Advanced Operations Research

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Lecture-7

Dantzig Wolfe Decomposition Algorithm

Primal Dual Algorithm

We continue the discussion on the decomposition algorithm.

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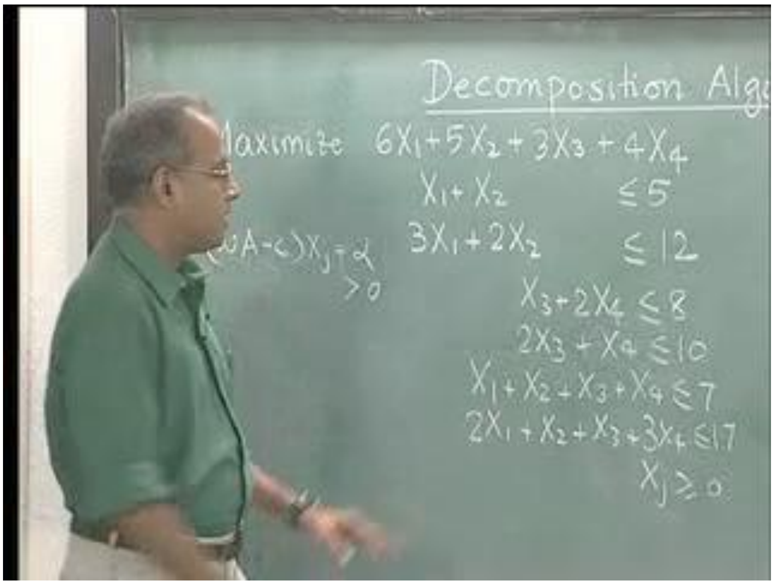

The problem that we are solving is maximize, $6X_1$ plus $5X_2$ plus $3X_3$ plus $4X_4$ subject to these six constraints and X_j greater than or equal to 0. The underlying principle in the decomposition algorithm is that, when we relax a certain set of constraints the problem gets decomposed into more than one linear programming problem. Now for example in this particular problem if we leave out these two constraints, the problem is decomposed into two linear programming problems.

These constraints which we can take out, so that we can decompose belong to what is called as the master problem. We also said that if we remove this out, the problem becomes two sub problems. We also said in the earlier lecture, that if we add now these two constraints into the feasible region represented by this, then the optimum would change but the optimum can be represented as a convex combination of the corner points. What we did was, we first noted down all the corner points here and for the first set of constraints these four points are the corner points.

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For the second set here, these four points are the corner points. We do not explicitly store all the corner points. Instead we generate the corner points, as they enter into the basis by following a column generation procedure.

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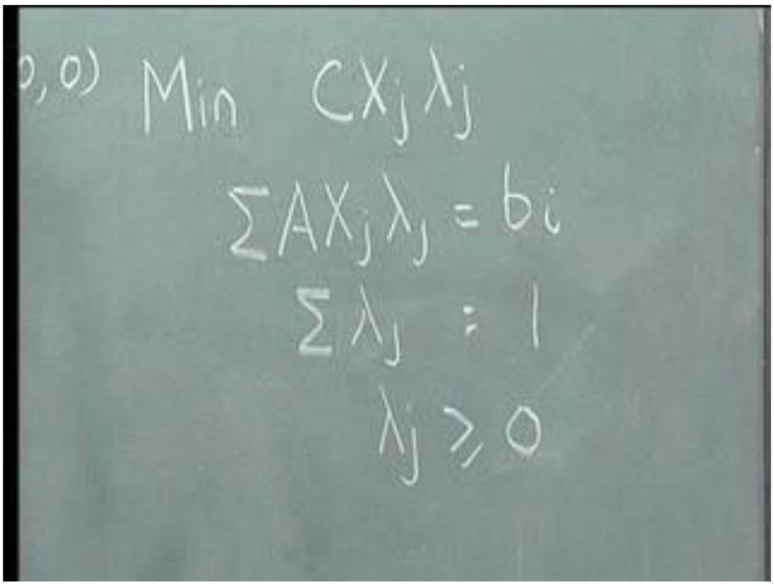

We converted this problem into this, where lambda$_j$, now becomes the variable; X_j is a known corner point that is there in the basis. The problem now becomes minimize C_j X_j lambda$_j$, where lambda$_j$ is the variable subject to AX_j lambda$_j$ equal to b_i, sigma lambda$_j$ equal to 1 and lambda$_j$ greater than or equal to 0.

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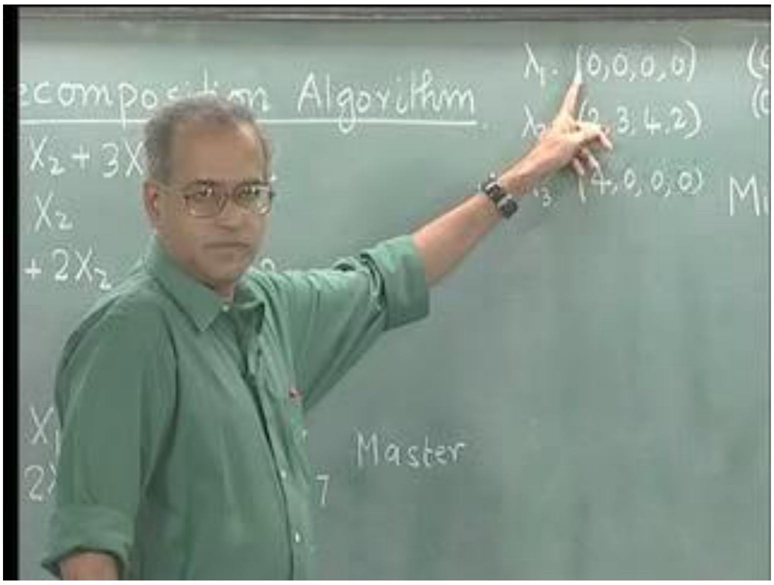

We started the problem by looking at the corner point (0, 0, 0, 0), which was the first basic variable. We also said, for example here, this is the master constraints. We have two master constraints which are here, so there will be two dual variables associated with this and there is one constraint which is sigma lambda$_j$ equal to 1. In our example the two master constraints were less than or equal to type, so we could start with S_1 and S_2 as the basic variables, along with lambda$_1$ equal to (0, 0, 0, 0), which is feasible to these four points. We started with lambda$_1$ equal to (0, 0, 0, 0) and started the simplex iteration. Then we found out that the basic feasible solution involving S_1 S_2 from here and lambda$_1$ was not optimal. So, we solved a sub problem, through which we identified that a corner point (2, 3, 4, 2), can enter the basis.

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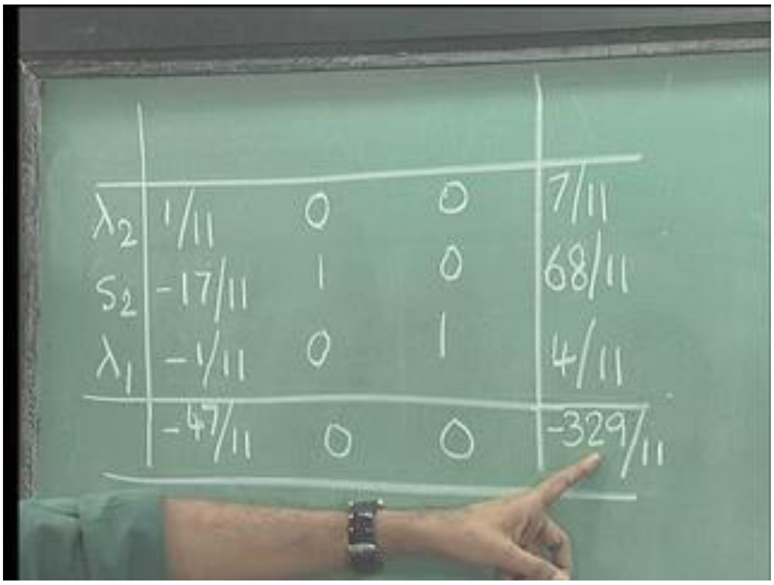

Then we performed one iteration of the simplex algorithm to get this solution at the end of one iteration, with λ_1 equal to 4 by 11, λ_2 equal to 7 by 11 and Z equal to minus 329 by 11. Now, we are solving a minimization problem as far as the simplex is concerned and since the given problem is a maximization problem, we are getting a minus sign here. So the objective function value is actually 329 by 11 for the maximization problem and minus 329 by 11 for the minimization problem. When we verified whether this is optimal, by solving another sub problem we found out that the corner point (4, 0, 0, 0), enters the basis with a positive value of 76 by 11.

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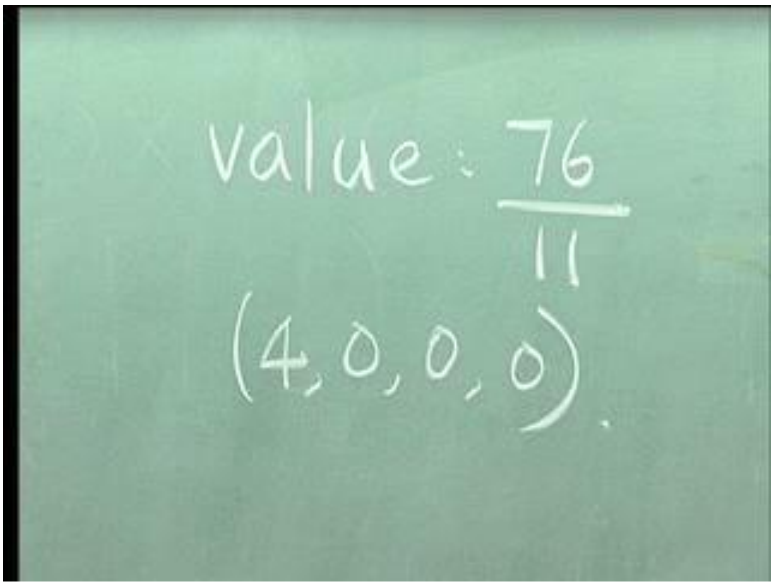

The sub problem that we solved, essentially tries to find out the maximum value of wA minus c into X_j plus alpha, subject to these two sets of constraints and when we did that we found that the point (4, 0, 0, 0) enters the basis with a positive value of 76 by 11.

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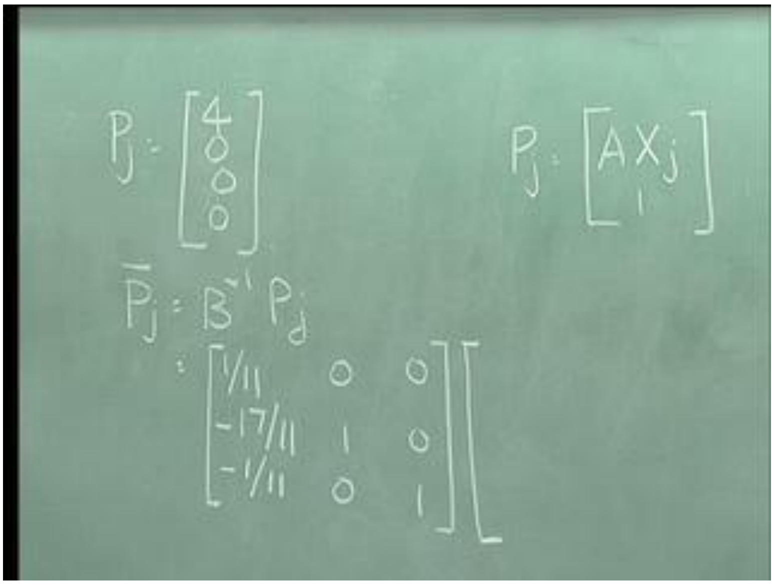

Now we realize that the point P_j equal to (4 0 0 0), enters the basis but then we also have to find out $Pbar_j$ which is the column corresponding to the entering variable. So we get $Pbar_j$ equal to B inverse P_j. B inverse is always found here under the starting basic variables. This is the B inverse which we are actually updating. So B inverse is 1 by 11, minus 17 by 11, minus 1 by 11, 0 1 0, 0 0 1. Before we compute this $Pbar_j$ equal to B inverse P_j, we know that this corner point (4, 0, 0, 0) actually enters but P_j which is the entering column is AX_j 1, which comes from this expression AX_j 1, for the corresponding $lambda_j$. The moment we know that the corner point (4, 0, 0, 0) enters that is not the corner point that is X_j; so X_j will be the point (4, 0, 0, 0). There is an associated $lambda_j$ that enters for this $X_{j,}$ and in order to find out $Pbar_j$ equal to B inverse P_j, we need to find out AX_j 1. To find out AX_j 1, we first find out AX_j. A is 1 1 1 1, 2 1 1 3 into X_j is 4 0 0 0. Now this A comes from somewhere here, sigma AX_j $lambda_j$ equal to b_i, which is from here, 1 1 1 1, 2 1 1 and 3. We first find out AX_j to get 1 into 4 plus 1 into 0 plus 1 into 0 plus 1 into 0, which is 4 and 8. So now P_j is equal to AX_j 1, which is 4 8 and 1.

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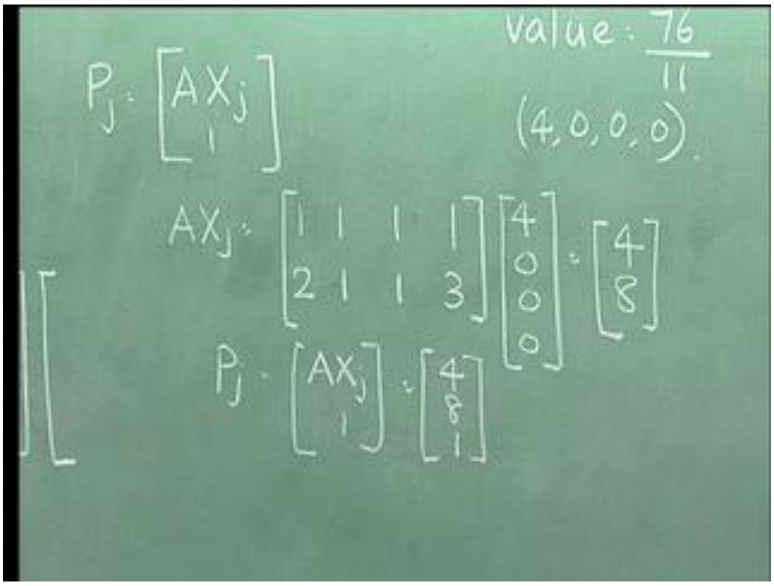

We have to find out Pbar$_j$ equal to B inverse P_j. So P_j is 4 8 1, from which Pbar$_j$ is equal to 4 by 11 plus 0 into 8 plus 0 into 1, which is 4 by 11, minus 17 by 11 into 4 is minus 68 by 11 plus 8 plus 0. This is 8 minus 68 by 11, which is 20 by 11 and the last one is minus 4 by 11 plus 0 plus 1, which is 1 minus 4 by 11 which is 7 by 11. So, now Pbar$_j$ is 4 by 11, 20 by 11, 7 by 11. Therefore, another corner point whose variable name is lambda$_3$ which is the corner point 4 0 0 0, which is shown here, lambda$_3$ which is 4 0 0 0 enters the basis with Z_j minus C_j value of 76 by 11, a positive value, and the entering column has values 4 by 11, 20 by 11 and 7 by 11.Now we perform one simplex iteration, by entering this.

In order to find the leaving variable we need to find out theta, which is the ratio between right hand side value and the entering column. 7 by 11 divided by 4 by 11 is 7 by 4, 68 by 11 divided by 20 by 11 is 68 by 20, which is 34 by 10 which is 17 by 5 and this is 4 by 7; 4 by 7 is the smallest of the three. Therefore, this variable leaves the basis and this becomes the pivot element.

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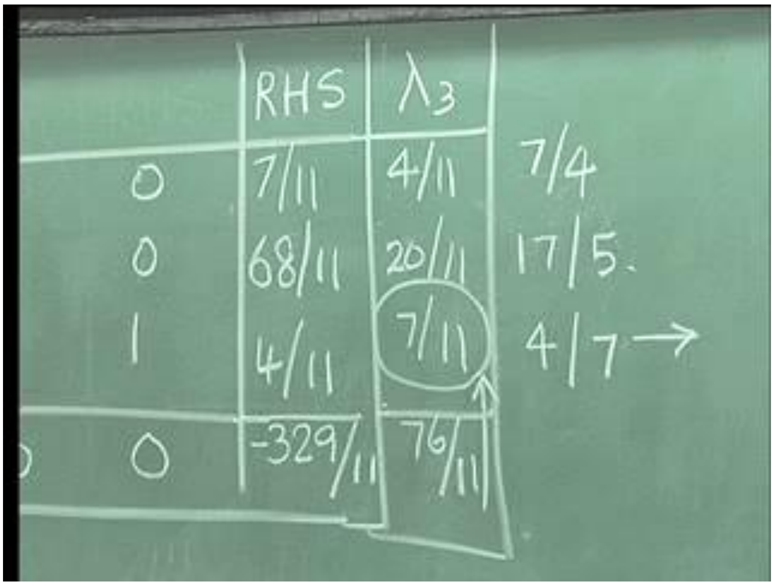

Now we perform the simplex iteration by replacing lambda$_1$ with lambda$_3$. So we get lambda$_{2,}$ S$_2$ and lambda$_3$ in the solution. Divide everything by 7 by 11 to get minus 1 by 7, 0, 11 by 7 and 4 by 7 here; this will become 1. So now we need a 0 here. This minus 4 by 11 times 1 will give 0. This (Refer slide time: 11:05) minus 4 by 11, so 1 by 11 plus 4 by 77 is 11 by 77, which is 1 by 7. This minus 4 by 11 times this is 0. This minus 4 by 11 is minus 4 by 7; this minus 4 by 11 into this, so this is 16 by 77, this is 33 by 77 which is 3 by 7. This would become a 0. Now, we need another 0 here so this minus 20 by 11 times 1 will give 0. So this minus 20 by 11; this is plus 20 by 77, this is minus 17 into 7 is 119; minus 119 plus 20 is minus 99 by 77, which is minus 9 by 7.

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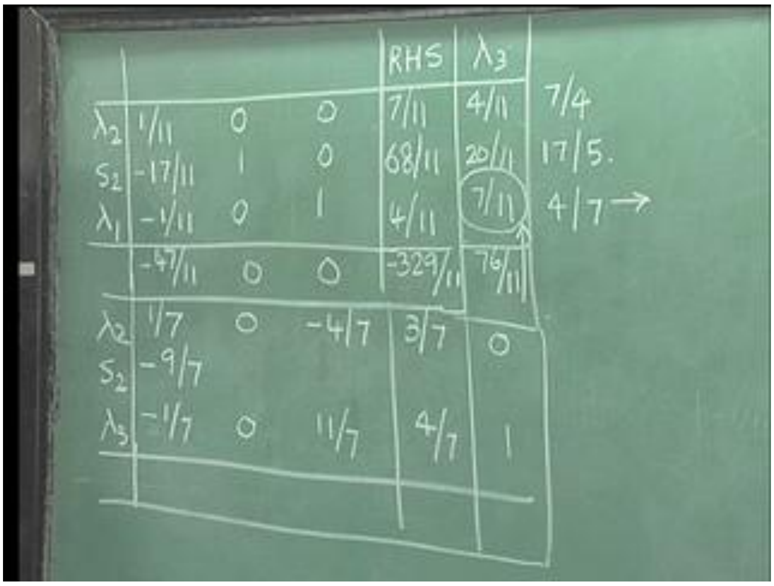

This is 1; this minus 20 by 11 times this, so this minus 20 by 11 will give minus 20 by 7. This minus 20 by 11, this is 80 by 77, so this is 68 into 7 minus 80 is 396 by 77, which would mean that this is 36 by 7; so we would get 36 by 7 here. We also know that we can do the same row operation to get the other one here. So in order to get a 0, this minus 76 by 11 times will give 0. So this minus 76 by 11 times so, minus 47 by 11 plus 76 by 77; so minus 47 into 7 is 329 plus 76 is 253 by 77; 23 by 7; so we would get minus 23 by 7. This will become 0; this minus 76 by 11 into 1. So this minus 76 by 11 into 1 is minus 76 by 7.

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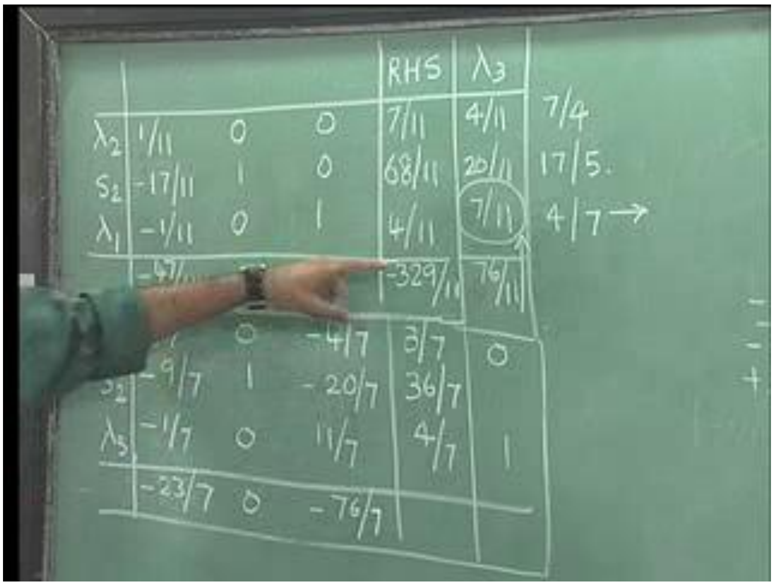

This will be minus 329 by 11 minus 76 by 11 into 4 by 7. So it is minus 304 by 77 minus 329 by 11. This would give 2303 plus 304 is 2607 by 77; 237 by 7, so minus 237 by 7 is what we get at the end of the iteration. Let us quickly make a few checks to ensure that we are doing it right.

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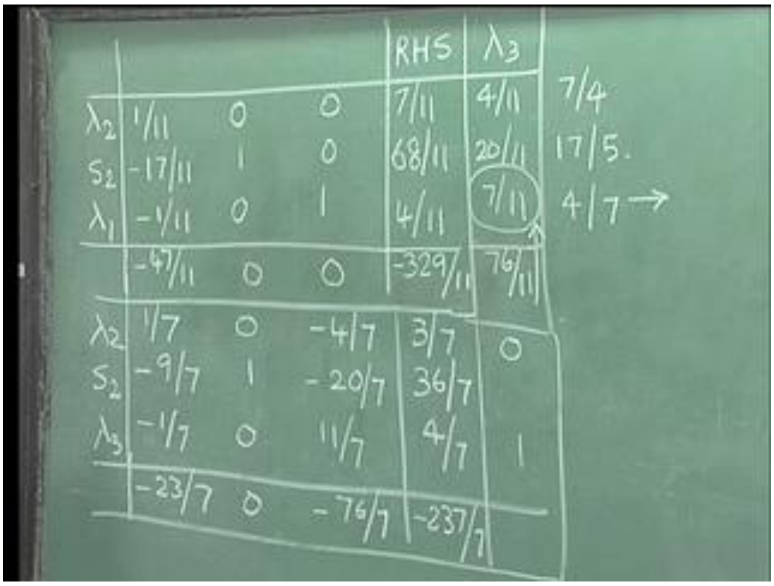

Sigma lambda$_j$ is equal to 1, so 3 by 7 plus 4 by 7 is 1. The actual basic feasible solution to the entire problem at this stage is given by lambda$_2$ into X_2 plus lambda$_3$ into X_3, which is 3 by 7 into (2, 3, 4, 2) plus 4 by 7 into lambda$_3$, which is (4, 0, 0, 0), which is 6 by 7 plus 16 by 7 is 23 by 7, rest of them are 0. This is 9 by 7, this is 12 by 7 and this is 6 by 7. This is the corner point that we actually have as basic feasible solution to the entire problem.

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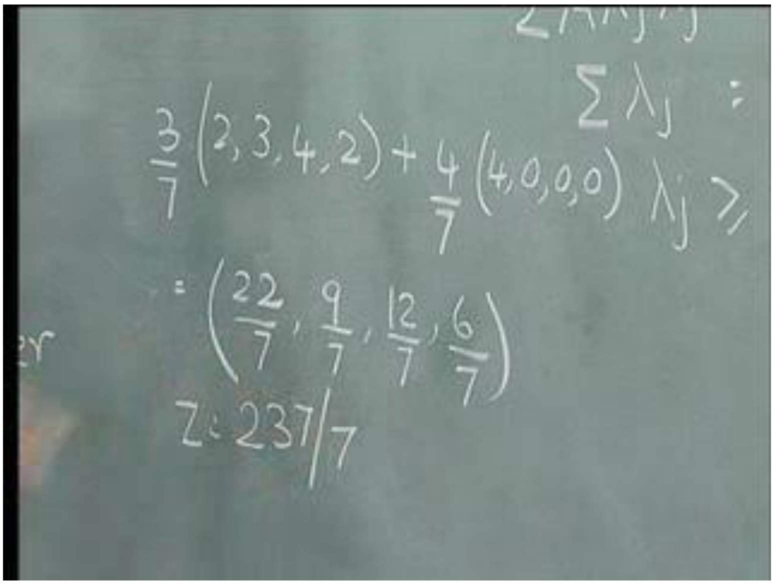

Now the corresponding value of the objective function is, $6X_1$, 23 fives are 115, 23 six are 138. So this is 4 into 4 is16, plus 6 is 22 by 7, 9 by 7, 12by 7 and 6 by 7. 22 into 6 are 132; 132 plus 45 is 177, 177 plus 36 is 213, 213 plus 24 is 237 by 7, which is what we have here. We are now looking at this basic feasible solution, whose objective function value is given by this. We have to check whether this basic feasible solution is optimal and if this basic feasible solution is not optimal, then there will be another entering corner point which is obtained by solving a sub problem which maximizes wA minus c into X_j plus alpha. The w and alpha are the dual variables that can be found from this solution; so w is 23 by 7, 0 and alpha is minus 76 by 7.

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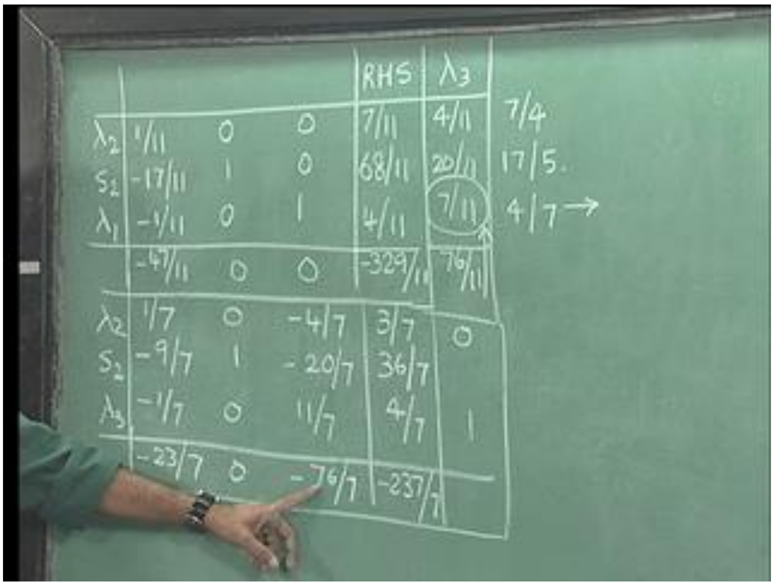

The value of w is always the negative of the value that we have here; w is 23 by 7, 0 so w is minus 23 by 7, 0 and alpha is minus 76 by 7. Because we have values Z_j minus C_j here, they directly give us the value of the dual associated with the problem. So w is minus 23 by 7 0, and alpha is minus 76 by 7. We now find out wA minus c, and then X_j. So wA is now minus 23 by 7 0, into X_j is 1. A is 1 1 1 1, 2 1 1 3 which is minus 23 by 7, minus 23 by 7, minus 23 by 7 and minus 23 by 7. Now wA minus c is equal to minus 23 by 7, minus 23 by 7, minus 23 by 7, minus 23 by 7, minus of minus 6, minus 5, minus 3 and minus 4.

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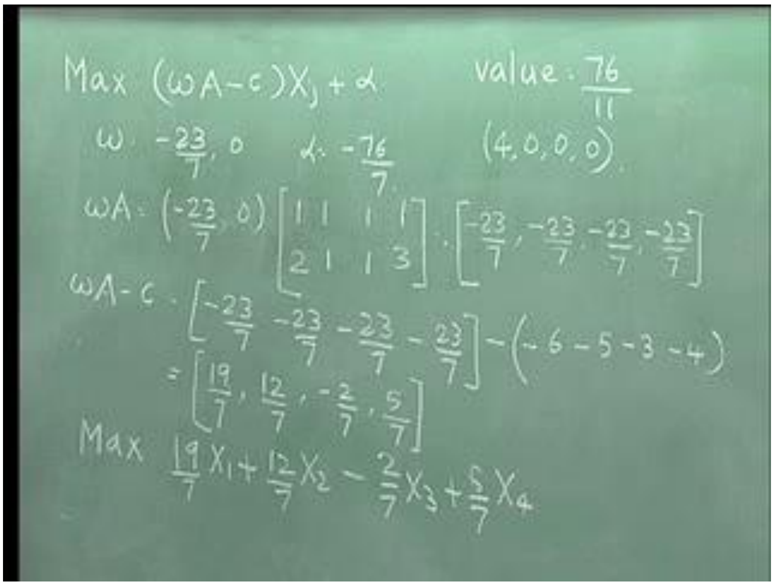

c becomes minus 6, minus 5, minus 3, minus 4, because this problem is a maximization problem; so when we convert it to a minimization based on which this theory is developed, c will become minus 6, minus 5, minus 3 and minus 4. We now compute wA minus c; this will be 6 minus 23 by 7, so 42 minus 23, which is 19 by 7. This will be 5 minus 23 by 7 which is 12 by 7, third will be 3 minus 23 by 7 which is minus 2 by 7 and the last will be 4 minus 23 by 7 which is plus 5 by 7. Now, we have to find out and solve the sub problem which maximizes 19 by 7 X_1 plus 12 by 7 X_2 minus 2 by 7 X_3 plus 5 by 7 X_4, subject to the condition these two sets of constraints are satisfied and X_1, X_2, X_3, X_4 greater than or equal to 0.

Now that gives us two linear programming problems, where the first problem will be to maximize the X_1, X_2, term, 19 by 7 X_1 plus 12 by 7 X_2 subject to this set, and the second will be to maximize minus 2 by 7 X_3 plus 5 by 7 X_4 subject to this set. These two linear programming problems have to be solved but as already mentioned, since this problem is a smaller numerical example, we have now listed all the corner points associated with these two constraints. We simply substitute the corner points into it and get the solution, though theoretically it is not the best way to solve an LP.

We substitute this, so we have 19 by 7 X_1 plus 12 by 7 X_2. This gives 0; this would give 19 into 4, 76 plus 0. This will give us 60. This will give us 19 into 2, 38 plus 12 into 236 which is 74. Therefore, the point 4, 0 is optimal here, with 76 by 7 as the objective function value. Second one is minus 2 by 7 X_3 plus 5 by 7 X_4; so (0, 0) will give 0. This will give a negative value; this will give us a value 20 by 7 and this would give as a value minus 8 by 7 plus 10 by 7, so this is optimal. (0, 4) is optimal, the objective function value here is 19 into 4 is 76 by 7. The other one is 20 by 7, so the total objective function value is 96 by 7. But we also have to consider this alpha. wA minus c into X_j is 96 by 7, alpha is minus 76 by 7; so wA minus c into X_j plus alpha is 20 by 7, which is positive. Therefore, the corner point (4, 0, 0, 4) enters the basis with the contribution of 20 by 7.

We write the fourth corner point lambda_4 is equal to (4, 0, 0, 4), which enters the basis. Now we write this here. So the next point that enters the basis is the point; let us call this as lambda_4, so lambda_4 will enter the basis with value here equal to 20 by 7. We do not need this one, now this can be removed; so enters with 20 by 7. In order to find out this entering column corresponding to $lambda_4$, we now need to find out first P_j equal to AX_j 1.

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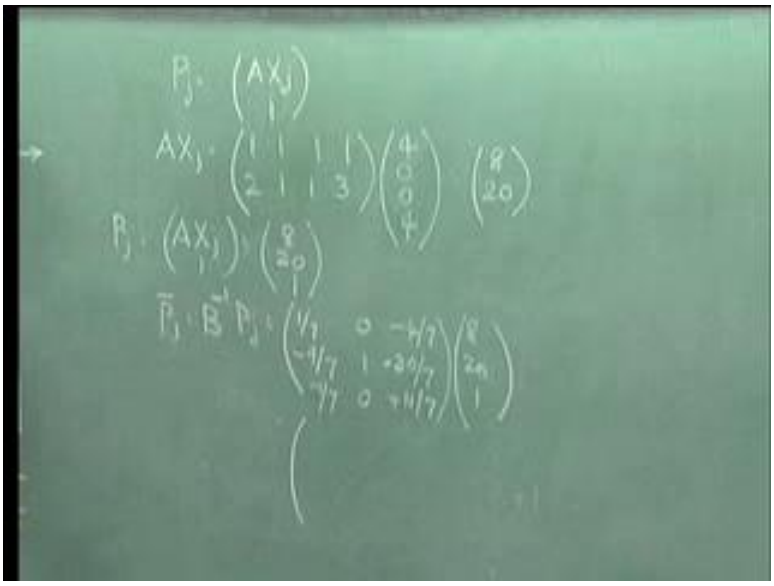

AX_j will be, A is 1 1 1 1, 2 1 1 3, into X_j is 4 0 0 4, which is 4 into 1 plus 0 plus 0 plus 4 into 1, which is 8, 4 into 2 is 8, 0, 0, plus 12, is 20. So AX_j 1, which is equal to P_j will be 8 20 and 1. Now $Pbar_j$ equal to B inverse $P_{j,}$ and B inverse can be obtained from here. So B inverse is 1 by 7 minus 9 by 7 minus 1 by 7, 0 1 0 minus 4 by 7 minus 20 by 7 plus 11 by 7 into 8 20 1, which is given by 8 by 7 minus 4 by 7 is 4 by 7, minus 72 by 7 plus 20 minus 20 by 7 is 20 minus 92 by 7, which is 48 by 7, 140 minus 92 is 48 by 7, minus 8 by 7 plus 11 by 7 is 3 by 7. So the entering column is now 4 by 7, 48 by 7 and 3 by 7. Now we need to find out theta. Theta is the right hand side divided by the entering column, so this variable will enter the basis.

Now theta is 3 by 7 divided by 4 by 7 which is 3 by 4, 36 by 7 divided by 48 by 7 which is 36 by 48, which is also 3 by 4, 4 by 7 divided by 3 by 7 is 4 by 3. So the minimum theta will leave the basis. Now there is a tie for the leaving variables. So when there is a tie for the leaving variables, it gives us a degenerate basic feasible solution. At present there is no conclusive rule that we can use, we could actually take any one of them to leave the basis and now we will choose to leave $lambda_2$ based on the fact that it occurs first. So, we enter the variable $lambda_4$ and the variable $lambda_2$ leaves the basis.

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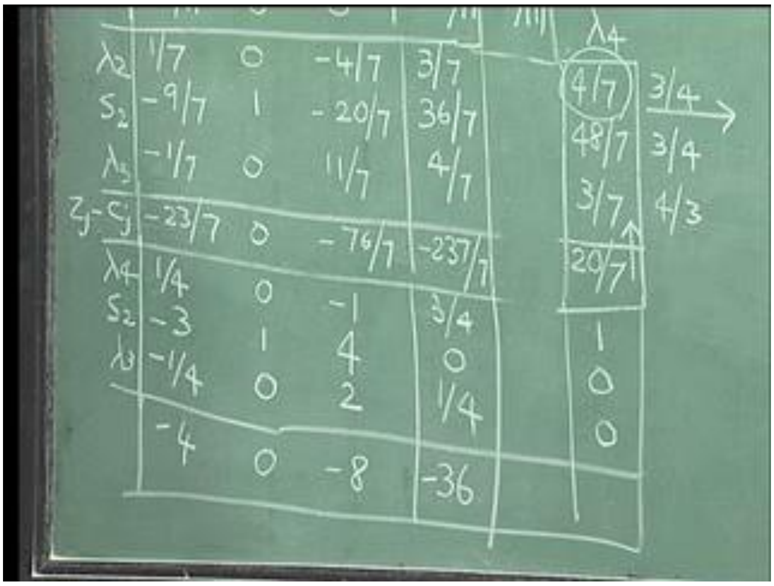

We perform one more simplex iteration with $lambda_4$ replacing $lambda_2$ and we have S_2 and we have $lambda_3$. Now this is pivot because this is the leading element. So divide every element of the pivot row by the pivot to get a 1 here. So 1 by 7 divided by 4 by 7 is 1 by 4, 0 minus 1. This is the right hand side value, so you get 3 by 4 and we will get a 1 here. Now we need a 0 here so this minus 48 by 7 times this is 0. This minus 48 by 7; so this is 48 by 7 into 1 by 4 is 12 by 7. So minus 9 by 7 minus 12 by 7 is minus 21 by 7 which is minus 3, so you get a minus 3. This minus 48 by 7 times will give us a 1. This minus 48 by 7 times this; so minus 20 by 7 plus 48 by 7 is plus 28 by 7 which is plus 4.

This minus 48 by 7is 36 by 7; so you will get a 0 here, because 3 by 4 into 48 by 7 is 36 by 7. 36 by 7 minus 36 by 7 is 0, which is understandable because we had a tie here and which resulted in a degenerate basic feasible solution. So automatically the next one will have value equal to 0. Now this would also be 0. We need another 0 here, so this minus 3 by 7 times 1 would give us a 0. So this minus 3 by 7, so minus 1 by 7 minus 3 by 28 is minus 7 by 28 which is minus 1 by 4. So this minus 3 by 7 would give us 0. This minus 3 by 7 is 11 by 7 plus 3 by 7 which is 14 by 7, which is 2. This minus 3 by 7 is 4 by 7 minus 3 by 28; this into 3 by 7 is 9 by 28. So 4 by 7 minus 9 by 28, 16 minus 9 is 7 by 28 which is 1 by 4. Now this will be 0. We also know that we can get this by a similar row operation. So this minus 20 by 7 times this, would give us a 0 here. This minus 20 by 7 minus 23 by 7 minus 5 by 7 is minus 28 by 7, which is minus 4, this minus 20 by 7, is 0. This minus 20 by 7 is minus 76 by 7 plus 20 by 7 minus 56 by 7, which is minus 8. This minus 20 by 7 minus 237 by 7 minus 60 by 28, which is minus 15 by 7. So this is minus 252 by 7. So minus 252 by 7 is, 7 threes are 21, 7 six are 42, minus 36. So we get a solution with minus 36 that comes here. The basic feasible solution here has λ_3 equal to 1 by 4. λ_4 is equal to 3 by 4. So λ_3 plus λ_4 is equal to 1, which satisfies sigma λ_j equal to 1 and then it has a solution with 36.

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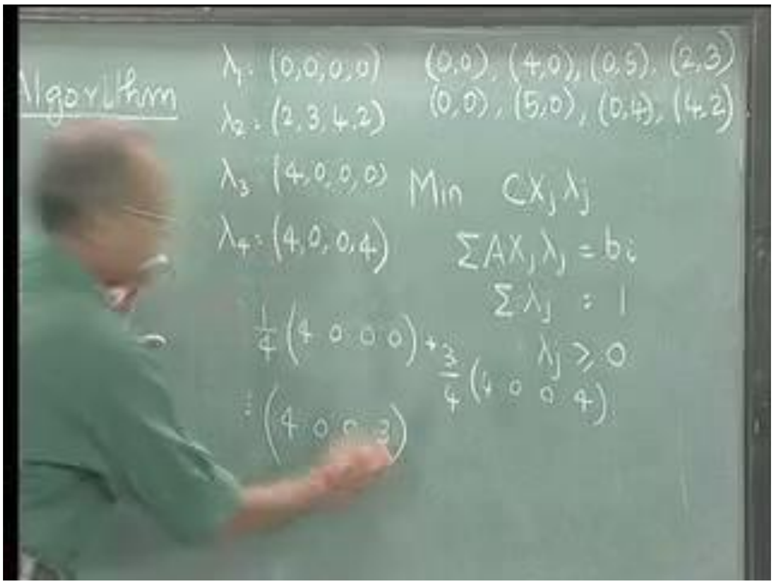

So the solution that we actually are looking at from here is λ_3 X_3 plus λ_4 X_4 which is 1 by 4 into X_3 (4 0 0 0) plus 3 by 4 into X_4 (4 0 0 4) which is 1 by 4 into 4 plus 3 by 4 into 4 is 4. You get 0, you get 0, and you get 1 by 4 into 0 plus 3 by 4 into 4, which is (4 0 0 3). This is the basic feasible solution to the original problem represented by this solution. Now the Z value associated with this is 4 into 6 is 24, plus 3 into 4 is 12 which is 36. We have to find out whether this is optimal. So, in order to find out that this is optimal, we need to check if there is any other entering column or entering corner point that we have.

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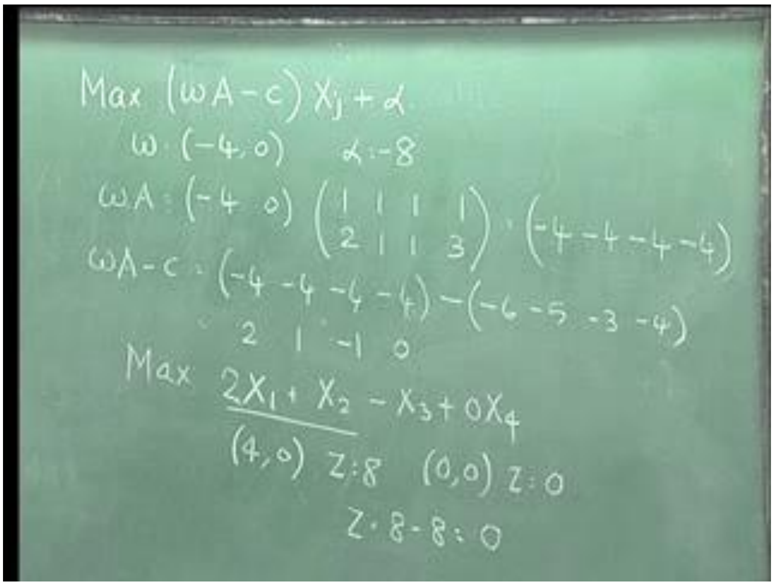

In order to do that, we again need to find out the maximum value of wA minus c into X_j plus alpha, where w and alpha are the dual variables that can be obtained from here. So w is (- 4 0) and alpha is equal to minus 8. We first have to find out wA minus c into X_j, so wA is equal to (- 4 0) into A is 1 1 1 1, 2 1 1 3. We know that A is the constraint coefficient corresponding to the master problem constraints, which can be got from here and therefore w into A is now minus 4 into 1 plus 0 into 2, which is minus 4, minus 4 into 1 plus 0 into 1, which is minus 4, minus 4 and minus 4. Now wA minus c will be (- 4 - 4 - 4 – 4), minus, (- 6 - 5 - 3 – 4); again the (- 6 - 5 - 3 - 4) comes from here. The original problem is a maximization problem, so when we convert it into a minimization problem, the coefficients take a negative sign. So c is (- 6 - 5 - 3 – 4), this will give us (2 1 - 1 0). We need to solve a problem that maximizes $2X_1$ plus X_2 minus X_3 plus $0X_4$, subject to these two sets of constraints and X_j greater than or equal to 0.

This problem can be decomposed into two linear programming problems, one that maximizes $2X_1$ plus X_2, subject to this and the other that maximizes minus X_3 plus $0X_4$ subject to this. Two separate LP problems have to be solved. But again, as explained earlier, since it is a smaller sized problem, we have written down all the corner points here. So by a simple substitution of the corner points, we can get the optimum, though it is not the best way to solve the LP problem. We substitute here; for $2X_1$ plus X_2, for (0, 0) the value is 0, for (4, 0) the value is 8, for (0, 5) the value is

5 and for (2, 3) the value is 7, so (4, 0) is optimal, so (4,0)with Z equal to 8. For the second one that maximizes minus X_3 plus $0X_4$, (0, 0) is 0, (5, 0) is minus 5, (0,4) is 0, and (4,2) is minus 4. Therefore the value will be (0, 0) with 0 will be the optimum. So this is (0, 0) with Z equal to 0. The best solution at this point is the corner point (4, 0, 0, 0) with Z is equal to 8 plus 0 which is 8. But then we have to go back and look at this. We have solved only this part and said there is a corner point 4 0 0 0 with Z equal to 8, but alpha is another minus 8. Therefore, when we include this alpha into the objective function, the Z value actually is 8 minus 8, which is 0.

We do not have the corner point with a positive value of wA minus c into X_j plus alpha. The best we can get is 0 and also that corner point (4 0 0 0) is already in our solution as lambda_3, which is here. We do not have entering corner point, therefore the algorithm will terminate saying that the current solution is optimal.

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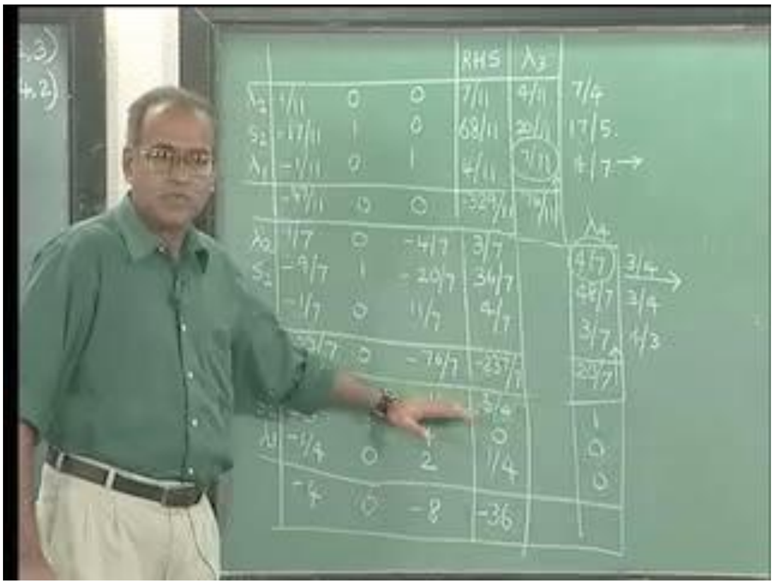

We have already found that the solution which is given by lambda_3 equal to 1 by 4, lambda_4 equal to 3 by 4, gives us on substitution and simplification the corner point (4 0 0 3), which means X_1 equal to 4, X_4 equal to 3 with Z equal to 36 as the optimum solution to the given linear programming problem. This is how the decomposition algorithm actually works, so very quickly let us go back and summarize what we have done.

Even though the procedure that we have followed looks a little laborious, involves a lot of iterations, but for a larger sized linear programming problems this one is far superior in terms of computation time, far superior compared to treating it as one set of six constraints and solving it as a linear programming problem. This decomposition that we have seen is called the Dantzig Wolfe decomposition principle or it is called the Dantzig Wolfe decomposition algorithm, where if the problem can be decomposed into smaller sized linear programming problems by leaving out a set of constraints, we exploit that property and solve smaller sized linear programming problems. We identify the master constraints, and then we also add a linking constraint which is sigma lambda$_j$ equal to 1 and it also borrows the principle that because of the convexity property of the feasible region of a LP, any point inside the feasible region can be represented as a convex combination of corner points.

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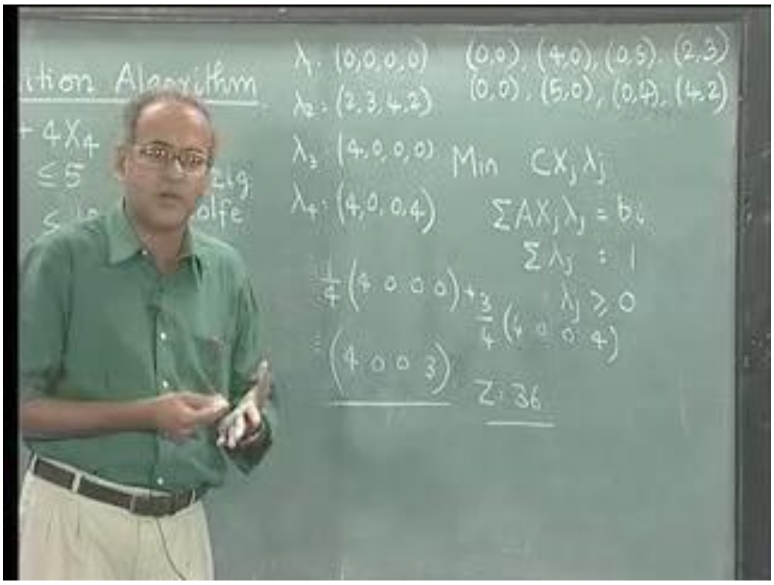

If we leave this out and get a set of corner points after we put this in, the optimum solution can be a point that is inside or on the feasible region given by this. So if it is inside it can be represented as a convex combination of corner points which we exploit. At every iteration of the simplex, where we store only the B inverse, we solve a sub problem which tries to maximize wA minus c into X_j plus alpha. If there is a corner point that can enter the basis, then such a corner point will have a positive value of wA minus c into X_j plus alpha, which means it essentially has a positive value of Z_j minus C_j for a minimization problem. Through a process of column generation and decomposition into sub problems, we are able to solve a larger decomposable linear programming problem very efficiently, using the Dantzig Wolfe decomposition principle or the Dantzig Wolfe decomposition algorithm.

What we do next is we will look at one more approach to solve linear programming problems, which is called the Primal-Dual algorithm. We look at the primal-dual algorithm through a numerical example which is as follows.

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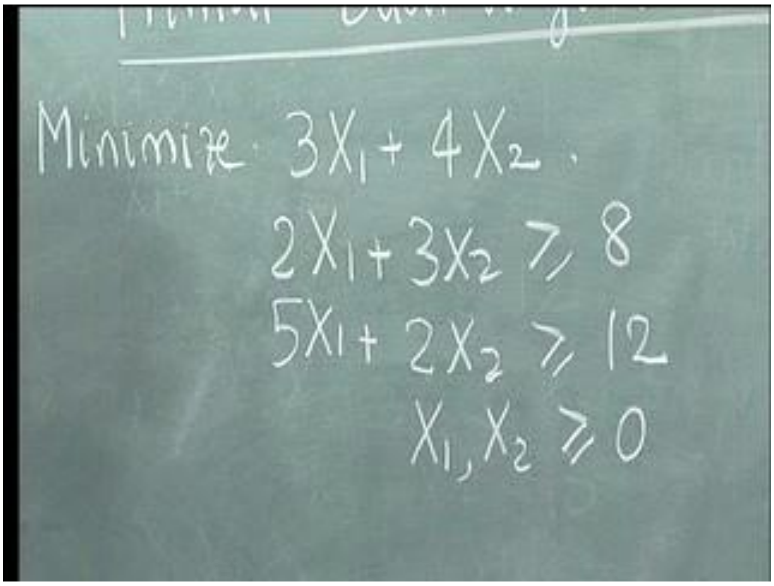

$2X_1$ plus $3X_2$ is greater than or equal to 8 and $5X_1$ plus $2X_2$ greater than or equal to 12; X_1, X_2 greater than or equal to 0. We have this linear programming problem which is given by this one. We have already seen how to solve this linear programming problem by the simplex algorithm. Now what we are going to do is, we are going to introduce another technique, which is somewhat related to the simplex algorithm. It borrows ideas from linear programming theory and then we are going to solve this problem. At the end of it we will see how efficient this method is.

The first thing that we observe is that this problem is a standard minimization problem with two constraints. Both the constraints are at the greater than or equal to type. Now, first let us write the dual to this problem by also considering these two explicitly as constraints. We will write this as X_1 greater than or equal to 0 and X_2 greater than or equal to 0.

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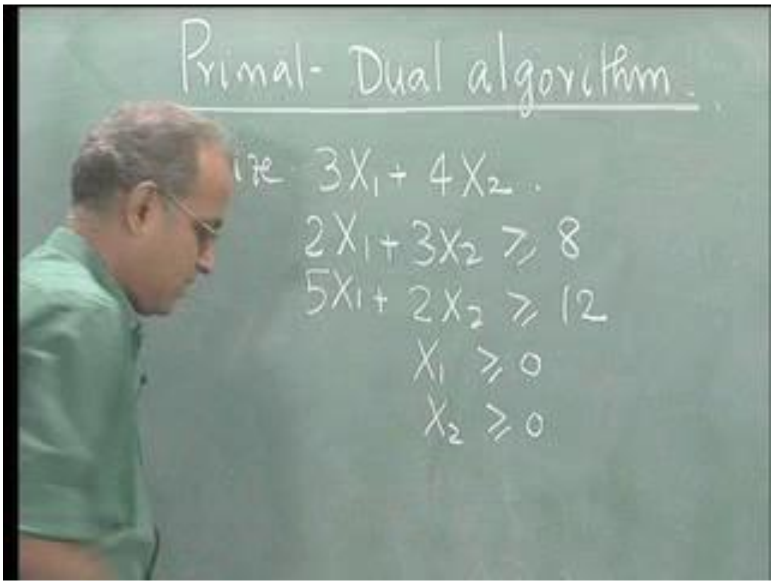

Let us write it this way, and let us also treat these two as some kind of constraints to begin with. What we will do is we will now introduce dual variables. We will introduce dual variables y_1 and y_2 into this. We consider this with X_1 X_2 greater than or equal to 0. So, as always, what we do is we convert these inequalities into equations by adding slack variables. So we get minus X_3 equal to 8, minus X_4 equal to 12.

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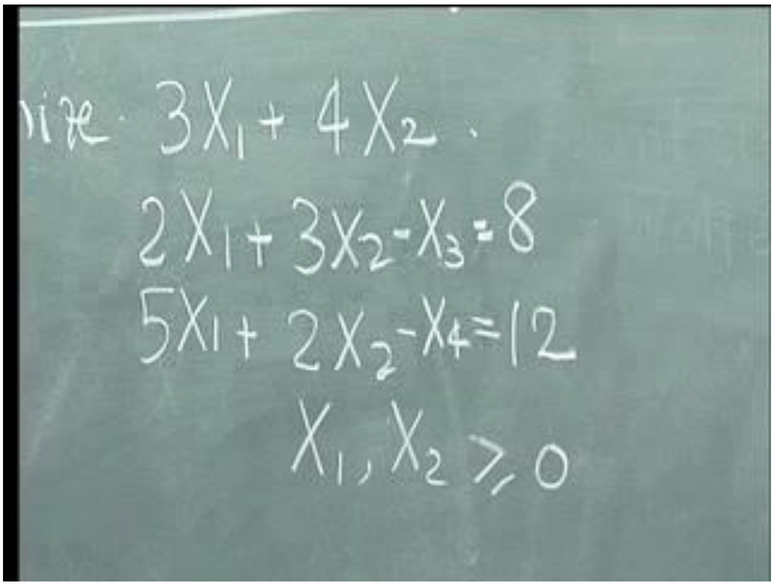

Now let us write the dual associated with this problem. We introduce two dual variables which we call as y_1 and y_2. The dual will be a maximization problem, so maximize $8y_1$ plus $12y_2$, subject to the condition $2y_1$ plus $5y_2$ from here, $2y_1$ plus $5y_2$ is less than or equal to 3. 3 y_1 plus $2y_2$ is less than or equal to 4 from here. This is y_1; minus y_1 less than or equal to 0, and minus y_2 less than or equal to 0. 4531This is $0X_3$ plus $0X_4$. You will have minus y_1 less than or equal to 0 and minus y_2 less than or equal to 0 and we will have y_1, y_2 unrestricted in sign.

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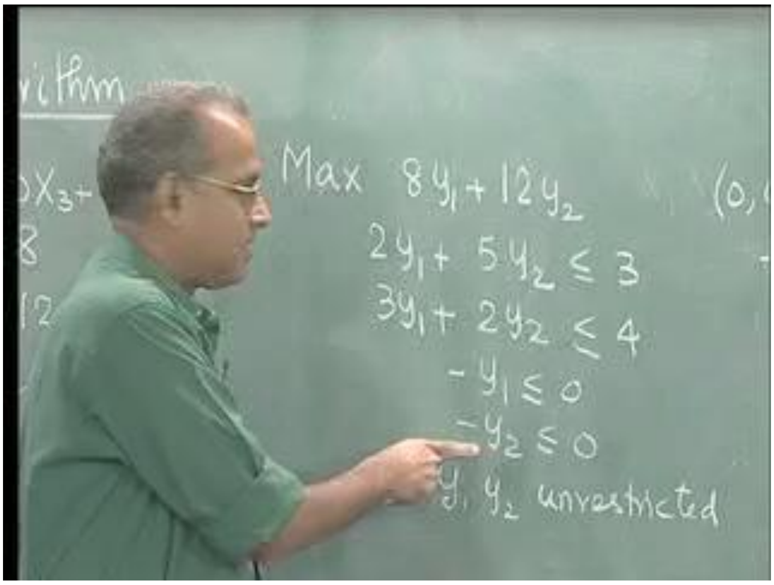

y_1, y_2 unrestricted in sign because, the two primal constraints are equations and therefore, the two dual variables will become unrestricted in sign. Now if we look at this dual, there is one thing that we can do, is to say that the solution (0, 0) is feasible to the dual. So the solution (0, 0) is feasible to the dual and it will always be so. Because, it satisfies this condition, it satisfies this condition, it satisfies this, it satisfies this and gives a solution 0, so (0,0) is feasible to the dual. If the primal is a minimization problem, with all greater than or equal to constraints then we will certainly add minus X_3 and minus X_4 and therefore we will have a constraint of the type minus y_1 less than or equal to 0, minus y_2 less than or equal to 0.

Once again, if all these coefficients are strictly positive, then these right hand sides will be strictly positive or greater than or equal to 0. Therefore, the solution (0,0) will always be feasible to the dual, if the primal is a minimization problem with objective function coefficients greater than or equal to 0 and both the constraints greater than or equal to type which is what this standard problem is.

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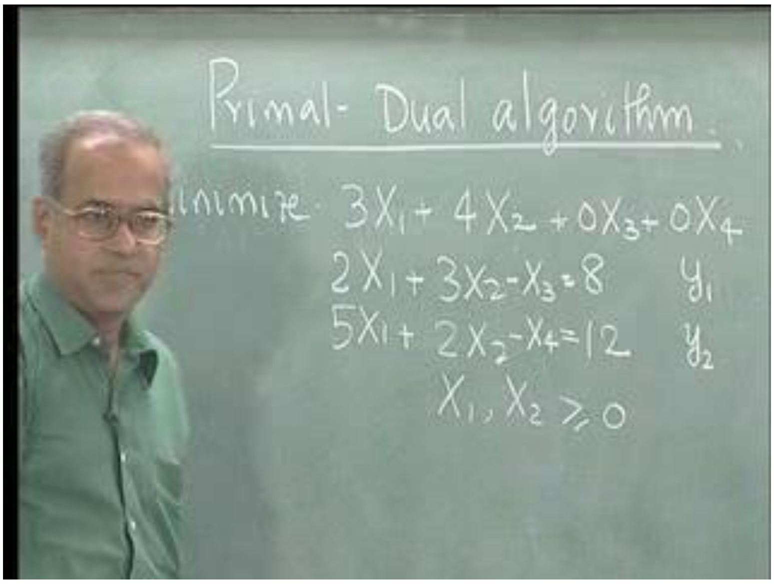

We can always say that (0, 0) is feasible to the dual. Now let us also find out more than this being feasible to the dual, which are the constraints it satisfies as an equation and which are the constraints it satisfies as an inequality. If we look at this, these two are the constraints it satisfies as an equation, because y_1 is equal to 0, y_2 equal to 0 is satisfied as an equation. Now y_1 equal to 0, y_2 equal to 0 gives us 0 less than or equal to 3, which is satisfied as inequality. Similarly, 0 less than or equal to 4 is also satisfied as an inequality. Now we go back and say that we have a feasible solution to the dual and we apply complementary slackness and go back to the corresponding primal. Let us see what happens when we apply complementary slackness and look at the corresponding primal. Now what happens to that?

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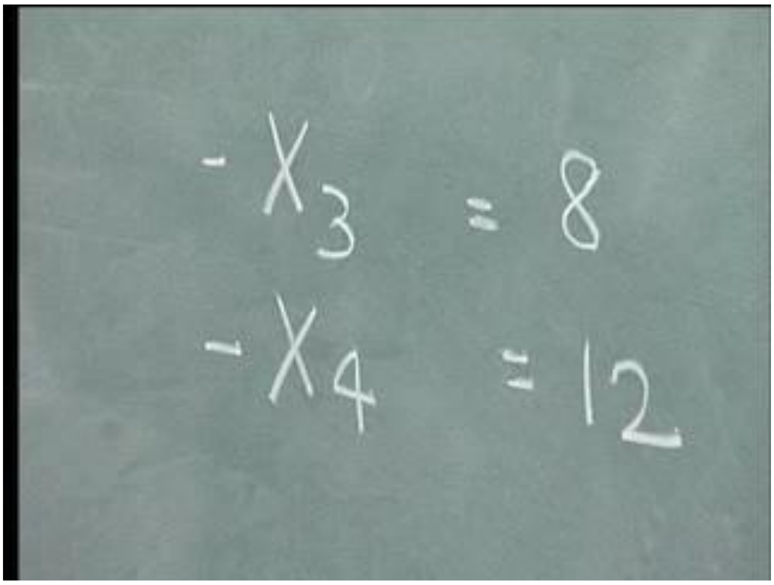

This is satisfied as an equation therefore, X_3 is in the basis. This is satisfied as an equation, so X_4 is in the basis. These two are satisfied as inequalities, so X_1 and X_2 are non basic. If we apply complementary slackness to this solution, then we end up actually getting to solve minus X_3 is equal to 8, minus X_4 is equal to 12. This is what we actually solve when we apply complementary slackness to this. Because when we apply complementary slackness, it tells us that X_3 and X_4 are basic variables, X_1 and X_2 are non basic variables. It is enough to solve for this.

We have already seen in the fundamentals of OR course, that a system of equations, though this is a very trivial solution, with X_3 equal to minus 8 and X_4 equal to minus 12, in general a system of equations can be solved as a linear programming problem by simply adding two artificial variables here, plus a_1 plus a_2 and by minimizing a_1 plus a_2, and saying a_1, a_2 greater than or equal to 0.

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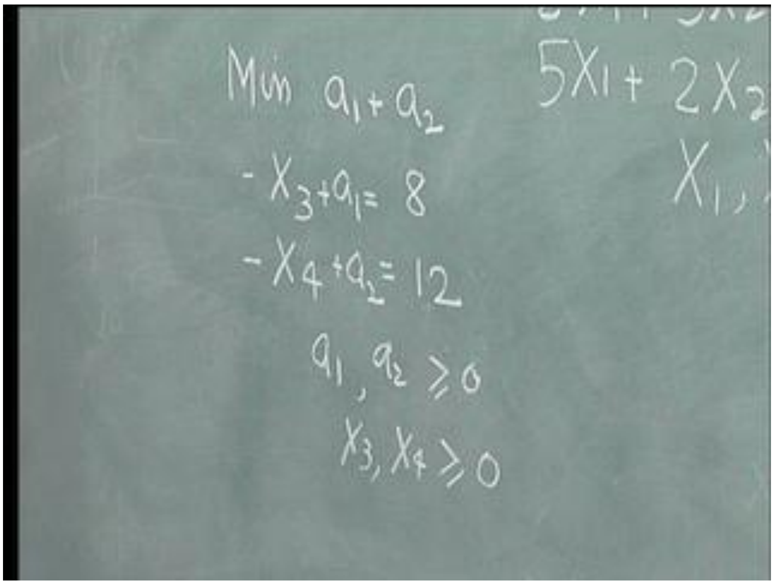

Now this is one way to solve for minus X_3 equal to 8, minus X_4 equal to 12 and $X_{3,}$ X_4 greater than or equal to 0. Now, this is converted into a linear programming problem, which will solve for minimizing a_1 plus a_2 subject to minus X_3 plus a_1 equal to 8, minus X_4 plus a_2 equal to 12, $a_{1,}$ a_2 greater than or equal to 0 and $X_{3,}$ X_4 greater than or equal to 0. This is a linear programming problem and this can be solved by what is called a 2 phase method. What we do here is very similar to, what is called the 2 phase method. If you remember correctly, when we introduced the 2 phase method, what we will do is, we actually solve for two phases. Since these are artificial variables, we will not put a minus big M, which is called the big M method.

In a 2 phase method, the artificial variables will have coefficients 1, the rest of the variables will have objective function coefficient 0, and we solve this problem. If we get Z equal to 0 then the system has a solution. If it gets a positive value of Z, it means there is one artificial variable that is lying in the basis. At the end of the first phase, we would want all the artificial variables to leave, which will be given by a solution Z equal to 0. When we solve this as a linear programming problem, we can actually solve using the simplex, and let us just do that for a movement to try and see how we solve this using the 2 phase method of simplex.

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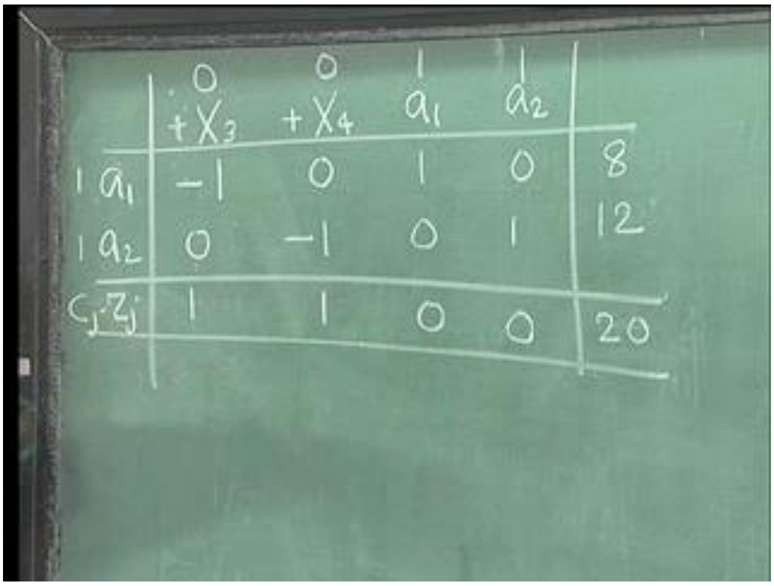

Minus X_3, minus X_4, a_1, a_2; it is a minimization problem, so 1 and 1. We have a_1 and a_2 as the variables. We have minus X_3 plus X_4 plus a_1 is equal to 8; minus X_4 plus a_2 equal to 12. This value is 1,1, this is 0, 0. So 1 into minus 1 plus 1 into 0 is minus 1, so 0 minus 1 is plus 1; plus 1 0 0 dash because these variables are here or you could also put a value 20; 1 into 8 plus 1 into 12 is 20. This value is C_j minus Z_j, it is a minimization problem, so negative C_j minus Z_j will enter and positive C_j and Z_j will not enter. So there is no negative C_j minus Z_j, so the algorithm terminates with the optimum solution a_1 equal to 8, a_2 equal to 12 and Z equal to 20. This terminates with the solution a_1 is equal to 8, a_2 equal to 12 and Z is equal to 20.

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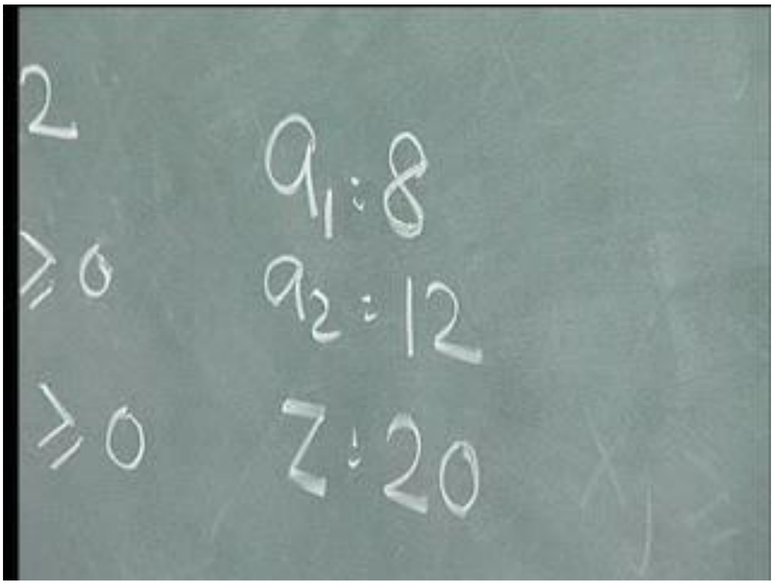

So now we have a_1 equal to 8, a_2 equal to 12 and Z equal to 20. What we have actually done is the following. We wrote the dual, we picked a feasible solution to the dual. Incidentally we picked a solution; the dual has two variables here, so we picked a solution for y_1 and y_2. Now the dual had two variables here, because the primal had two constraints here. We picked a value for y_1, y_2, which is dual feasible and then we apply complementary slackness to see whether the corresponding primal is feasible. In this particular instance the corresponding primal was not feasible, because the artificial variables are lying in the basis. If the corresponding primal were feasible, then the dual feasible solution is optimal by the duality theorems, because we could go back and say that there is a feasible solution to the dual, there is feasible solution to the primal which satisfies complementary slackness and therefore, it will be optimal to the primal and dual respectively. We would also say in the case that, the objective function values would also be equal. So the basic idea is to have a feasible solution to the dual, apply complementary slackness and then try to find out the corresponding solution to the primal. The corresponding solution comes because the dual has two variables. The dual originally had two variables because the primal has two constraints. Now the feasible solution has values for y_1, y_2 greater than or equal to 0.

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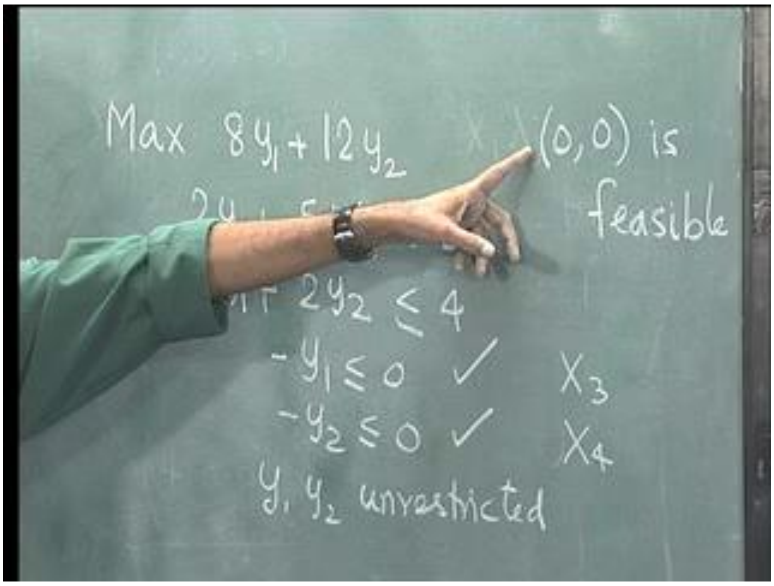

When we apply complementary slackness to this, this will result in two variables here becoming basic when we do this. So two variables will become basic or less than two variables will become basic. Therefore we can solve the problem after we obtain the complementary slackness conditions to check whether we have a feasible solution. Now we have found that this is not so. What we have to do now is to go back and find out another dual feasible solution from this preferably and then apply the same complementary slackness and see whether a primal is feasible. So we repeat such a process. What we do next we will see in the next lecture.