manufacturer has to do is to decide or determine how many units of A, and how many units of B, he or she is going is to produce.

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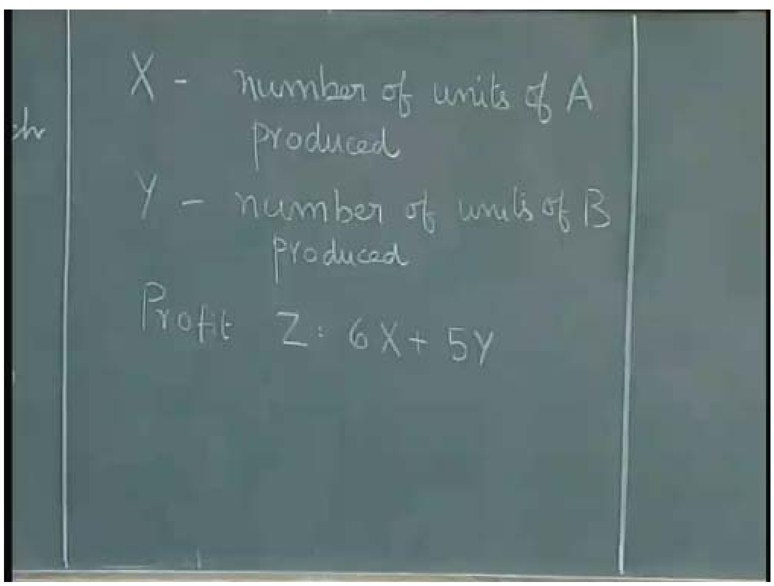

So we first call these as X and Y. So let X be the number of units of A produced and Y be number of units of B produced. So if the manufacturer produces X units of A and Y units of B then the money that the person makes or the profit that the person makes, which we call as Z, will now be 6X + 5Y. Now the manufacturer would ideally like to produce as much of A and as much of B possible.

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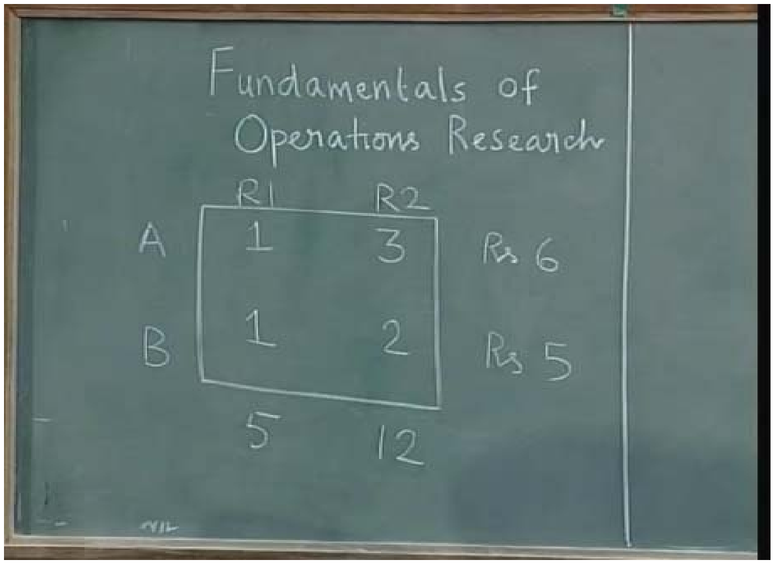

But what happens is the availability of these resources will now try and restrict the amount of A and B i.e., X and Y that the manufacturer can produce. So we look at the resource requirement to produce X of A and Y of B.

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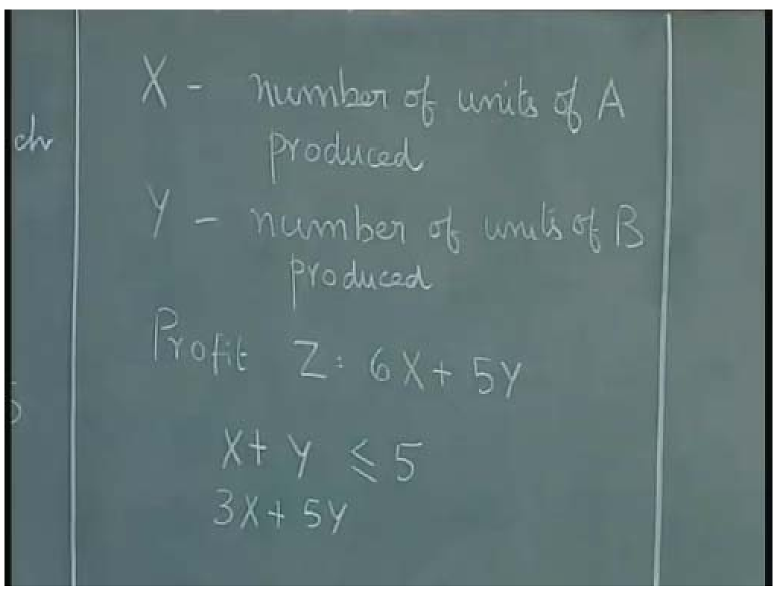

So since each unit of A requires one of R1 and each unit of B requires one of R1, the total amount of resource R1 required is X + (Refer Slide Time 08:42) X + Y. What is available is 5. This X + Y should not exceed 5 or X + Y is less than or equal to 5. It cannot be more than 5. Similarly 1 unit of A requires 3 of R2 and 1 unit of B requires two of R2. So the requirement to produce X units of A and Y units of B is given by 3X + 5Y.

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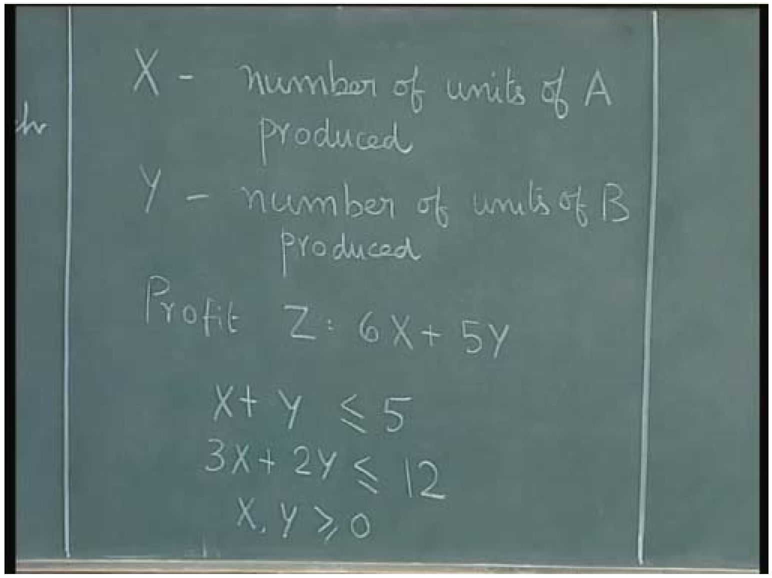

This (Refer Slide Time: 9:27) has to be 3X + 2Y and this has to be less than or equal to 12 this cannot be greater than 12. Also we need to explicitly state that this manufacturer will make X and Y which are greater than or equal to zero that is the person cannot produce negative quantities of both X and Y. And you can see this here in this power point slide.

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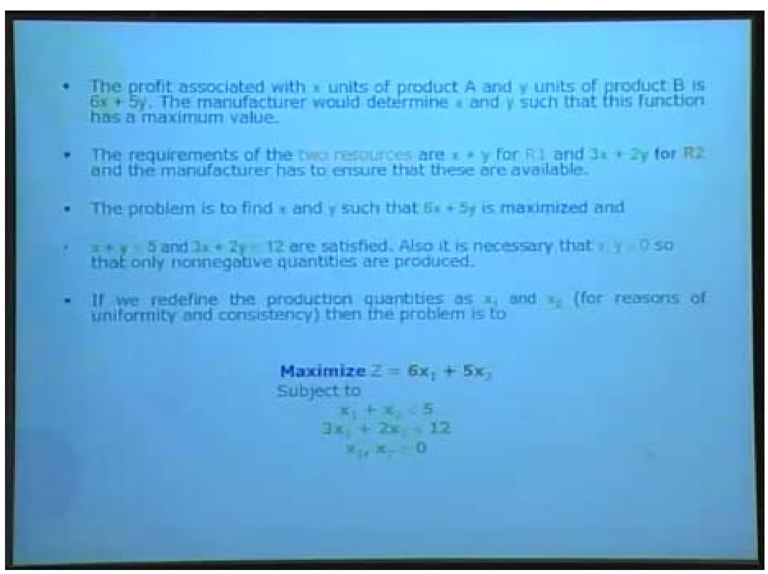

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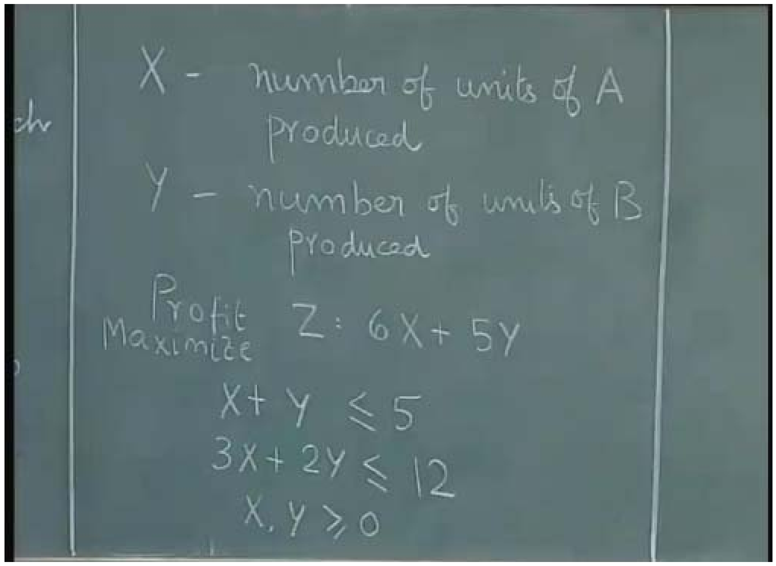

Also since the person wants to maximize the profit, the profit function Z is to be maximized. So the person would like to find out X and Y in such a way that the function 6X + 5 Y is maximized and then X and Y has to satisfy these conditions,

(i) X + Y is less than or equal to 5 and

(ii) 3X + 2 Y less than or equal to 12

And explicitly stating that X and Y has to be greater than or equal to zero which is what is shown here.

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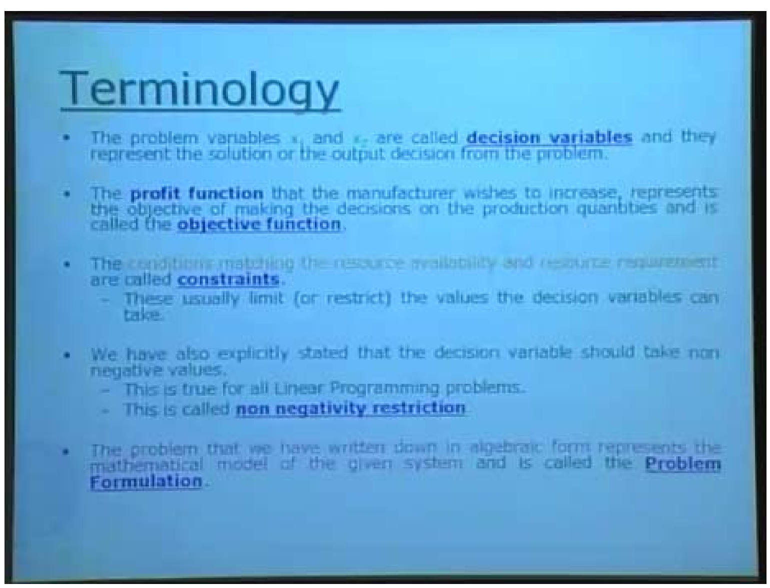

This is called formulating a linear programming problem. What we have done now is we have converted the descriptive portion that was shown here earlier into a mathematical form and this mathematical form is called as formulation. Now let us define some amount of terminologies and we will try to maintain these terminologies consistently in all our formulations. The first thing that we define is called (Refer Slide Time: 10:58 min) decision variables. The two variables that we solve this problem X and Y are called Decision variables and they represent the output. Whatever comes out as a solution is going to imply that we make some amount of A and some amount of B, X amount of A and Y amount of B. The values of X and Y is the output or the outcome after having solved this problem. So the variables that we want to solve are called Decision Variables and then we have a function which represents what we are solving this problem for. And as far as this problem is concerned, we want to solve this problem to maximize the profit or to get maximum profit and this is the objective with which the manufacturer works.

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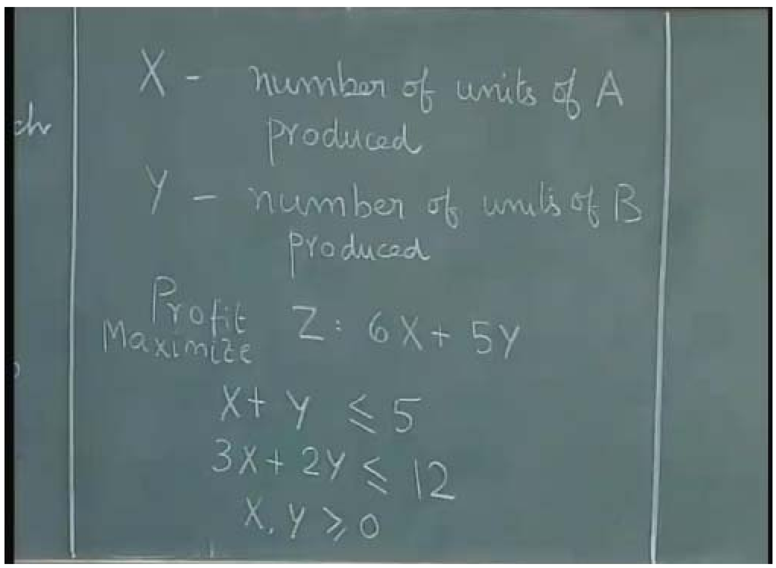

This function 6X + 5Y represents the profit function which is to be maximized and this is called the Objective Function. So the second term that is used is called Objective Function which is precisely the purpose of working on this problem.

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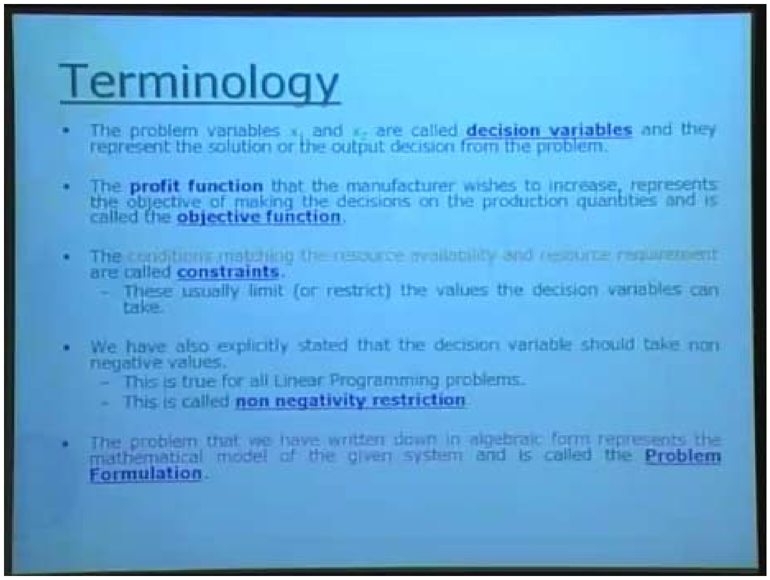

The third set of things is the Condition. The conditions actually represent the resources that are available and the resources that have to be used efficiently.

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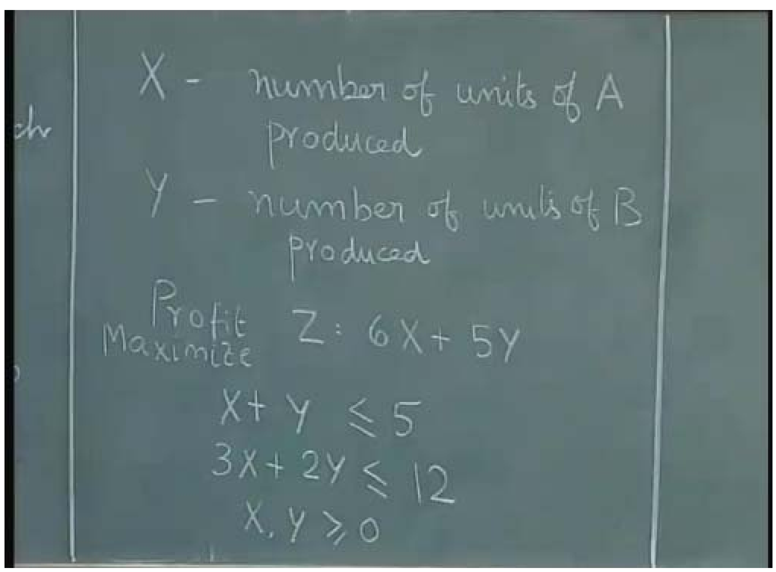

So let these conditions that we have to satisfy namely,

(i) X + Y less than or equal to 5 and

(ii) 3X + 2Y less than or equal to 12 are called constraints because they constrain the decision variable from taking as larger value as we would ideally like the variable to take. Because if we simply want to maximize 6X + 5Y without these constraints, then X and Y can take as higher value as possible. Now these constraints restrict the decision variables form taking an unlimited or a very high value and so they are important and we have to explicitly state in all linear programming problems (we will define what a linear programming problem is formally) that these decision variables will have to be greater than or equal to zero.

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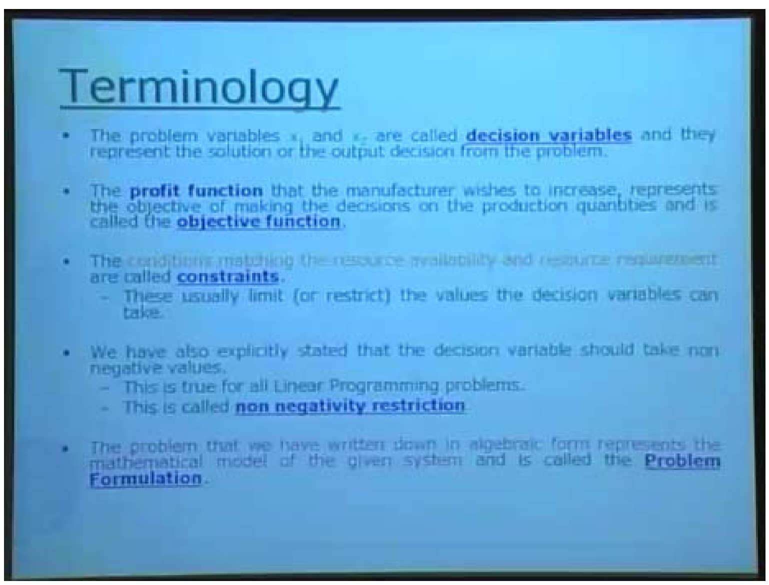

So the four terms that we have tried to introduce here as you can see some there (Refer Time Slide: 13:15) are

(i) The decision variables
(ii) The objective function
(iii) Constraints and
(iv) The non-negativity restriction

So in a problem in where we define all the four and write it in the mathematical form it is called formulating a linear programming problem or formulating an operation research problem. (We will come to the (Refer Slide Time: 13:37) definition a little later).

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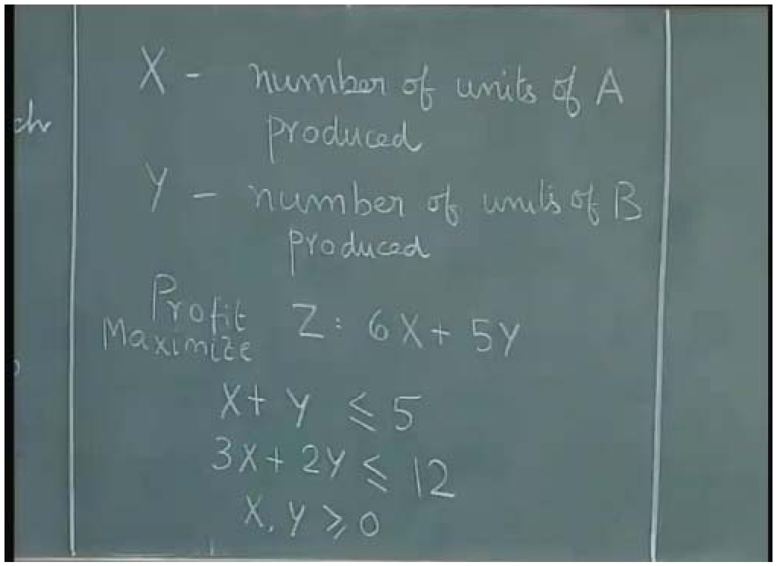

Now as far as this problem is concerned, X and Y are the Decision variables. The functions 6X + 5Y that is to be maximized is called the Objective function.
X + Y less than or equal to 5 and
3X + 2Y less than or equal to 12 are the Constraints.
X, Y greater than or equal to 0 are the non negativity restriction

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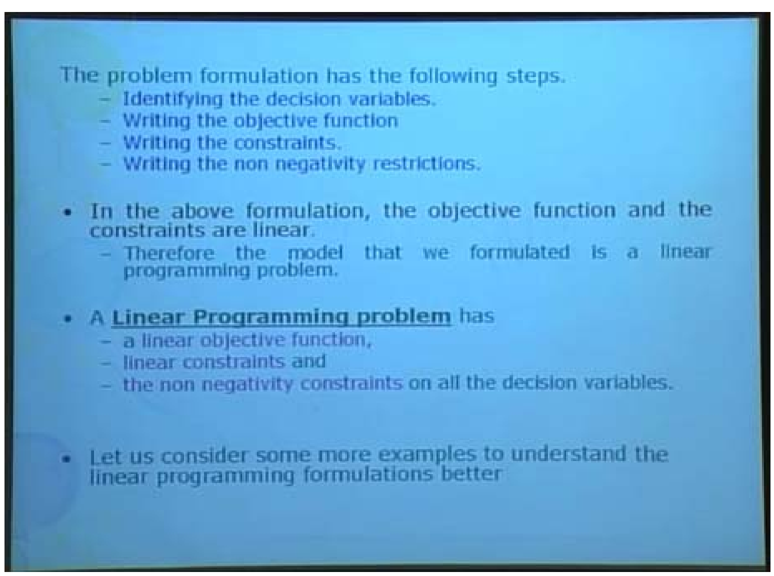

So the problem formulation has four steps.

- Identifying the decision variables
- Writing the objective function
- Writing the constraints and
- Explicitly stating the non negativity restriction on the variables.

Next, ((Refer Time Slide: 14:17)) in this formulation, we realize that the objective function is a linear function of the decision variables and all the constraints are linear functions of the decision variables.

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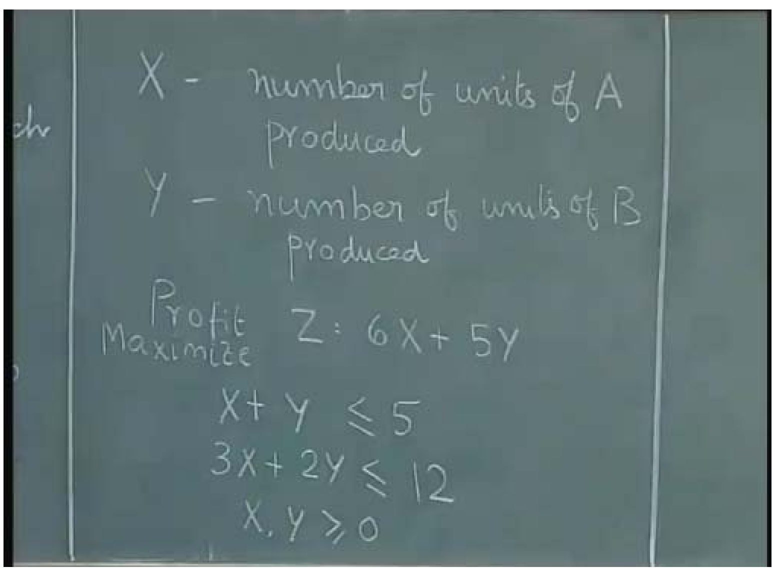

So if you write a formulation such that the objective function is linear and the constraints are linear then such a formulation is called a Linear Programming Formulation.

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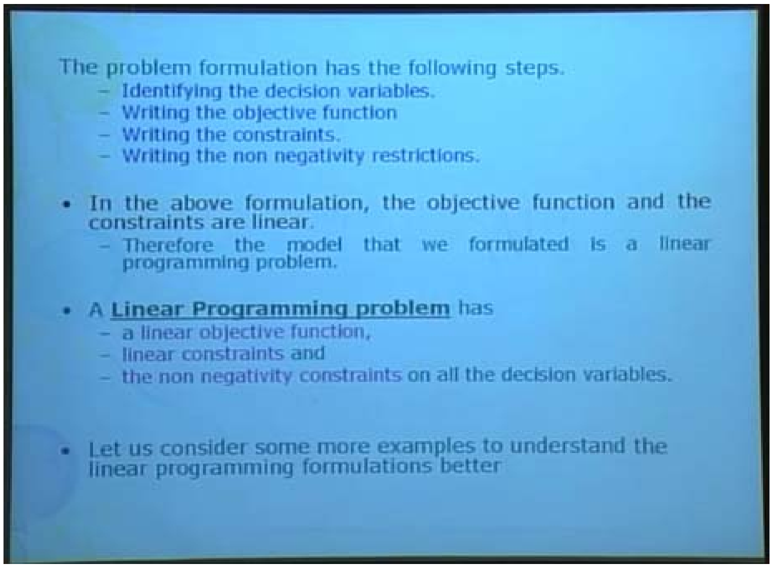

So the important requirements of a linear programming formulation are that the objective function is linear, the constraints are linear and we explicitly state that the non negativity restriction is held.
So these are the three important aspects in a linear programming problem.

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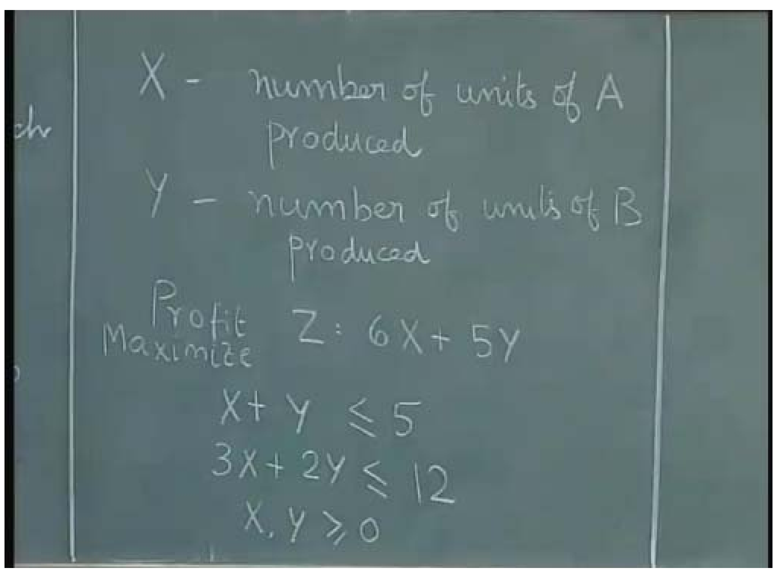

If either the objective function or the constraints are Non linear or the case maybe that this does not exist then it is not a linear programming problem.

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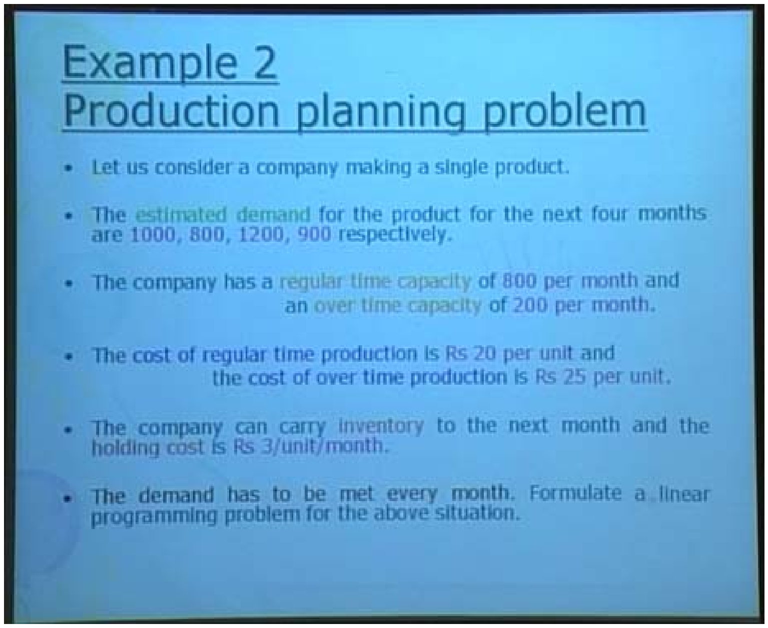

Now shall look into some more examples to understand the linear programming situation. So this is the simple formulation where we have looked at two variables, two constraints, and a maximization objective function.

Now we shall look at more formulations to understand various other aspects of problem formulation in Linear Programming problems.

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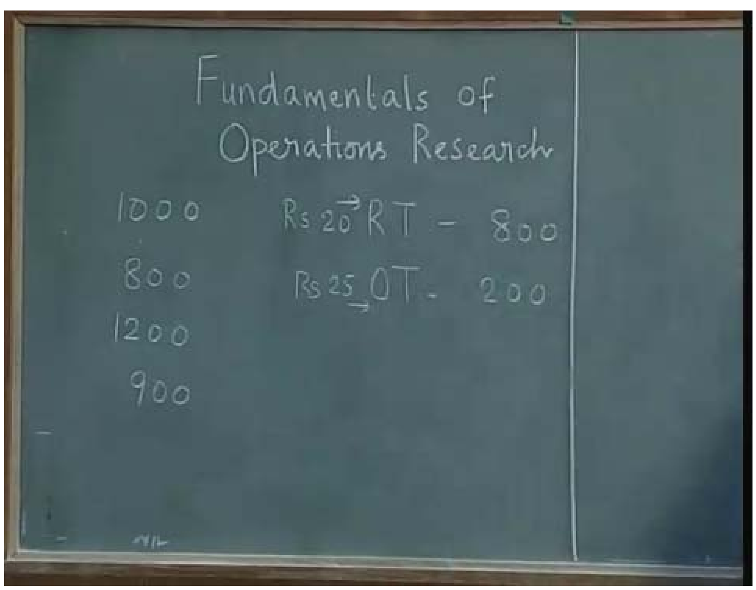

Let us look at the second example which is called as Production Planning Problem and the problem is as follows:

Let us consider a company making a single product demand of 1000, 800, 1200 and 900 respectively for 4 months. Now the company wants to meet the demand for the product in the next 4 months. The company can use two modes of production. There is something called as

Regular Time Production and Overtime Production. Now the regular time capacity is 800/month and overtime capacity is 200/ month. In order to produce one item in regular time, it costs Rs 20 and to produce overtime it costs Rs 25. The company can also produce more in a particular month and carry the excess to the next month.

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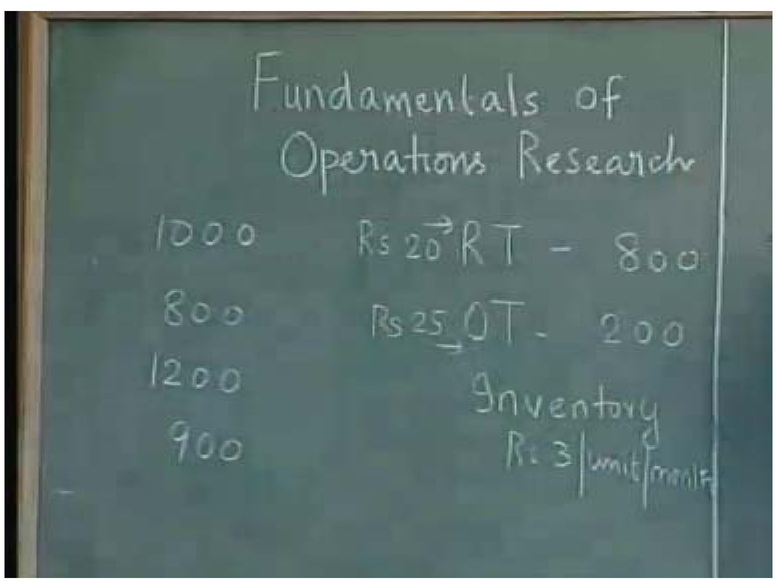

Such a carrying cost is rupees 3. What we call the inventory cost or carrying cost is Rs 3 per unit per month.
The important condition is that the demand has to be met every month. We cannot afford to have shortages.

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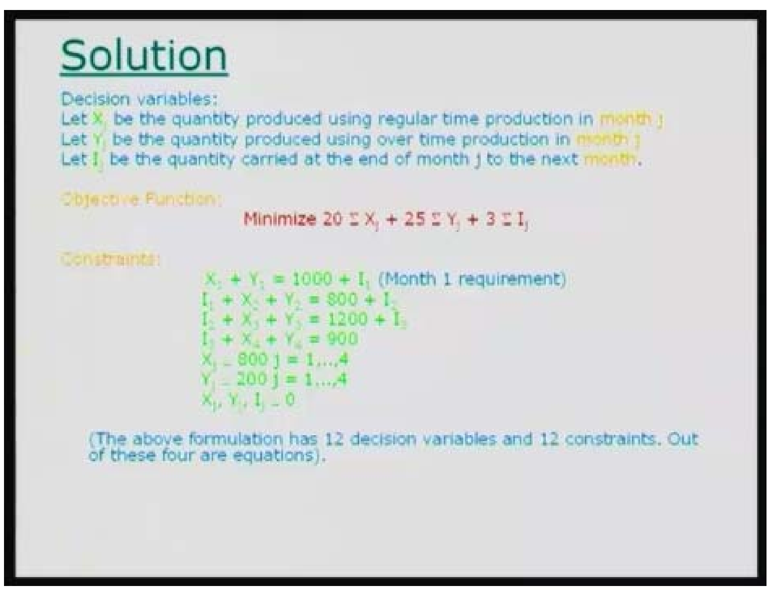

Now let's try to formulate a problem that represents this situation and also try to understand a few more things about problem formulation as we proceed. Now the first thing we do here is that we define X_1, X_2, X_3, and X_4.

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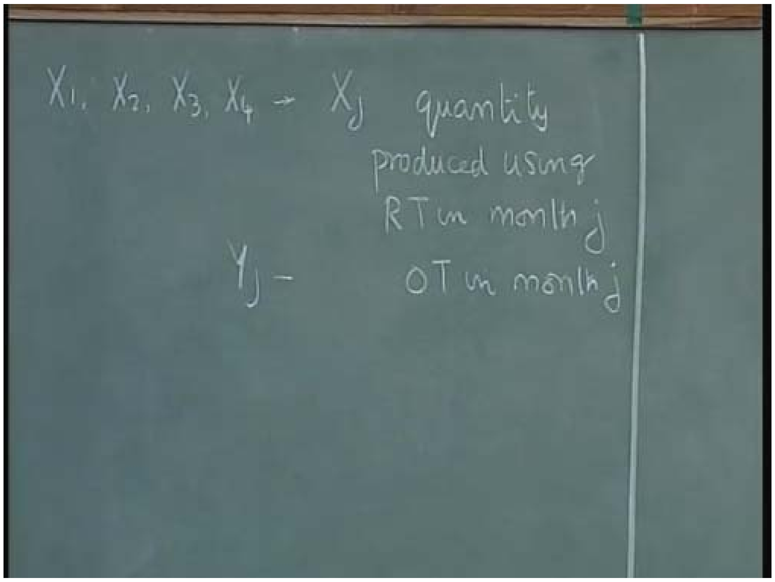

Or in general $X_{j.}$ as quantity produced using regular time production in month j and Y_j as quantity produce using overtime in month j. Now these are our decision variables, X_j and Y_j (quantities that are produced).

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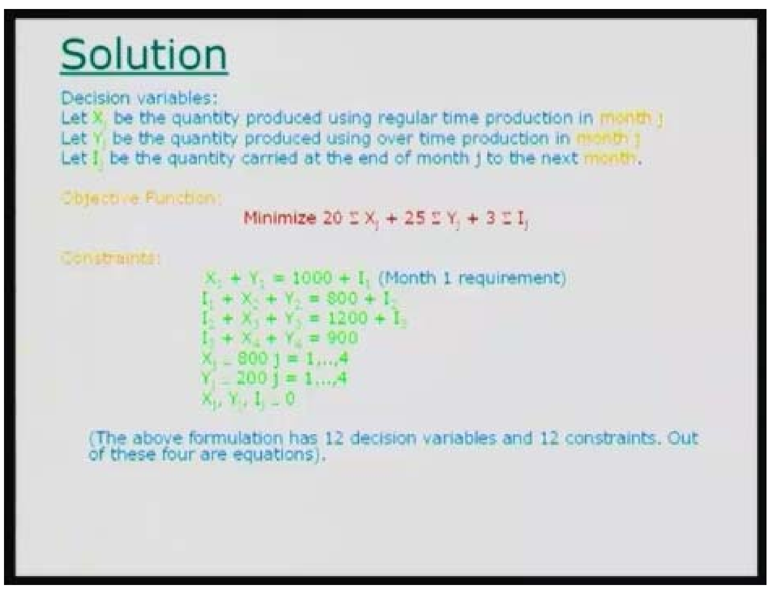

Now let us first write the constraints. In fact it is important first to define the decision variables then to write the objective function & the constraints. Sometimes the constraints can be written first and then the objective function or sometimes the Objective function & then the constraints and then explicitly state the non negativity. So in this example we will try the write the constraints first.

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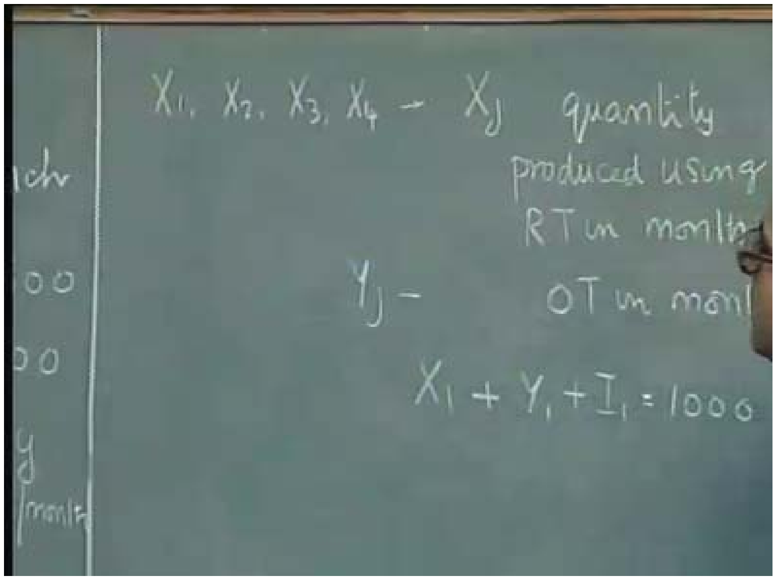

Now as far as the first month is concerned, we produce $X_1 + Y_1$. We assume that we do not have any initial inventory or any stock to begin with. So whatever is produced in the first month is $X_1 + Y_1$. We have to meet the first month's demand which is 1000. So we write $X_1 + Y_1$ is greater than or equal to 1000. We need to produce more than the first month's demand of 1000. In fact another alternative is that we can either state $X_1 + Y_1$ is greater than or equal to 1000, which implies that $X_1 + Y_1$ is exactly 1000 or more than 1000. If $X_1 + Y_1$ is more than 1000 then the balance is carried to the next month. So we can define another term called I_1 which is the quantity that is carried to the next month.

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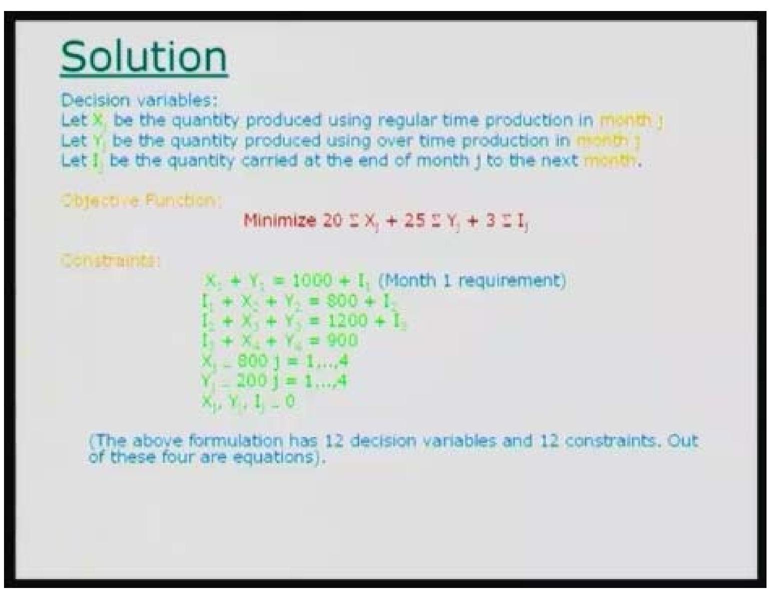
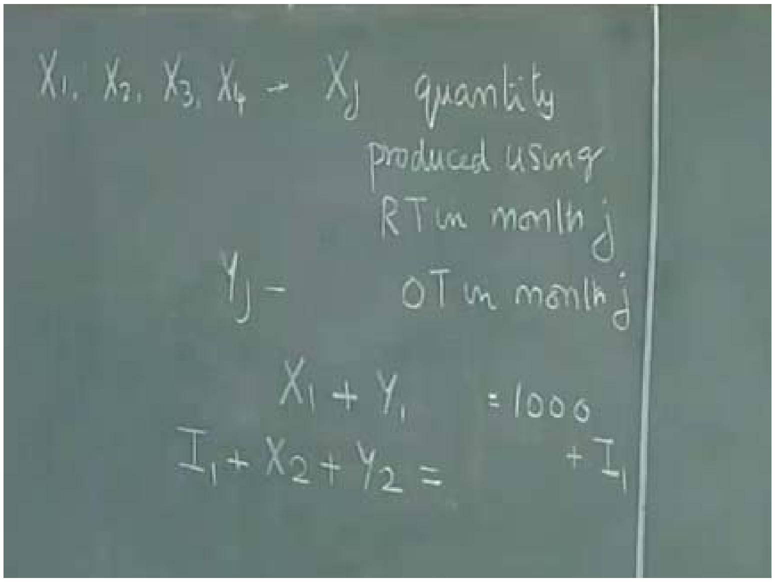


Therefore we can say $X_1 + Y_1 + I_1$ are equal to 1000. Now for the second month we have already carried an I_1 here (Refer Slide Time: 19:48).



So we start with an I_1 and then we produce X_2 in regular time. [(Refer Slide Time 19:54)]. $X_1 + Y_1 = 1000 + I_1$
The balance is carried i.e., so $X_1 + Y_1$-1000 is carried so that is $I_{1.}$ So we begin with $I_{1.}$ We produce X_2 by regular time in the second month and Y_2 by overtime in the second month and that has to be equal to the second month's demand of 800. Plus, what we carry at the end of the second month into the third month.

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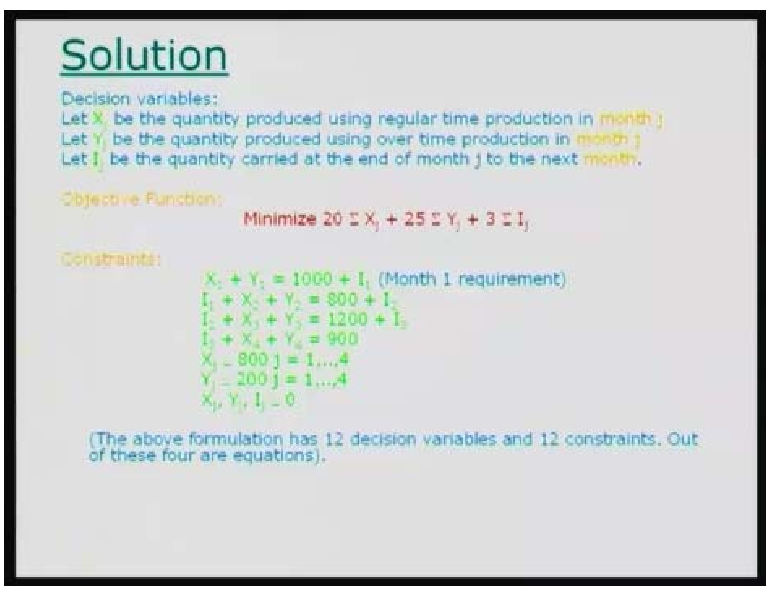

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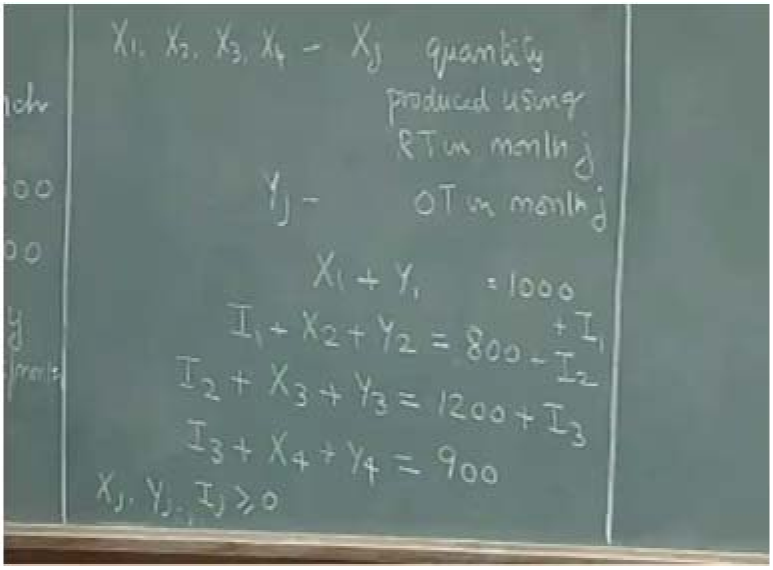

So, if this left hand side which is the quantity available at the end of the second month, is more than the demand which is 800, and if it exceeds 800 then the balance is carried to the third month.

So here (Refer Slide Time: 20:35), we have this I_1 and I_2 which are the quantities that are carried at the end of the month to the next month. I_j is the quantity that is carried at the end of month j to meet the demand of month j + 1. Similarly for the third one, we begin with I_2 to produce X_3 + Y_3 which will now be equal to 1200 which we have to meet and if it exceeds 1200, the balance I_3 is carried to the fourth month. Now for the fourth month, we begin with I_3 to produce X_4 + Y_4 and that is equal to 900. We are not using an I_4 here because the problem stops at the end of the fourth month and therefore we would adjust these quantities in such a way that we are able to get exactly 900 which is required to meet the demand of the fourth month. So we do not use an I_4 here. Moreover, we also have a situation here wherein all XY X_j Y_j and I_j are greater than or equal to 0.We have to write the objective function. We have not written the objective function as such.

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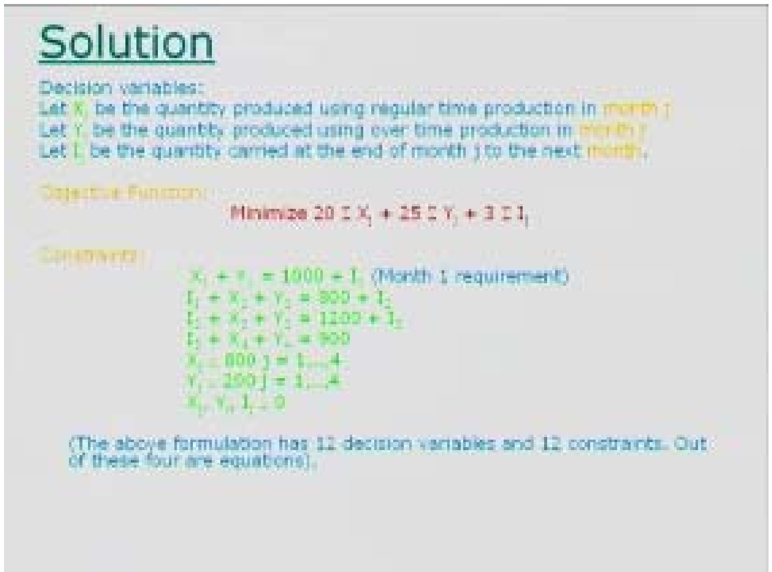

The objective function will be equal to the cost of regular time production. The cost of the regular time production we have is 20.

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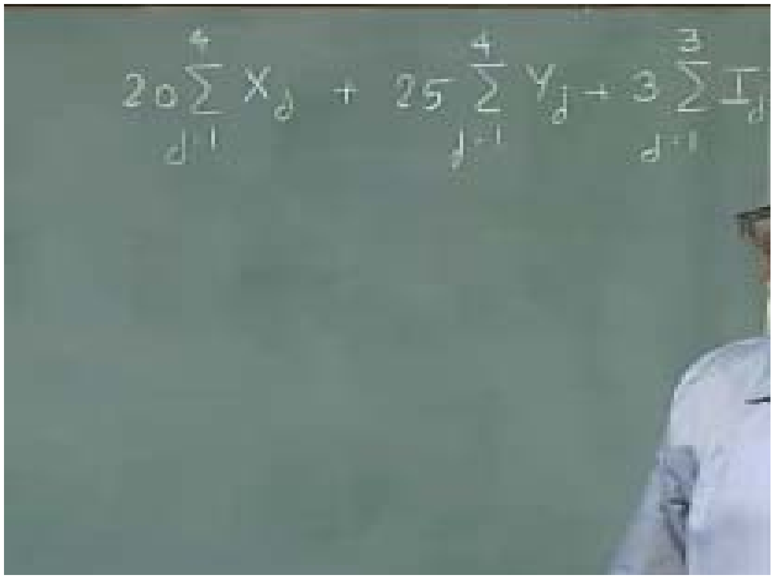

So 20 into sigma $X_{j.}$ Sigma X_j represent the total quantities produced in the four months using the regular time. So 20 into $X_1 + X_2 + X_3 + X_4$ + 25 into the overtime production $Y_{j,}$ if not, you could say J equal to one to four here (Refer Time Slide : 22:41) and j equal to one to four here. In addition, whatever quantity that is carried to the next month, we have an additional Rs 3 per unit per month we have + 3 into sigma $I_{j,}$ J equal to 1 to 3 and we don't have I_4 in this problem.

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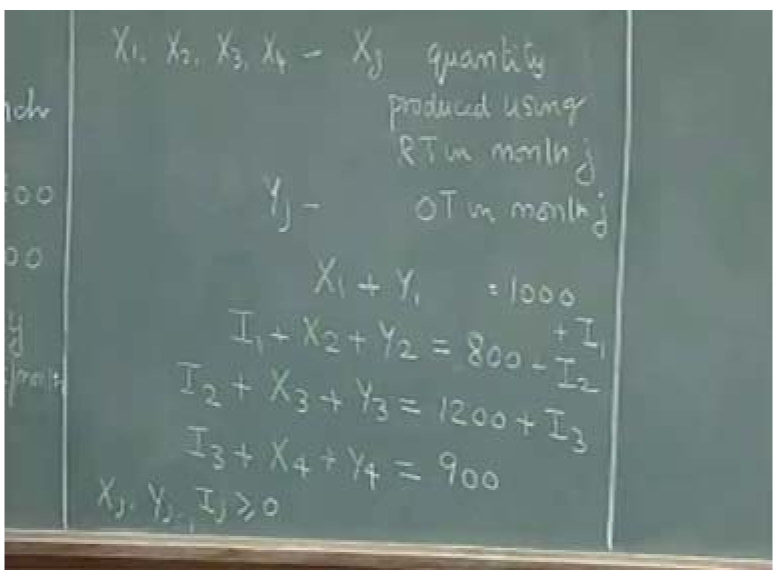

Plus, we have X_j, Y_j and I_j greater than are equal to 0.

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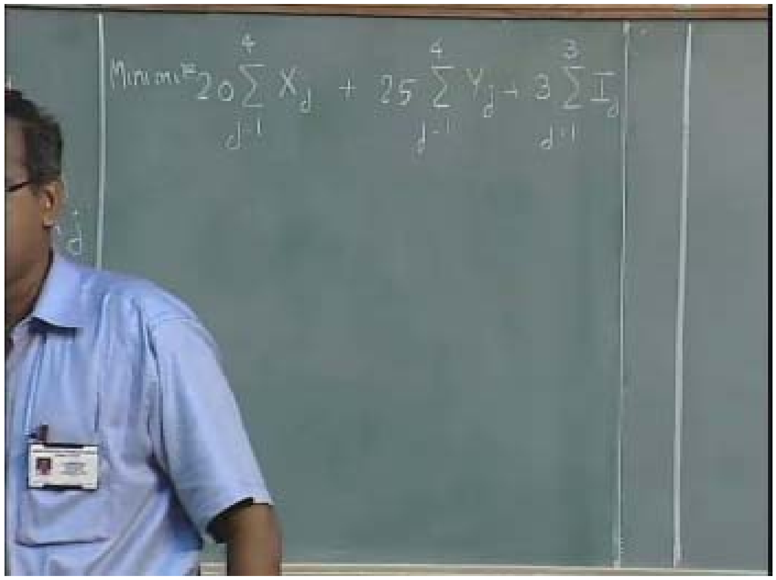

Now what is that we want to do? This is the cost function. And this being a cost function, we would like to minimize this cost function. We like to minimize this cost function.
So now this formulation is complete and this formulation finds out X_j.

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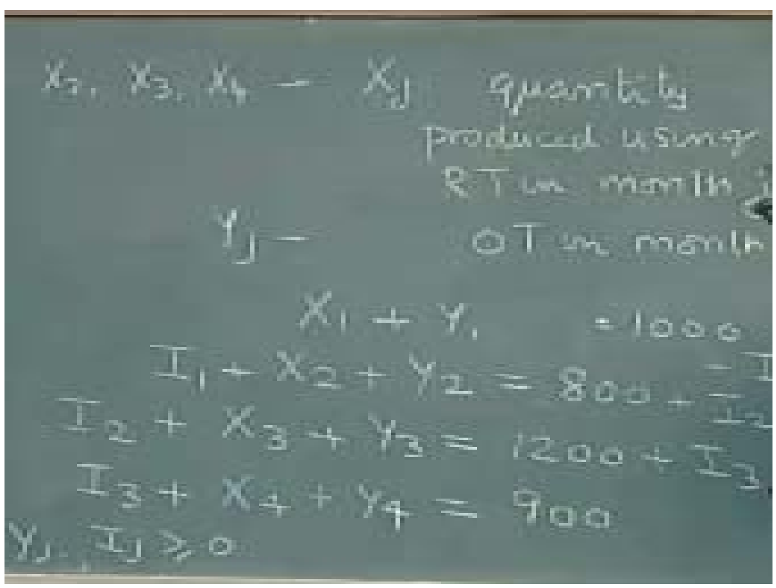

X_j, which is the quantity producing using a regular time in month j & Y_j which is the quantity produce using overtime in month j. Constraints are given here and the objective function now tries to minimize the cost function. So this is the formulation corresponding to the production planning problem

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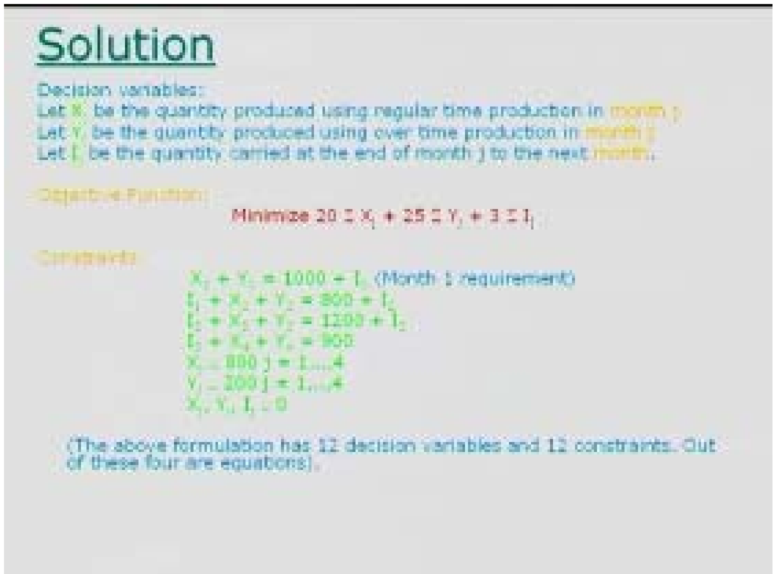

As we go back, as far as the problem is concerned, this has actually 11decision variables (we have left out the I_4).

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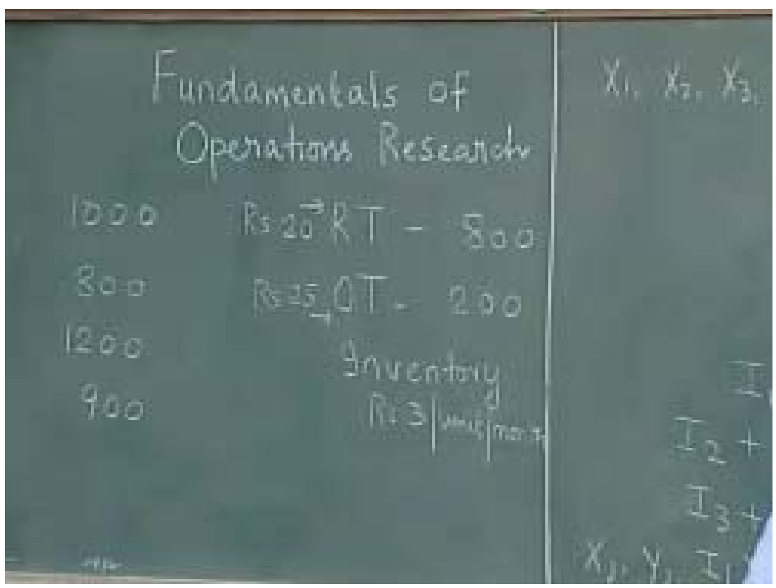

We need to write some more constraints which have not been written here. The regular time capacity and the overtime capacity are also given here (Refer Time Slide 24:07). So we need to write these constraints.

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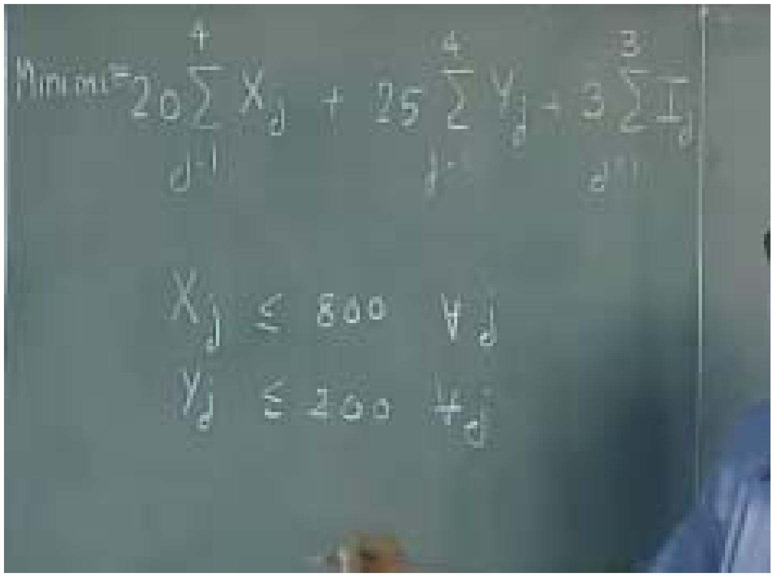

These constraints will now be

(i) X_j is less than or equal to 800 for every j

(ii) Yj is less than or equal to two hundred for every j

So we have four constraints here (Refer Slide Time 24:28) we have four more constraints here.

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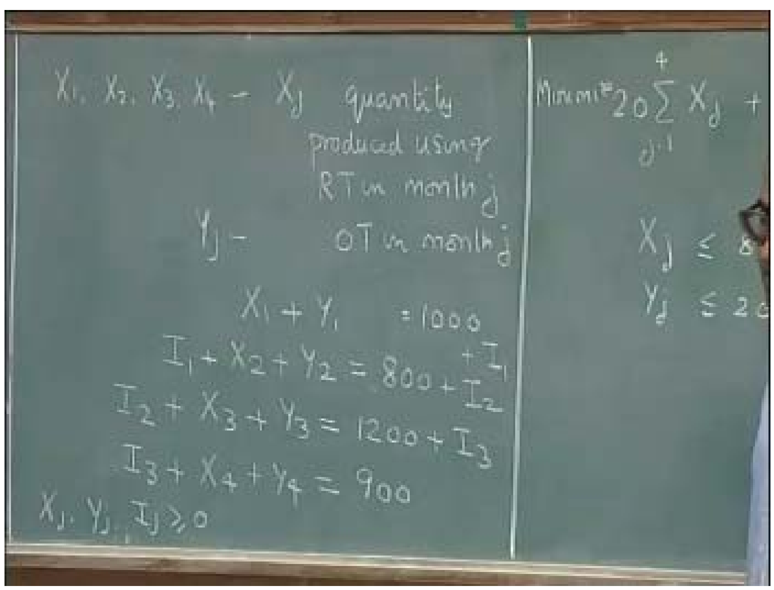
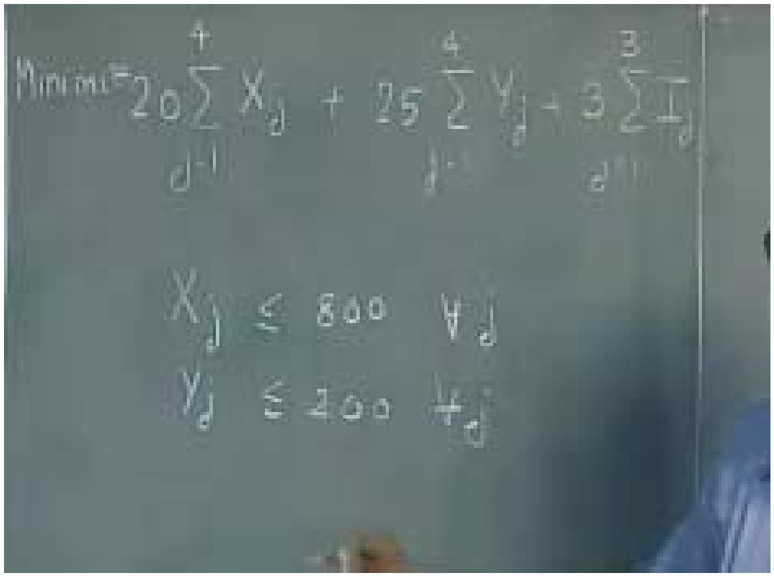


We also have 4 constraints for the 4 months. So we have 12 constraints and we have eleven variables, 4 variables (X_1 to X_4) and another four variables (Y_1 to Y_4) and another three variables (I_1 to I_3). So this formulation has a minimization objective which is different from the maximization objective that we saw in the earlier example. It has twelve constraints and eleven variables. It also has some equations here (Refer Slide Time 22:52) in the earlier formulation where we did not have any equation.

(Refer Slide Time: 25:11)



We also realize that there are constraints which are of the less than or equal to type here in the earlier formulation as well. We have some equations which are here.

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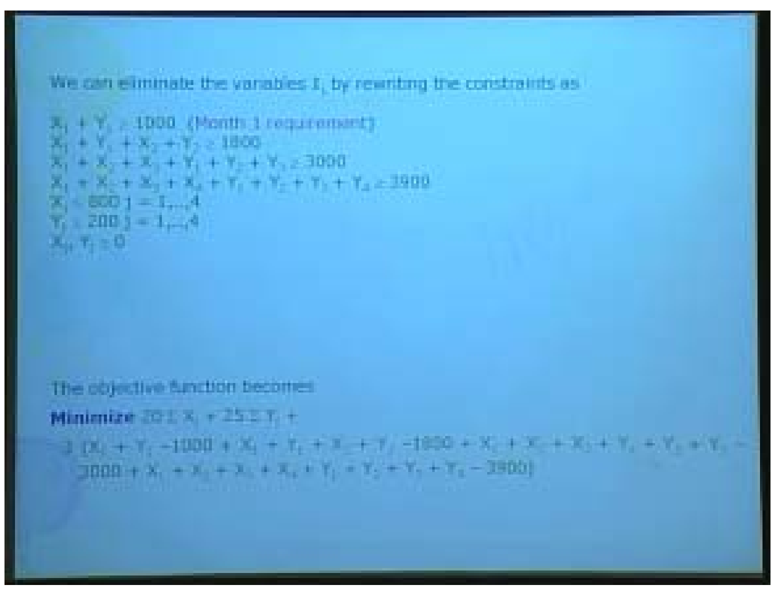

We can do a few more things. From the same formulation we can do a few more things.

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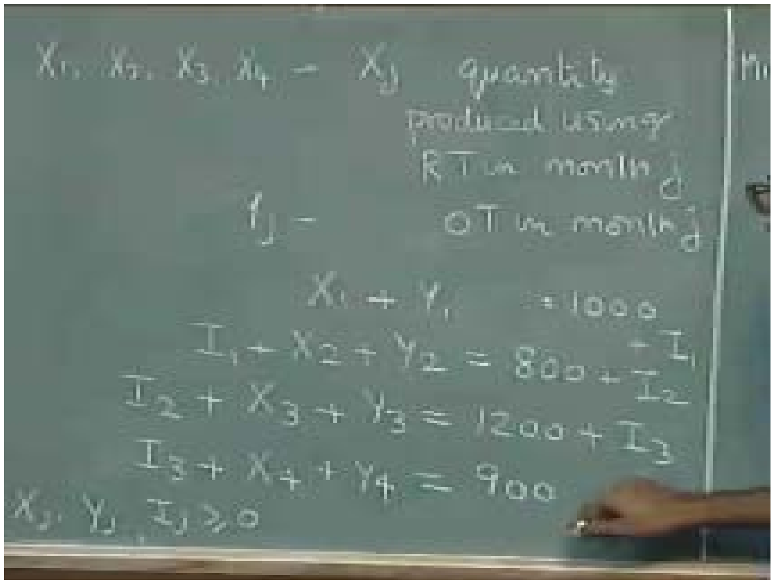

The first thing is that we can eliminate this I_1, I_2 and I_3 from this (Refer Slide Time 25:39) which can easily be done.

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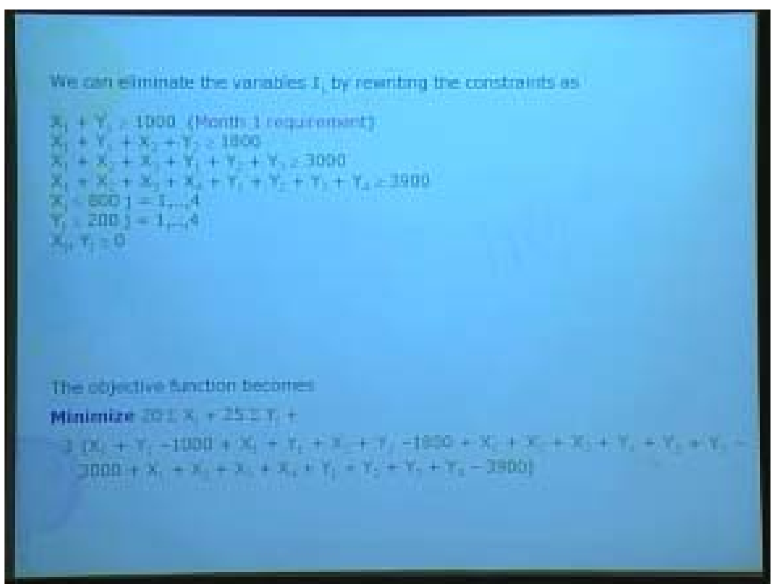
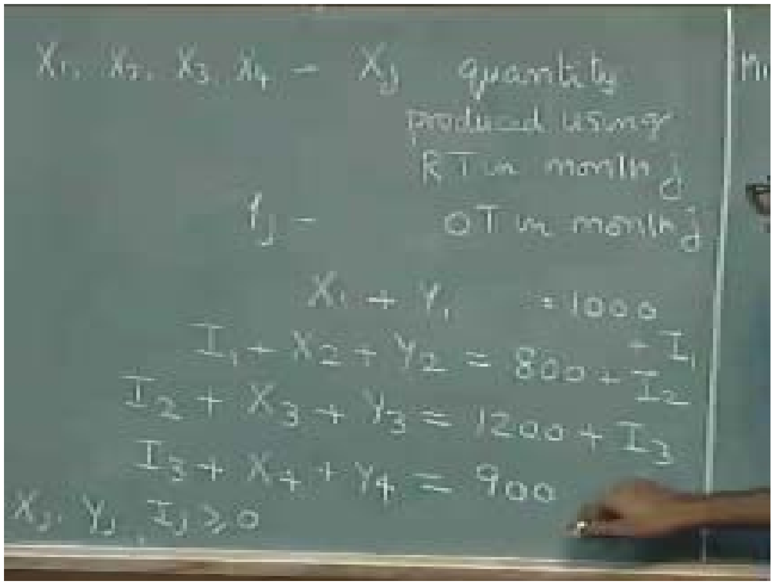


For example we can eliminate this I_1 by saying that $X_1 + Y_1$ should be greater than or equal to 1000.



Whatever is produced in the first month should be more than or equal to the demand from that month and if there is an excess it is carried to the second month.
So what could possibly be the excess? The excess will be $X_1 + Y_1 - 1000$.

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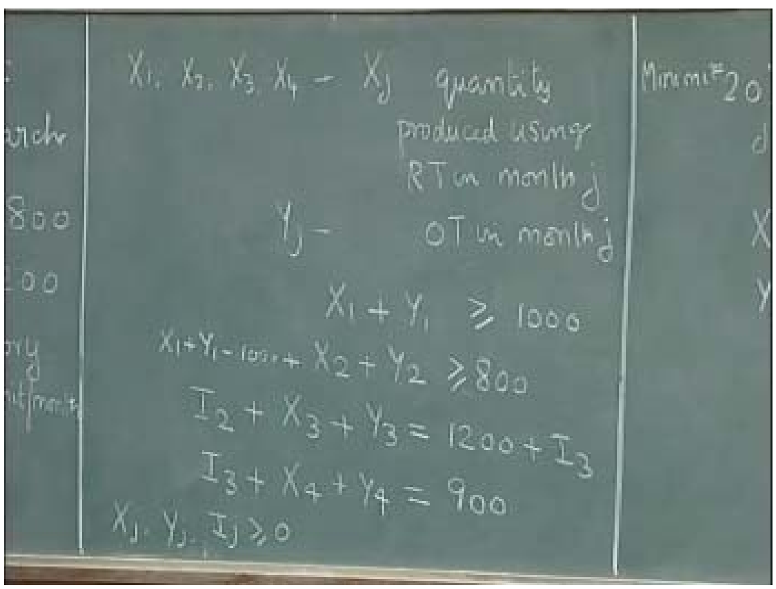

So this becomes $X_1 + Y_1 - 1000 + X_2 + Y_2$ should be greater than or equal to 800. So we eliminate this I_2 as well. Now this quantity is, what is available for the second month this is the demand for the second month. So if this (Refer Slide Time 26:28) quantity exceeds 800, then the balance is carried to the third month.

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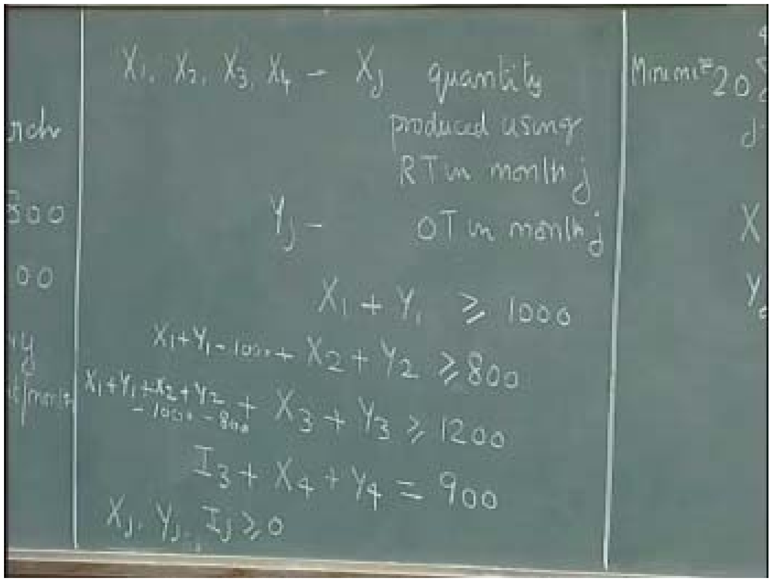

So this I_2 is nothing but $X_1 + Y_1 + X_2 + Y_2$ -1000-800 will be greater than or equal to 1200 and we eliminate this three because I_2 is nothing but this (Refer Slide Time 26:57) left hand side – 800. So we substitute for I_2 here (Refer Slide Time 26:52) to get this and we also eliminate this I_3. We now go back to write what is I_3, I_3 is nothing but this (Refer Slide Time 27:06) left hand side minus 1200.

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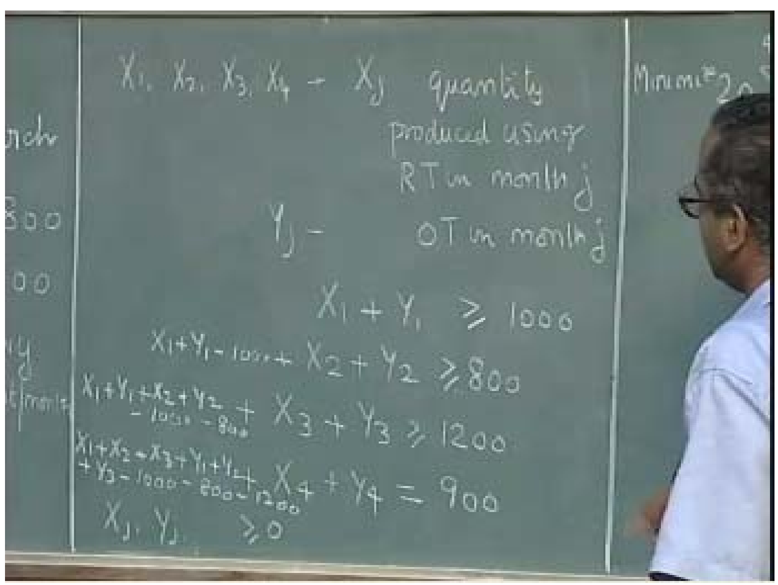

So we eliminate this I_3 here and write this as $X_1 + X_2 + X_3 + Y_1 + Y_2 + Y_3$ –1000–800–1200 = 900 and also we can eliminate these I_f's. Now we have the same number of constraints here. We have few more variables.

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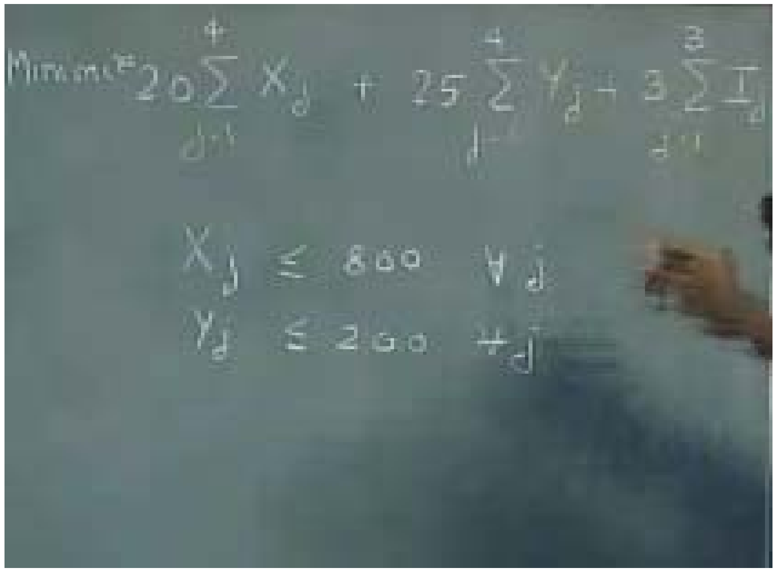

We have to rewrite the objective function. Also here this objective function has these $I_{f'}$s, so we have to write this I_j very carefully

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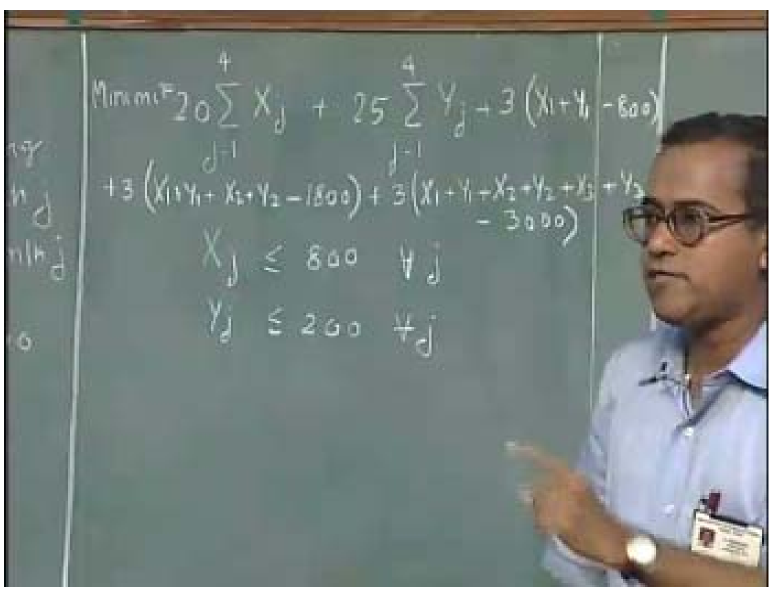

So this now becomes + three times (as far as the first month is concerned) $X_1 + Y_1 - 800$ + three times $X_1 + Y_1 – X_2 + Y_2 – 1800$ & (as far as the third month is concerned) $X_1 + Y_1 + X_2 + Y_2 + X_3 + Y_3 – 3000$.

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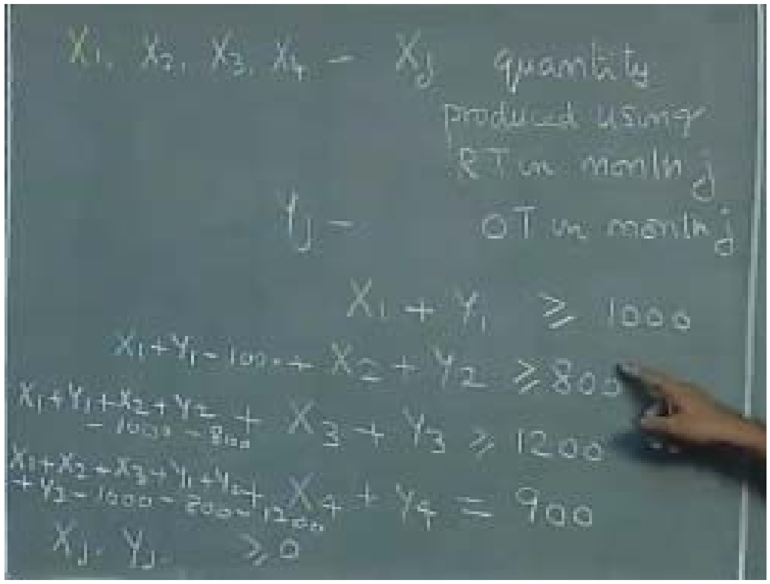

3000 is the sum of 1800 and 1200.

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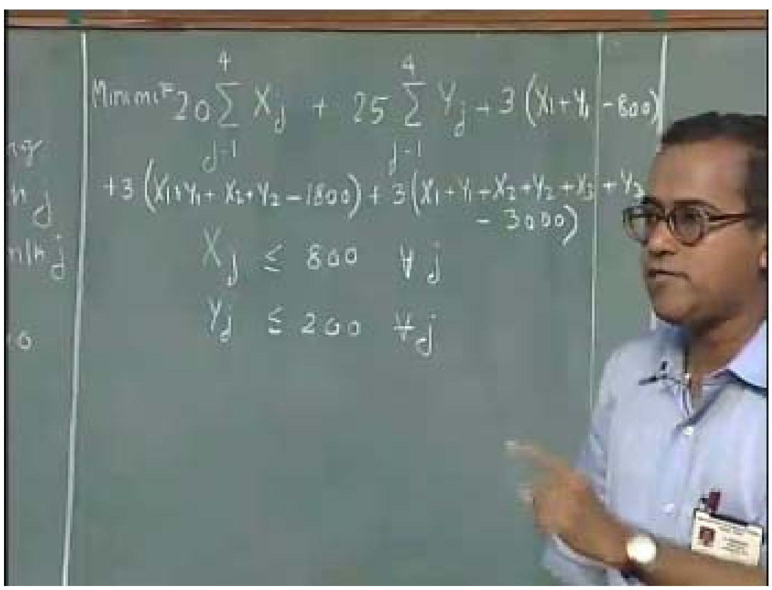

So this is what is being carried as I_3 which comes in. Right now we have not introduced an I_4 yet.

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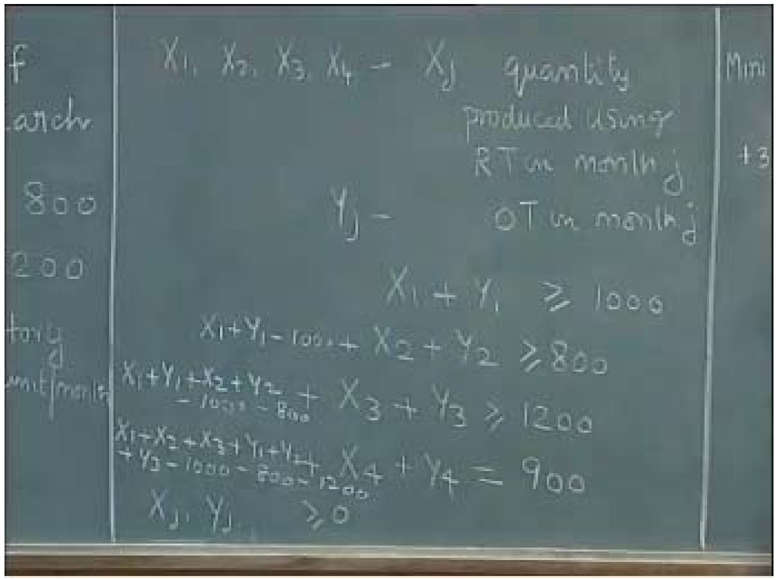
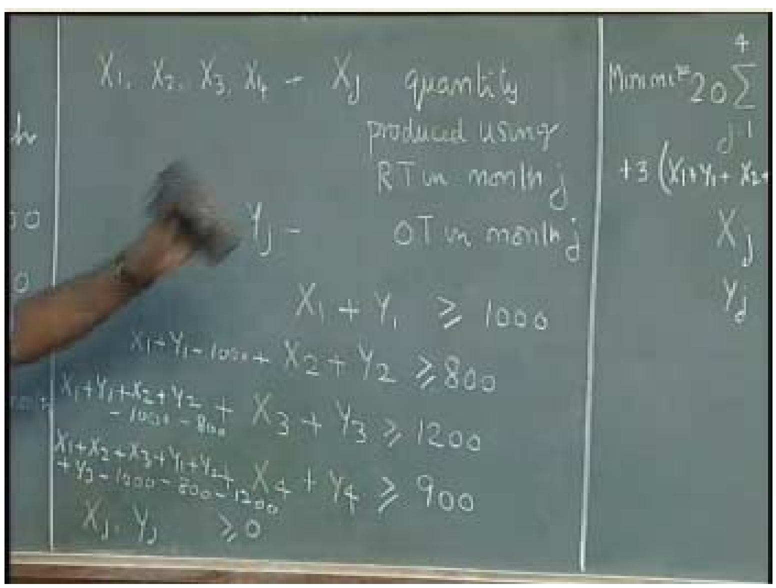


We have an equation here (Refer Slide Time 28:45).

(Refer Slide Time: 29:01)



We can even keep this as greater than or equal to just to be consistent with the other three constraints and if we choose to do that then what will happen is we have another term that comes in.

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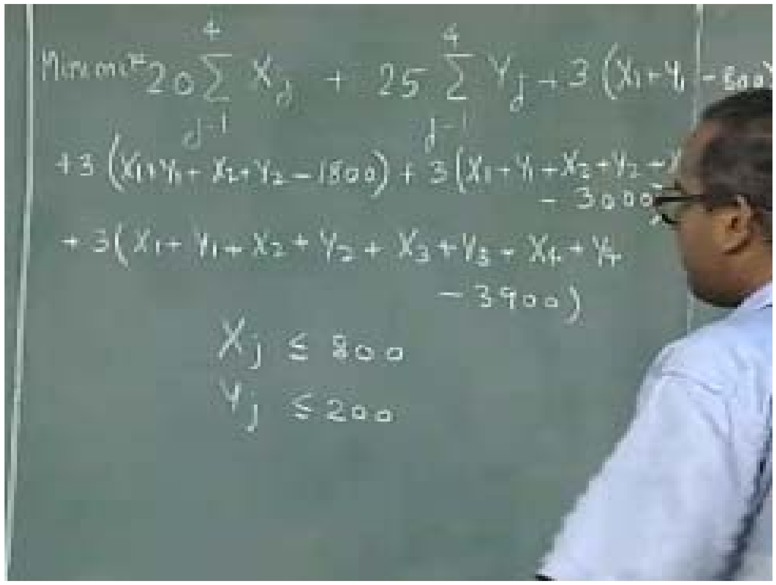

You will have a fourth term which is this $X_1 + Y_1 + X_2 + Y_2 + X_3 + Y_3 + X_4 + Y_4 – 3900$. We will continue to have these two constraints X_j less than or equal to 800 and Y_j less than or equal to 200. So you will continue to have these two constraints as well, but the only thing we have done is we have added another term into the objective function

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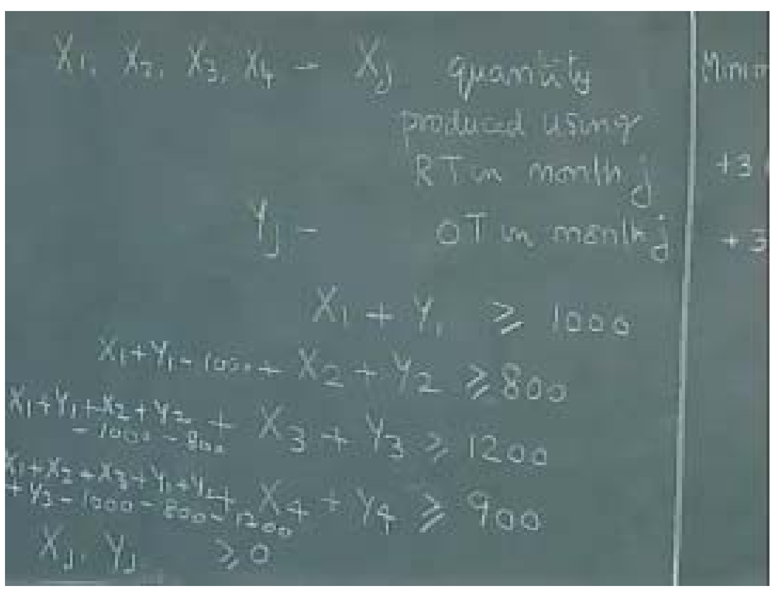

We have replaced this equation by an inequality just to be consistent with these four so we now have a second formulation for this problem. We can even have a discussion on this 900. We have a second formulation for this problem which is the same as what we saw in the first formulation.

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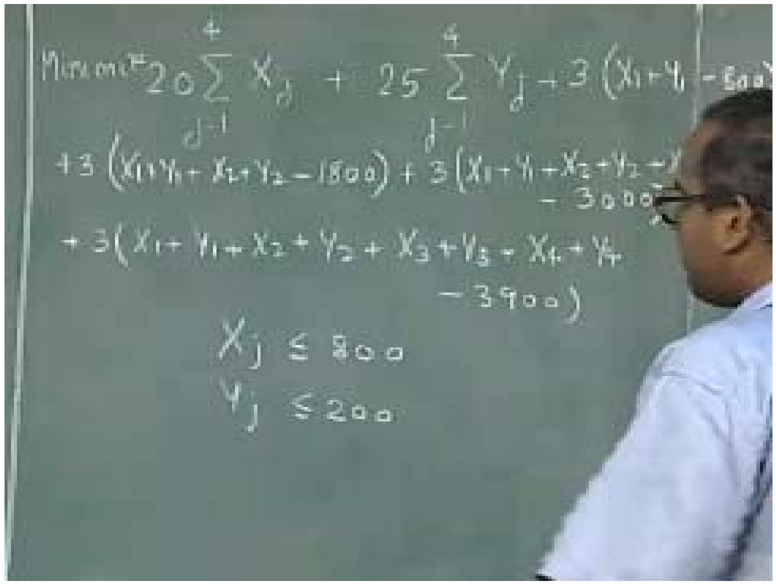

The only difference is in this formulation, we have fewer variables. We have 12 or even variables in the earlier one. We now have 8 variables in this formulation. We had equations in the first one and we now have all inequalities here and we also introduced the greater than or equal to inequity to this problem which we did not see in the earlier formulations also so this slightly different formulation but it also represents exactly the same problem that we are trying to model. Now let us come back to this 900.

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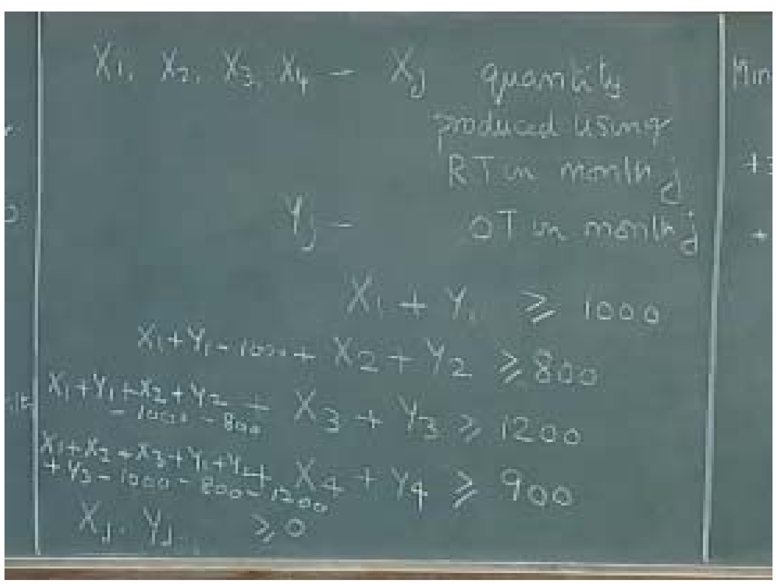

Now there are two ways of addressing this 900. One is to put an equation here (Refer Slide Time: 30:45) which will automatically correct the values of X_4 and Y_4 such that we do not have anything excess and if we put an equation here.

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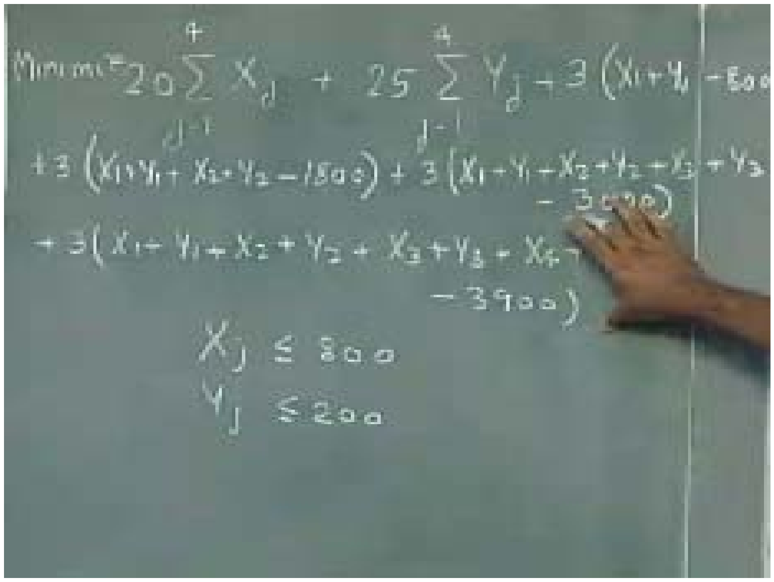

We can then you can leave out this term.

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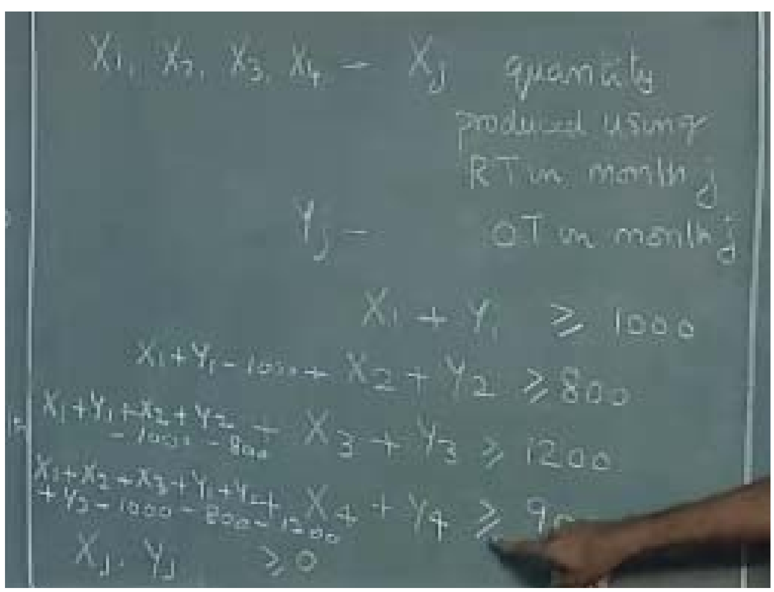

If you don’t put an equation here and put an inequality instead, it means you are allowing an I_4 to exist and if such an I_4 exists then it is carried into the objective function,

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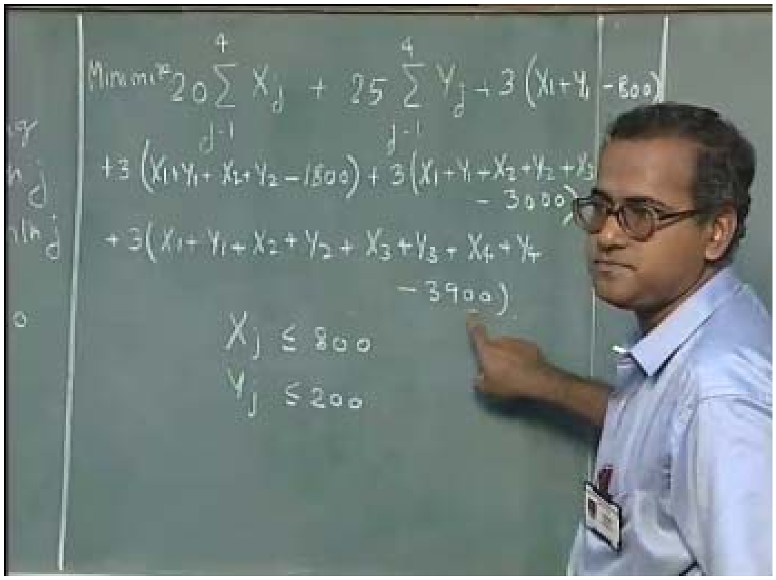

with the cost of this. Now both are correct.

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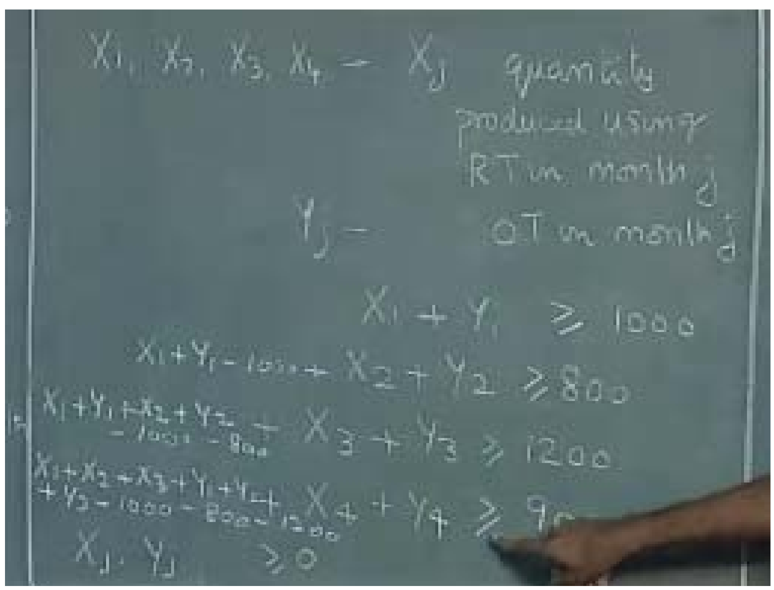

You can put an equation and leave out this (Refer Slide Time: 31:03) term. You can keep this as an inequality here and then add this. Both are correct simply because any I_4 that is carried from here comes to increase your cost.

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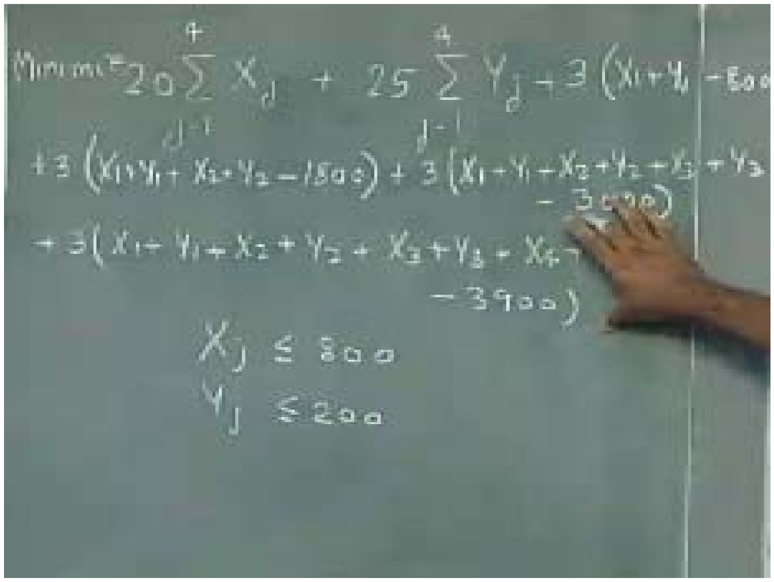

So the solution will be such that you will not carry an I_4.

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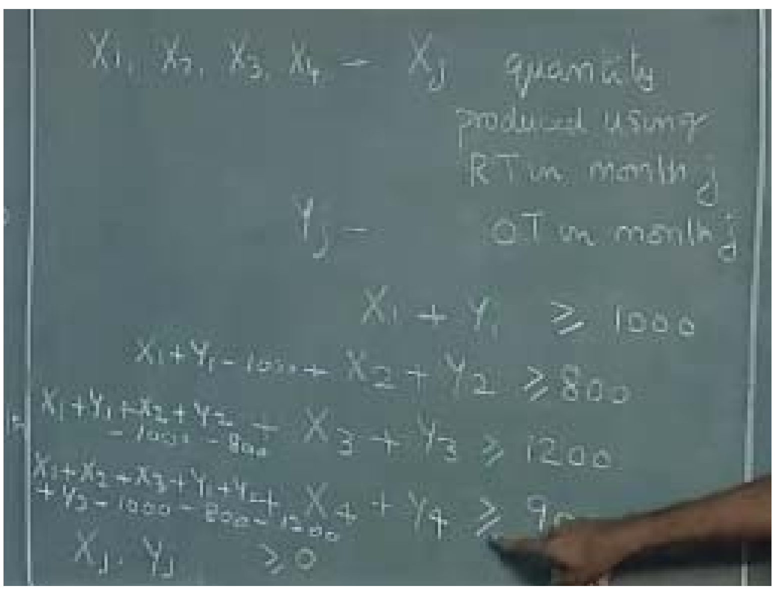

And so the I_4 will automatically be 0. So in the formulation stage, for the purpose of consistency you can still keep all these four as inequalities.

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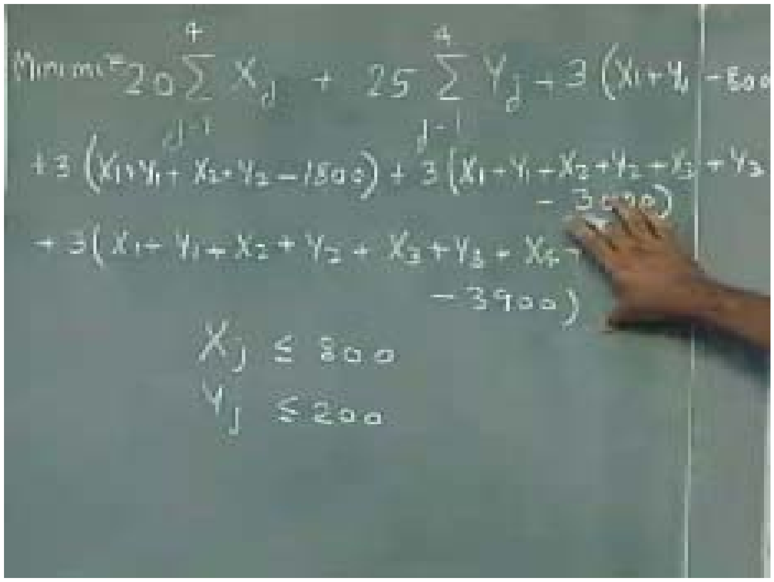

Then you can write this (Refer Slide Time: 31:36) term. Also, it is just that the objective function has an additional term that comes into play in this.

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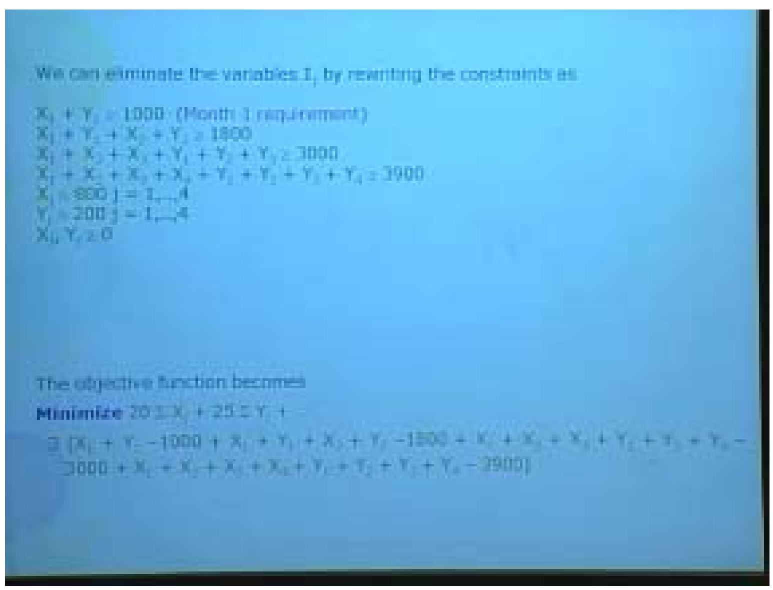

This is shown in this slide. The same formulation is shown in this slide. The requirements for the four months are the four constraints, except that the right hand side is simplified. You can see that these (Refer Slide Time: 31:55) two terms are taken to the right hand side (as you see the formulation there) and then you have the same objective function which is shown here (Refer Slide Term 32:04) including all the four terms,3900 and you will also find that all of them are inequalities.

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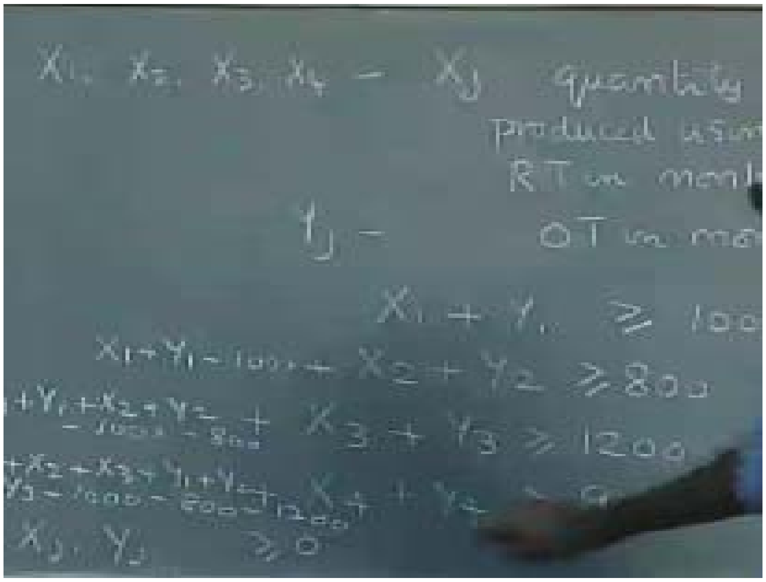

So this is the second way of formulating this problem. Now when compared to the first, the question arises as to which one is better? Which one is superior?
There are some very simple guidelines. For the same number of constraints (both these problems have twelve constraints) you can say that a formulation which has fewer variables is superior.

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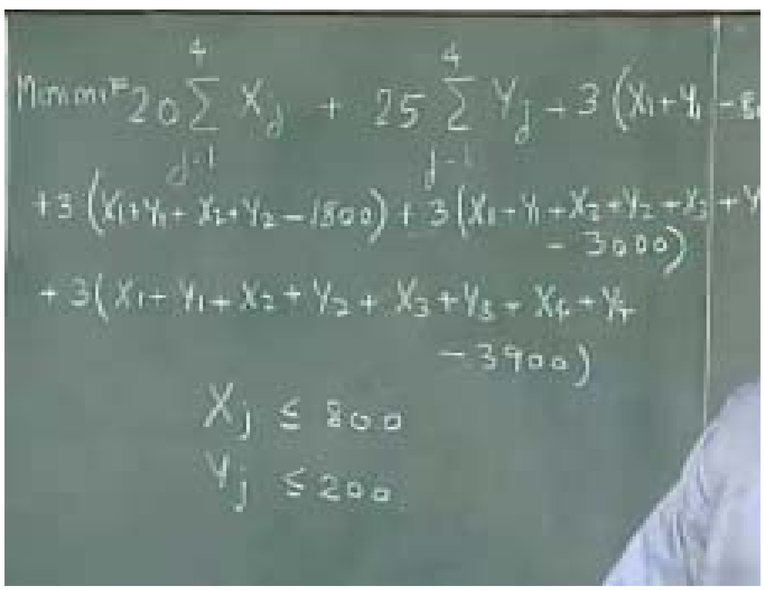

So this is seen to be the superior one when compared to the earlier. One, both has the same number of constraints but the number of variables is much less in this.

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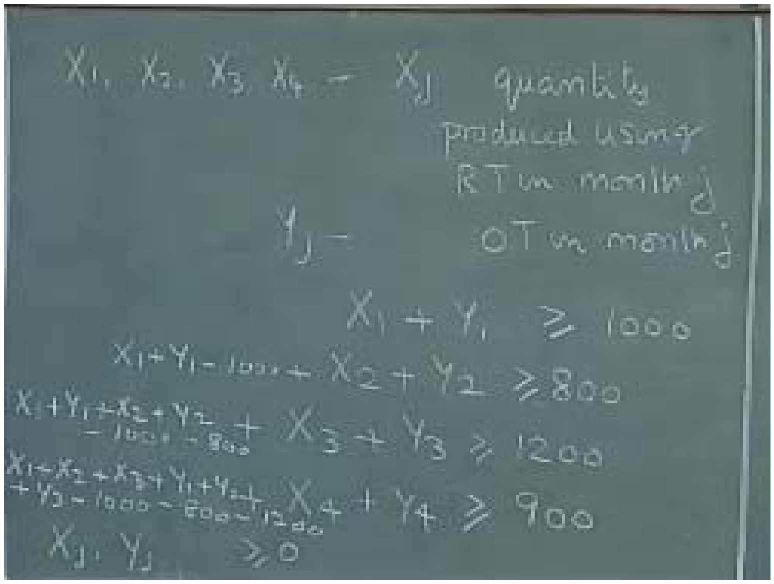

Also in general, a formulation that has inequalities is better than formulations that have equations. Most of us are used to solving equations though. Later we see that we convert these inequalities into equations and so on. But as a very general idea, one may assume that the formulation which has inequalities is better and preferred over formulations that have equations. So this is better than the earlier one.

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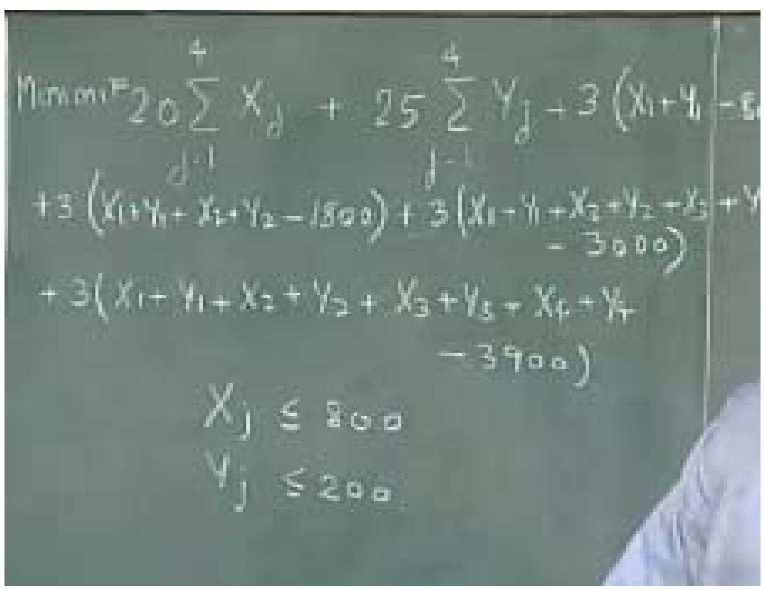

In this formulation, we have learnt a few things like, minimization objective function. The earlier formulation had a maximization of objective function.
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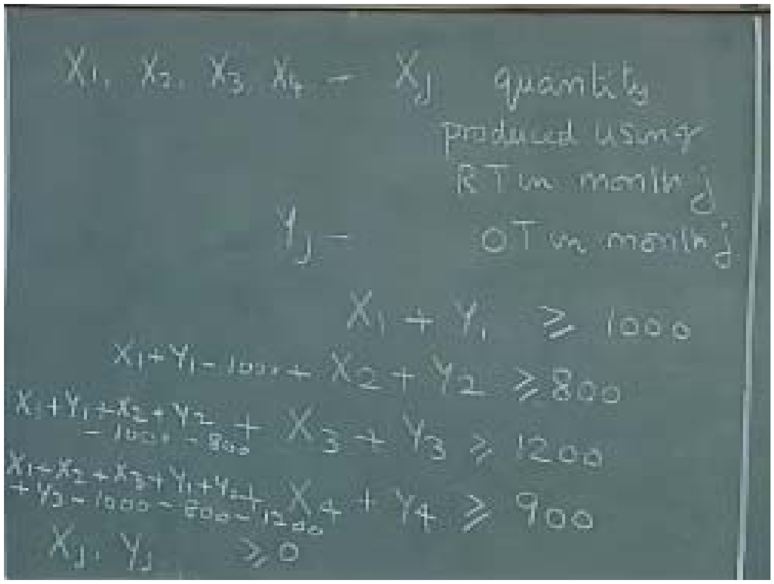

In this formulation we learnt to introduce greater than or equal to type inequalities, which we did not see in the earlier formulation.

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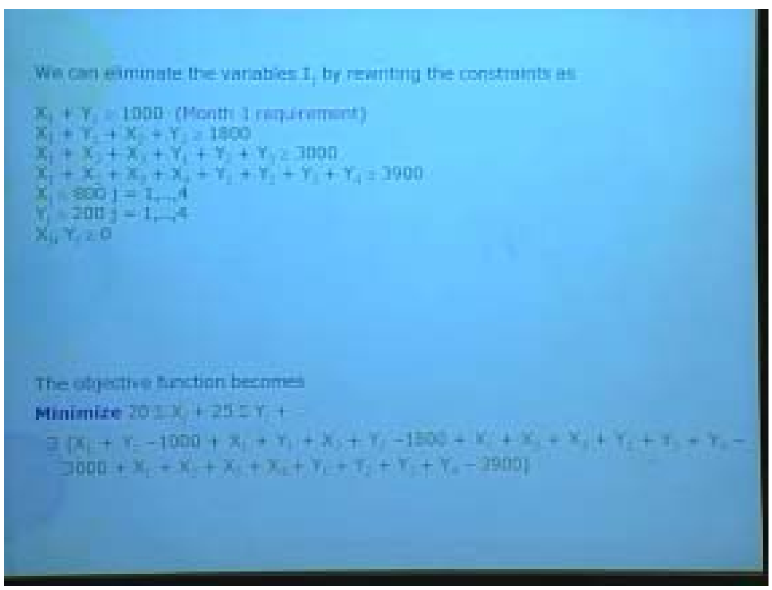

Now let us look at a third way to solve the same problem just to show you that the same problem can have different formulations depending on how you think about the problem.

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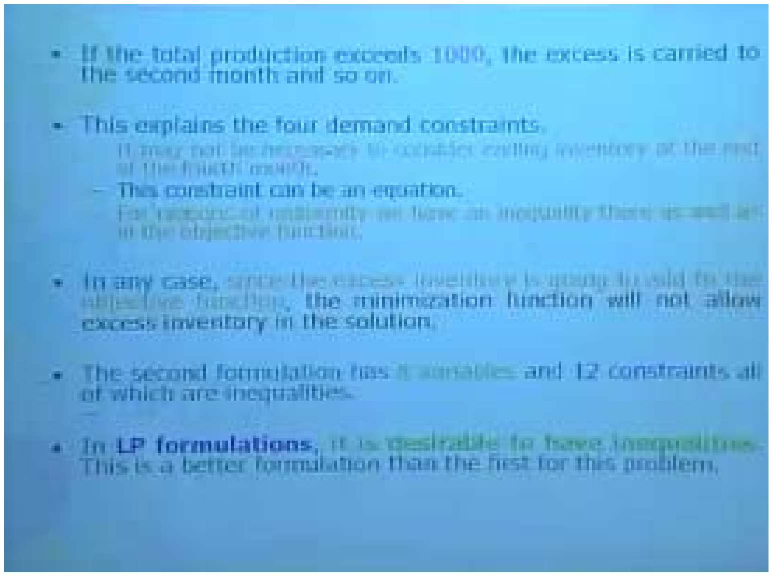

So let us look at the third way to formulate the same problem again. Now we introduced a slightly different set of decision variables.

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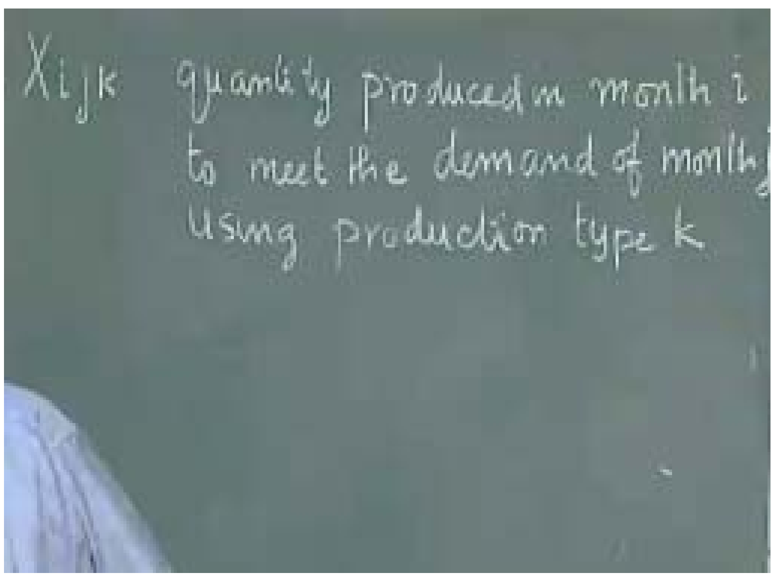

Now we are going to introduce I_{nk} as the quantity produced in month I to meet the demand of month j using production type K. Now we look at four months. So month I can take I equal to one to four, j can also take j equal to 1 to 4 and K has two types of productions, Regular time and Overtime. So K takes values 1 or 2. Now with this type of definition of decision variable, let us try to write the constraints and the objective function corresponding to this problem.

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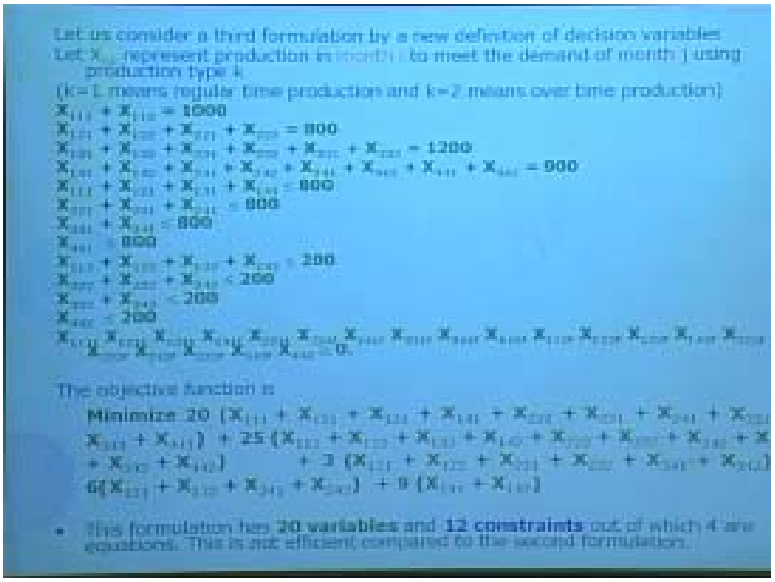

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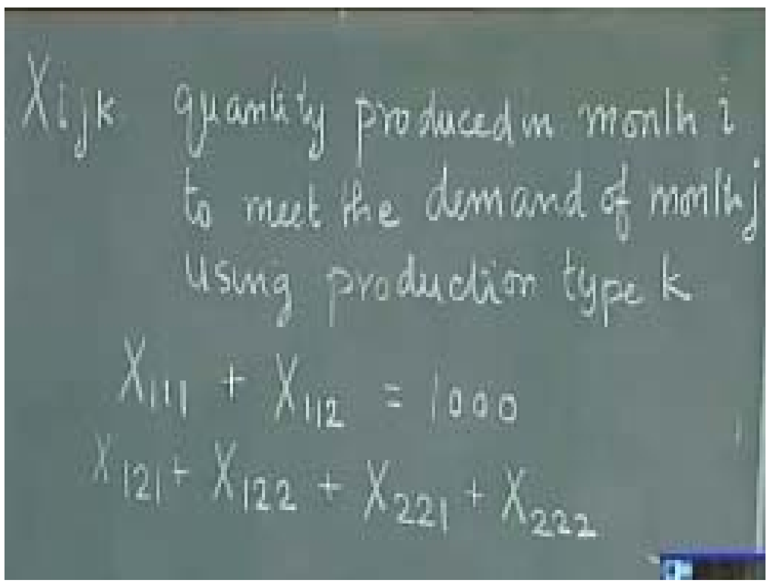

As for as the first month is concerned, first month we produced X_{111} indicates producing in regular time to meet the first month's demand using regular time production i.e., producing in the first month i.e., I equal to one to meet the first month's demand, j equal to one using regular time production. It is X_{111} + X produced in the first month to meet the first month's demand using overtime. So these are the only two variables that represent the production to meet the demand of the first month. So $X_{111} + X_{112}$ should be equal to the first month's demand which is thousand 1000.

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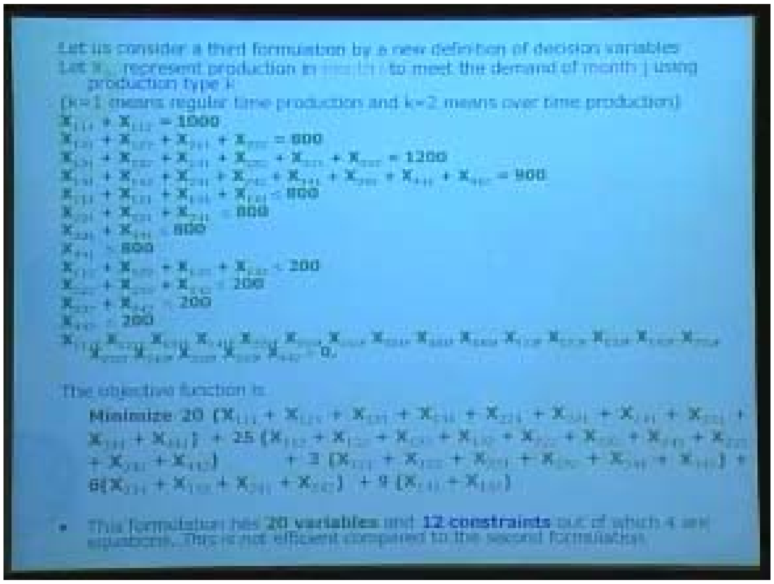

Here we are not looking at carrying anything.

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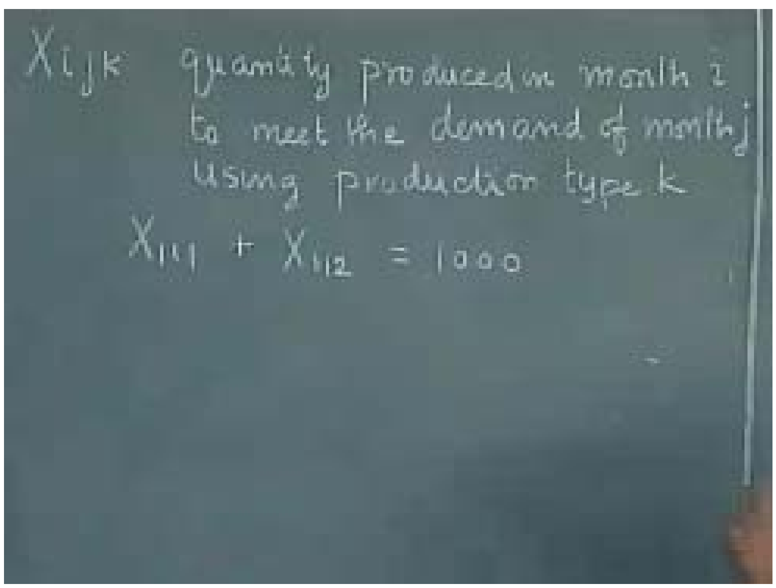

You see the way the variables are defined. Whatever quantity that is carried is also defined as a decision variable.

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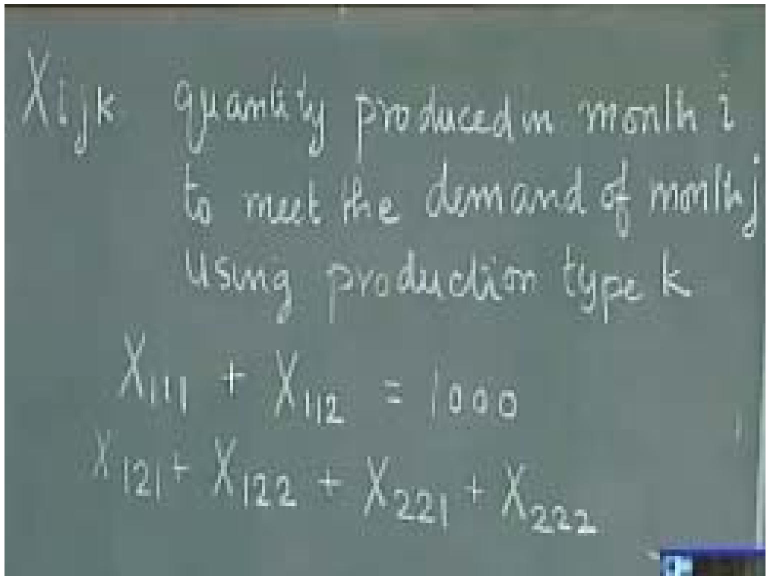

Now we try to meet the second month's demand. The second month's demand can be met into two ways:

(i) by production in the second month

(ii) by production in the first month

In this, we also make an assumption that you cannot meet the first month's demand by producing in the later months to be consistent with the earlier assumptions that we made, while you can actually produce in a month that is early, carry and then meet the demand of subsequent month. So, the second month's demand can be met by production in first as well as second. So you have X_{121} produced in the first month to meet the demand of the second month by regular time plus produced in the first month to meet the second month's demand by overtime plus produced in the second month to meet the second month's demand by regular time produced in the second month to meet the second month's demand by overtime

So these are the four ways by which you can meet the demands of the second month and these four should add to the second month's demand which is 800.

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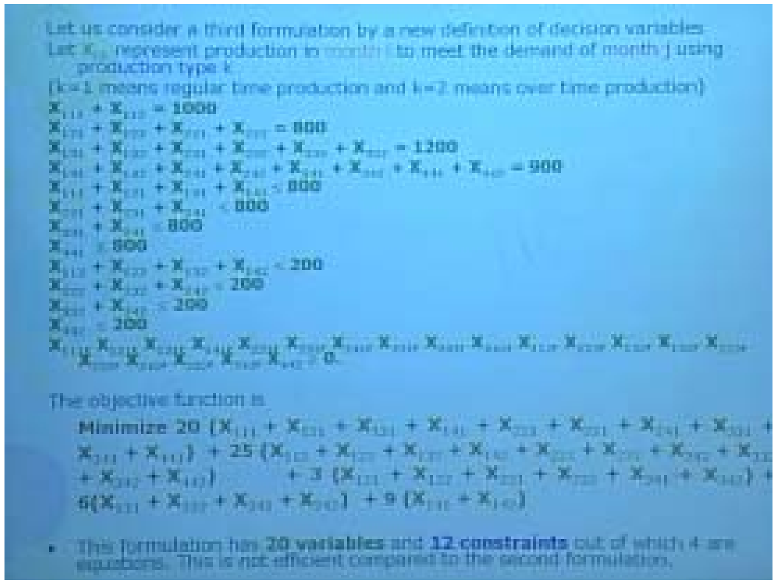

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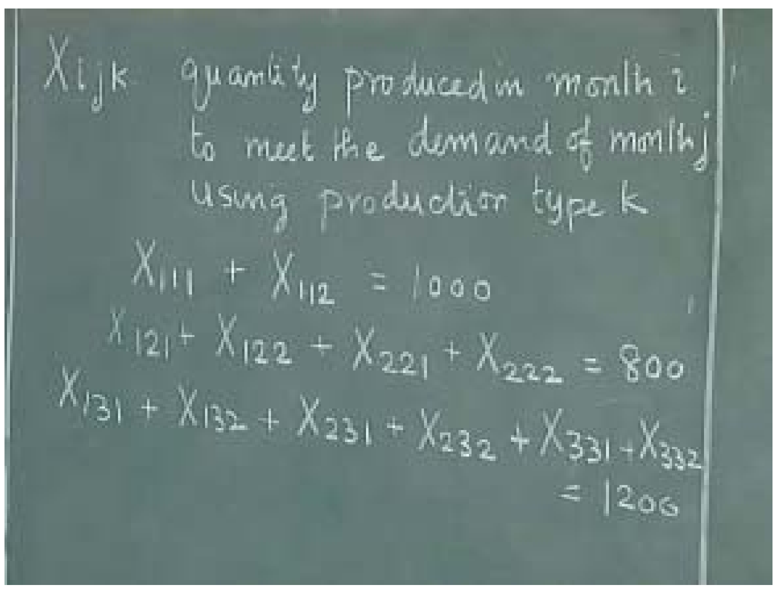

Now as far as the third month is concerned, we can meet the third month's demand by six ways.

(i) Producing in the first month, regular time and overtime

(ii) Producing in second month, regular time and overtime and

(iii) Producing in the third month, regular time and overtime.

So you end up writing X_{131} produced in the first month to meet the third month's demand with regular time + X_{132} produced in the first month to meet the third month's demand overtime, produced in the second month to meet the third month's demand regular time produced in the second month, meet the third month's demand overtime produce in the third month meet third month's demand regular time produce in the third month to meet third month's demand overtime. This should be exactly equal to 1200 which is the third month's demand.

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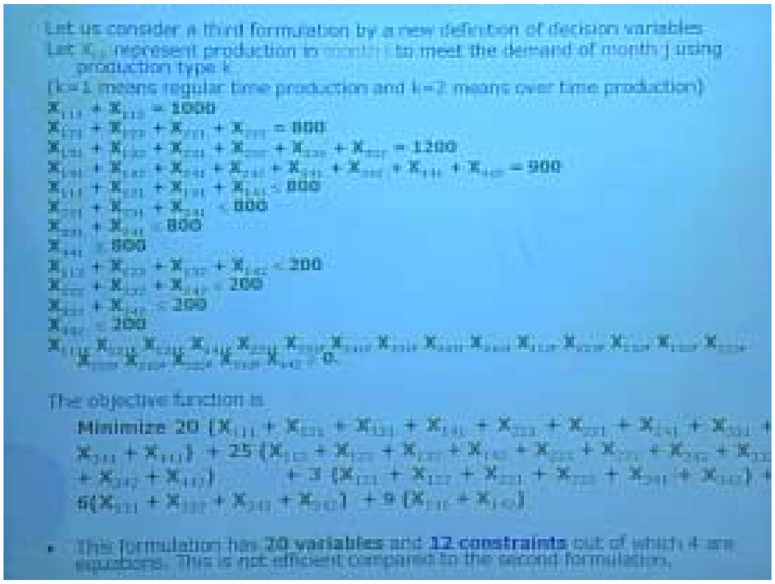

Now you extend it and write it for the fourth month.

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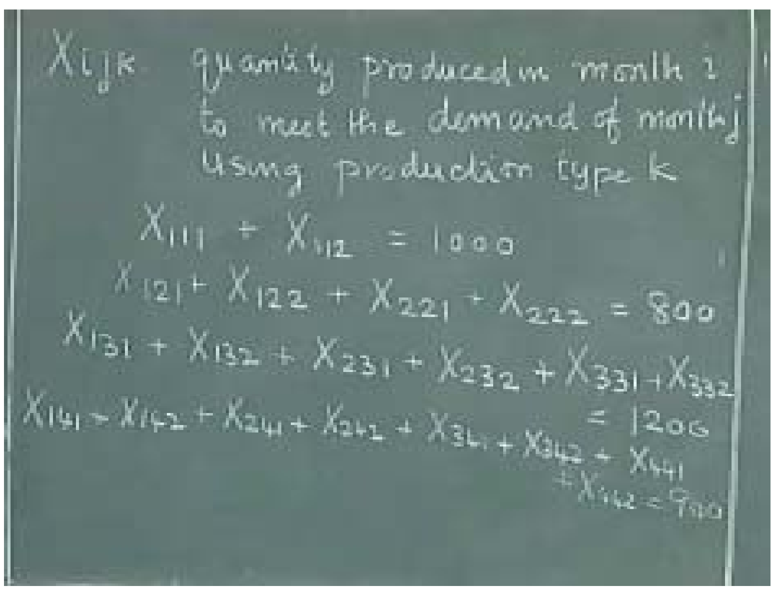

Now this can be done in 8 ways. You can produce in the first month by two ways, second month by two ways and third month and fourth month two ways. So you have X_{141} + X_{142}. This represents producing in the first month to meet the fourth month’s demand by two types. X_{241} + X_{242} + X_{341} + X_{342} + X_{441} + X_{442} are equal to 900. So we have written the four constraints corresponding to the requirement of the four months. Now we need to look at the constraints on the capacities.

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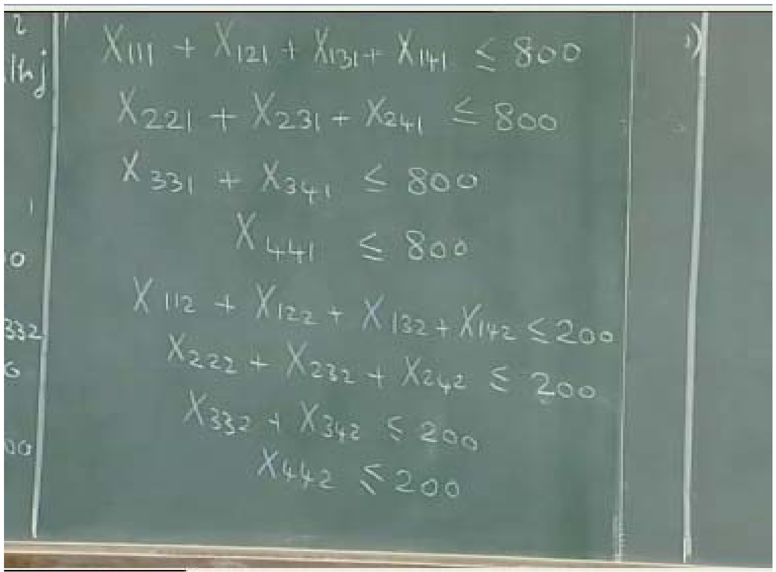

For the first month, the regular time, what we do is, $X_{111,}$ produced in the first month to meet the demand of the first month by regular time + $X_{121.}$ I produce in the first month to meet to the demand of the second month, once again by regular time. So when we are looking at the first month's regular time production quantity, I will be 1 and k will be 1 but j can be 1, 2, 3, and 4. So you will have X_{111} + X_{121} +X_{131} + $X_{141.}$ Now these four represent the quantity produced in the first month, using regular time, to meet the demand of 1, 2, 3 and 4 months. So this should be less than or equal to the regular time capacity of 800.

Now as far as the second month's regular time is concerned, you have X produced in second month to meet the second month's demand by regular time. So in this case I will be 2 because you are producing in the second month. k will be 1, since you are using regular time production. j will be 2,3 or 4 so you will have X_{221} + X_{231} + X_{241} should be less than or equal to 800. So as far as the third month is concerned you will get X_{331} + X_{341} because I is equal to 3 represents producing in the third month j can be 3 or 4. You can produce in the third month to meet the third month's requirement or the fourth month's requirement and k is 1 because you are using regular time So this should be less than or equal to 800 and as far as the fourth month is concerned you will have X_{441} to be less than or equal to 800. You have to write four more for the overtime production. The only difference being the third subscript will become 2. So the first month's overtime will now look like this X_{112} + X_{122} + X_{132} + X_{142} is less than or equal to 200. X_{222} + X_{232} + X_{242} is less than or equal to 200. X_{332} + X_{342} is less than or equal to 200 and X_{442} =200. 200 is the capacity that we have. So this (Refer Slide Time 42:38) will also be 200. So you have these capacity constraints written. Now let us go back and write the objective function.

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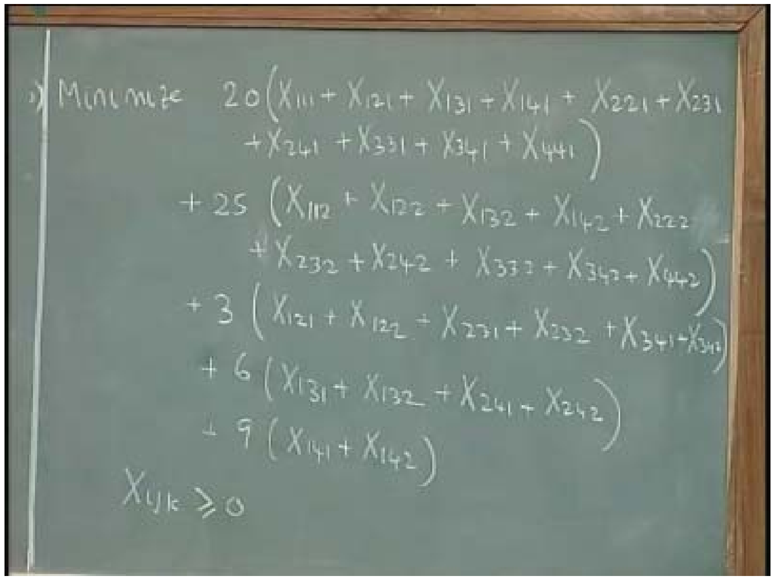

The objective function will look like this. I want to minimize the total cost of regular time, overtime and the quantity that is carried. So in the first month 20 is the cost of the regular time production. So I produce $X_{111} + X_{121} + X_{131} + X_{141.}$ Because I can produce by regular time to meet the demand of all the four. In the second month I produce $X_{221} + X_{231} + X_{241}$ because in the second month, I can produce only to meet the demands of months 2, 3 and 4. So I have three terms now. I have $X_{331} + X_{341}$ which is the thirds month's production to meet the demand of either 3 or 4 and $X_{441.}$ So these are my regular time production quantities. So it is 20 for all these. Now I have to write a similar term for the overtime production. The only difference is the third subscript k will become 2 instead of 1.

So I will have + 25 into $X_{112} + X_{122} + X_{132} + X_{142.}$ These four terms represent overtime production to meet the demands of months 1, 2, 3 and 4. k = implies overtime. Now for the second month, I will have $X_{222} + X_{232} + X_{242}$.

Once again, as we saw earlier in the second month, you can produce to meet the demand of months 2, 3 and 4 by overtime. You will have two terms for the third month $X_{332} + X_{342}$ and you will have one term for the fourth month which is $X_{442.}$ Now we also need to write the cost term corresponding to the quantities that we are carrying.

So what we do is + 3. 3 is the cost that we incur to carry a unit to one month. So all the quantities that are carried from the first month to second month will come here ((Refer Slide Time 00:46:15 min)) and the one produced in second month carried to third month will come here and produced in third month carried to the fourth month will come here. For example we will have $X_{121;}$ this is carried by one month. If j - I = 1, it means it is produced in a particular month and used in the next month. So it will incur 3. So $X_{121} + X_{122}$ are all carried by one month + $X_{231} + X_{232}$ produced in the second month to meet the third month produced in the second month to meet the third month by overtime and $X_{341} + X_{342.}$ So these are the quantities that are carried one month. There are quantities that are carried for 2 months. For example, $X_{131} + X_{132}$, produced in the first month to meet only the third months demand so carried to two months with a cost of 6. Similarly you will have $X_{241} + X_{242}$ and there are quantities which are carried to 3 months because you can produce in the first month and use it in the fourth month. So you will have another two terms that come in $X_{141} + X_{142.}$ So this is the objective function. Plus of course, all I or relevant X_{ijk} is greater than or equal to 0. So this is the third type of formulating the same problem.

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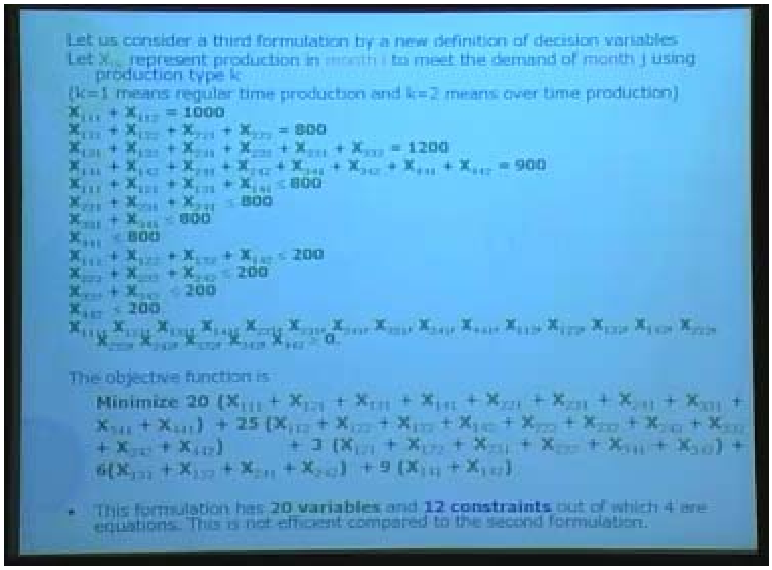

You can see that formulation here. You will also observe that this formulation has 20 variables, a little more than the previous. I_{nk} would allow you 4 into 4 into 2 = 32. But you have only 20 active variables (the other 12 are not active) 12 constraints, you also have some equations. So by the same discussion that we had had, you realize that this is not a very efficient formulation compared to the second one.

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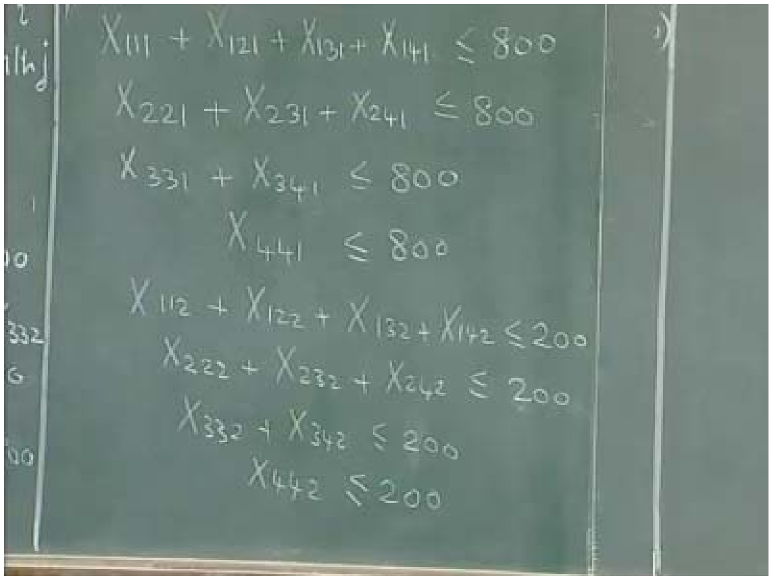

This is because this has more variables.

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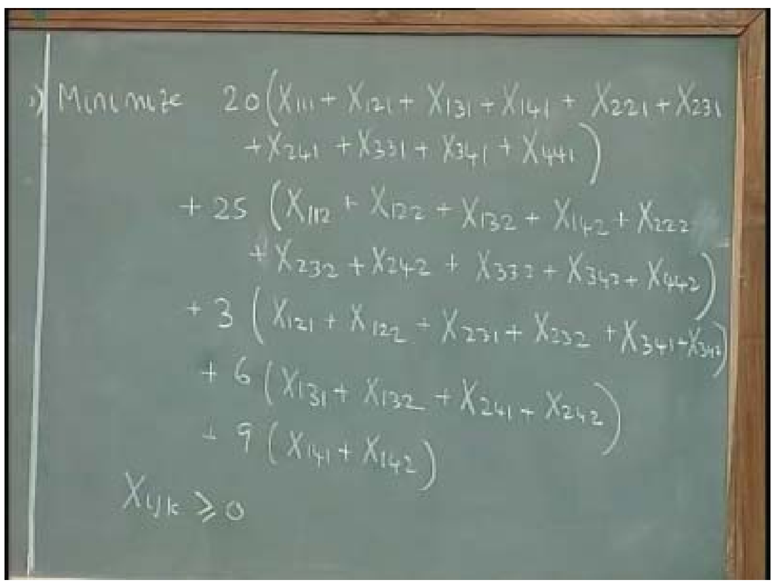

This also has some equations.

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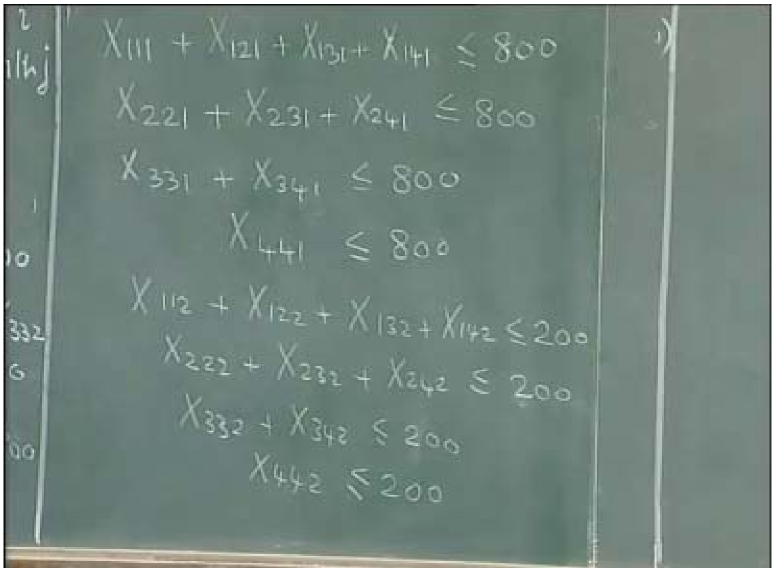

Nevertheless this is also a formulation that we can have for the same problem. We have learnt a few things in the second example which is called the Production Planning example.

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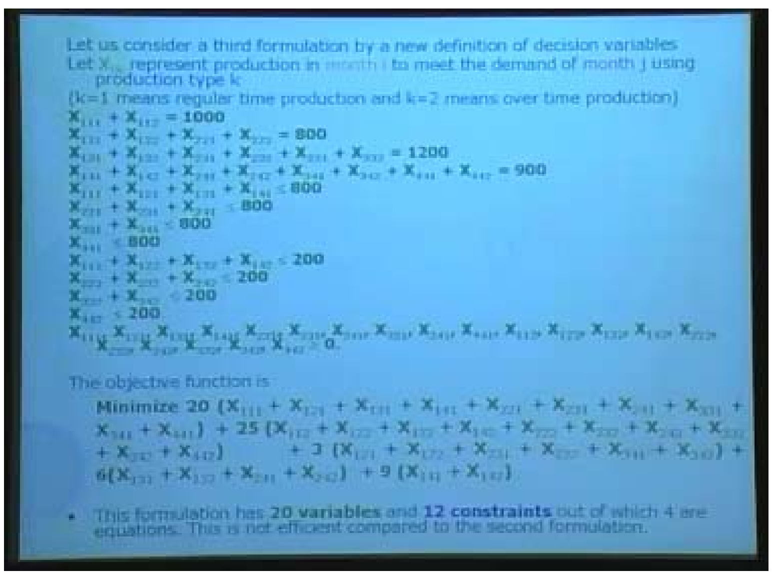

The first thing that we learnt is that, there are types of problems which can be formulated as Minimization problems. The first example was a maximization problem. Second example is the minimization problem. Objective functions are of two types, Maximization and Minimization. Both have linear objective functions. So we looked at the second type of objective function, which is minimization in this case. We also looked at the problem where we introduce the greater than or equal to constraints.

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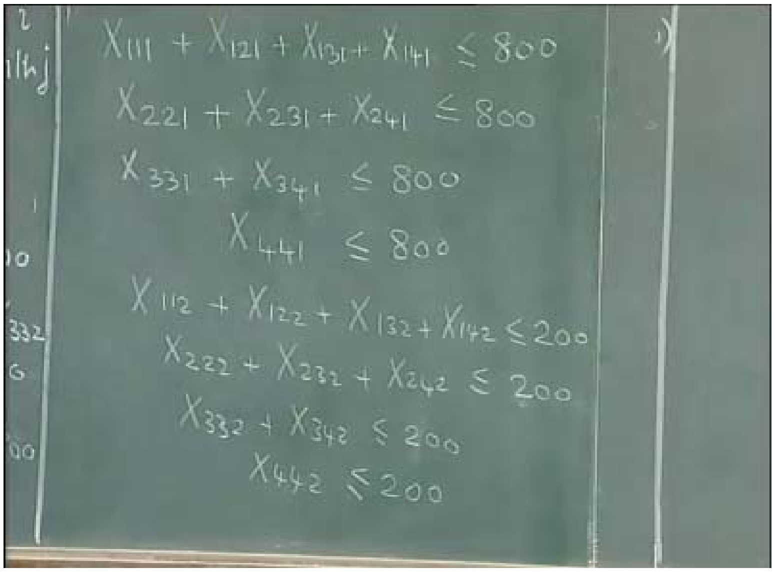

We do not see them here.

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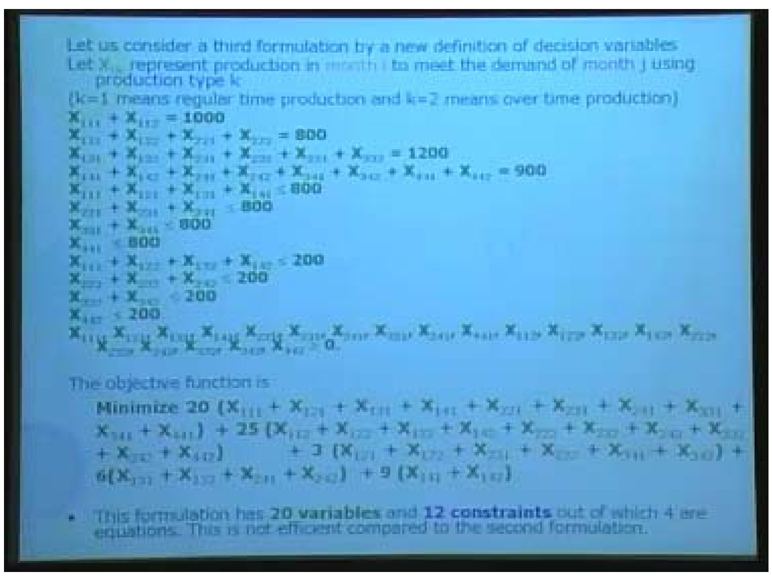

You could introduce greater than or equal to constraints when we eliminated the inventory that is carried.

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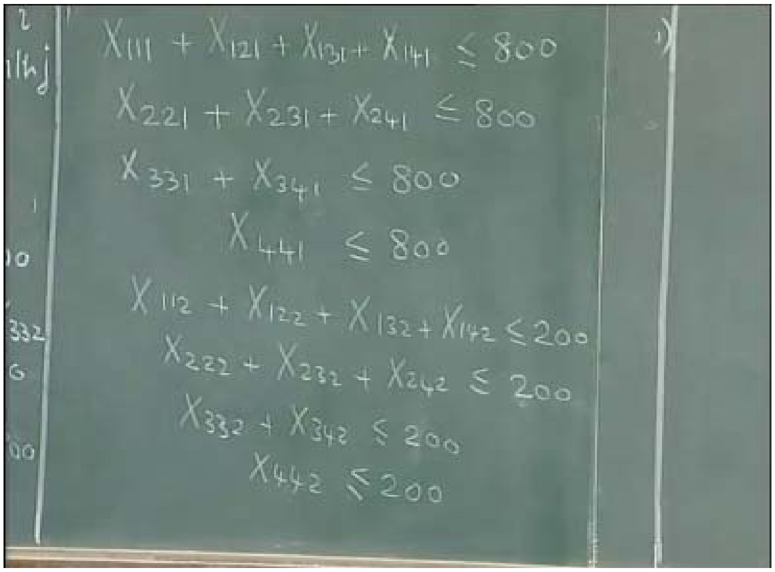

So constraints can be of three types,

(I) less than or equal to type that you see here
(II) Equations that you see here and
(III) Greater than or equal to type.

So we have seen three types of constraints in this. Decision variables of course, are also included. We have also learnt that a formulation is superior if it has fewer constraints, fewer variables. It happened that all the three types of formulation had the same number of constraints so we couldn't make out something superior based on fewer constraints. But in general a formulation that has fewer constraints is superior. For the same number of constraints a formulation that has fewer variables is superior. We also saw that the same problem can be formulated in more than one way depending on how the person formulating looks at the problem. Now we did see 3 different types of formulation and all the 3 formulations will give effectively the same solution if they are correct. Some of them may have more variables. Some of them may have fewer variables. So we have come across 3 formulations. We also saw the non negativity restriction that comes in. So in summary in these two examples that we have seen, we have seen how to formulate a linear programming problem, terminology in terms of Decision variable, Objective function, Constraints, Non negativity, different types of objectives, different types of constraints and different types of variables. We later go through two more formulations to understand some aspects of problem formulation that need to be covered which have not been covered in these two.