Advanced Operations Research

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Lecture – 5

One Dimensional Cutting Stock Problem (Continued)

We continue our discussion on the cutting stock problem.

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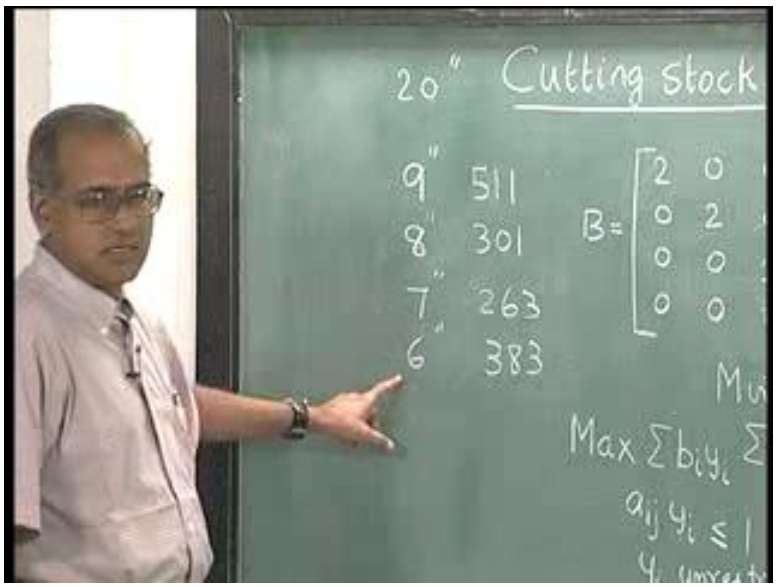

We were looking at this problem of cutting 511 sheets of 9-inch, 301 of 8 inch, 263 of 7-inch and 383 of 6-inch from 20-inch sheets. We also defined X_j as the number of sheets which are cut using the pattern j and the problem reduced to one of minimizing the total number of sheets that are being cut.

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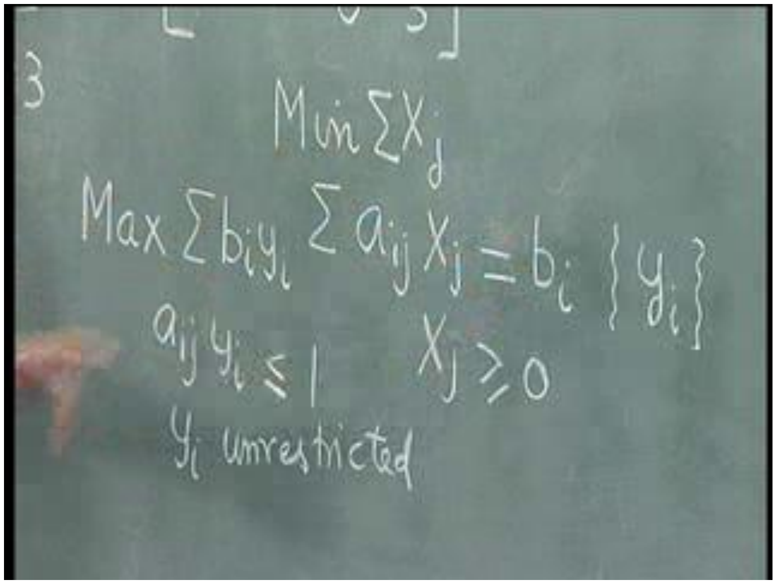

We also formulated this problem of this type by making this as an equation, by considering an exhaustive set of patterns where the waste can even be more than 6.

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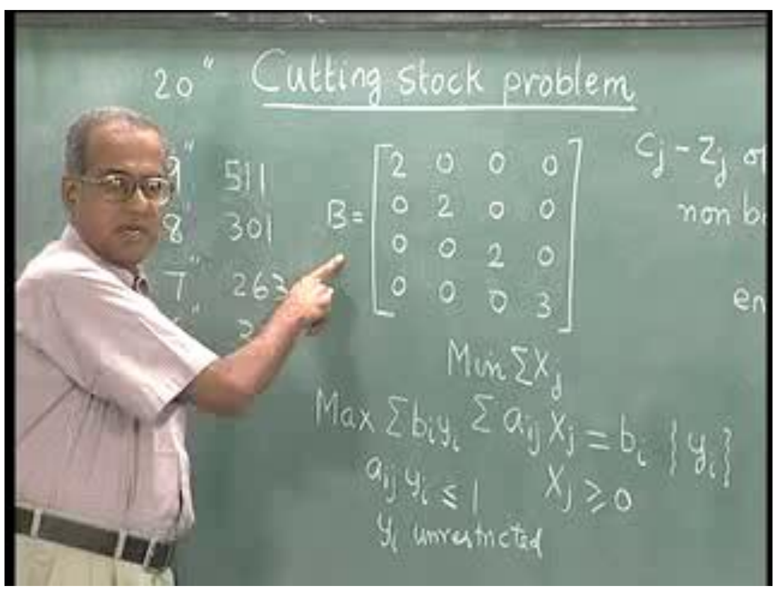

Now, we started the solution by considering this basis, which means by considering patterns that use only 9-inch, only 8-inch, only 7-inch and only 6-inch.

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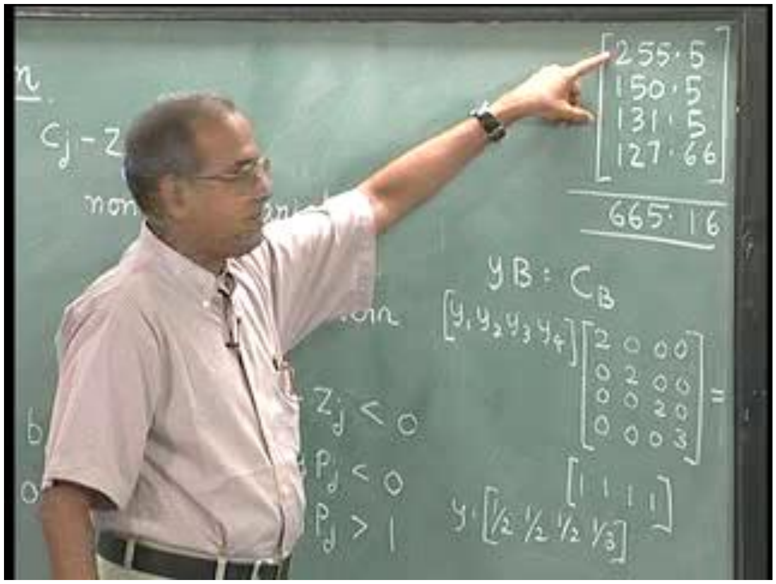

We said if we follow these four patterns, then we would require 255.5 of the first pattern; so much of the second, third and forth, which gave us 665.16 sheets.

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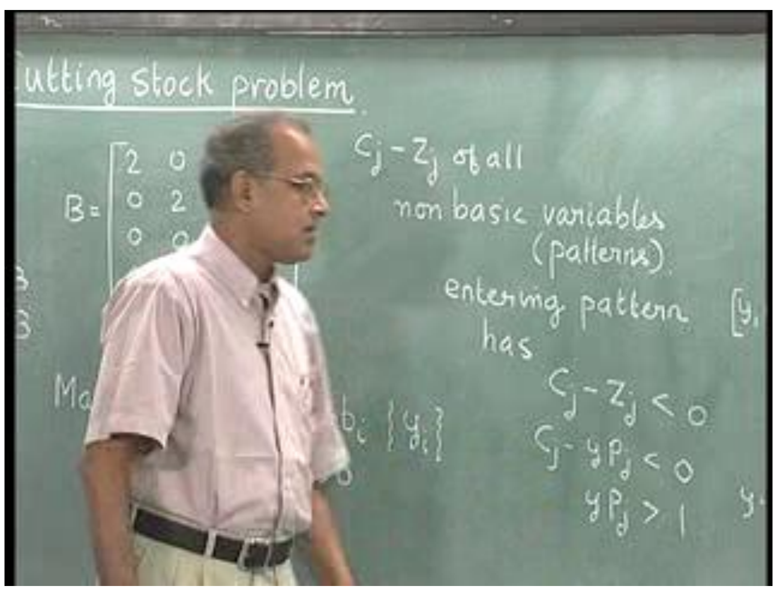

Next thing that we have to do is to find out whether this is optimal. Now, in order to find out whether this is optimal, since the problem is a minimization problem, any strictly negative C_j minus Z_j can enter the basis. So, if there is a pattern j such that C_j minus Z_j is less than 0, then such a pattern can enter the basis. This reduced to saying that if there is a pattern j such that yP_j is greater than 1, then such a pattern can enter. Now, y is the solution of the dual associated with this solution. So y was computed as 1 by 2, 1 by 2, 1 by 2 and 1 by 3.

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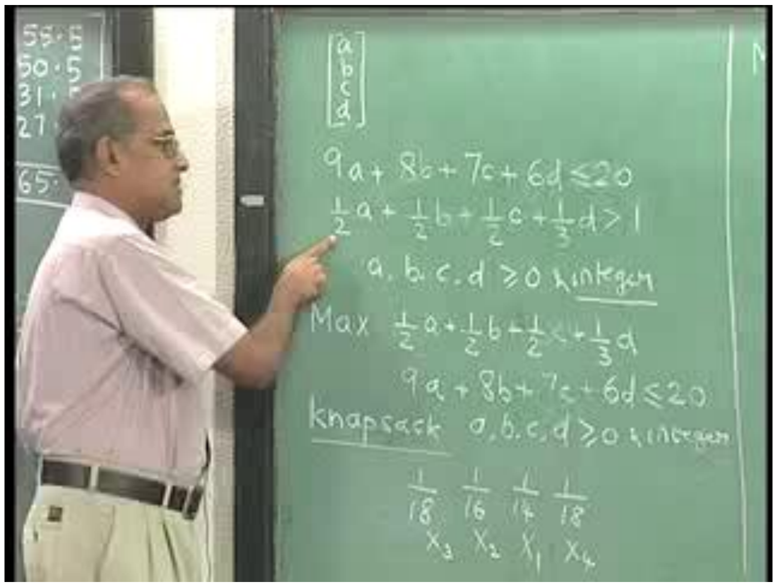

Any feasible pattern should satisfy the condition 9a plus 8b plus 7c plus 6d less than or equal to 20, where a, b, c and d are the number of 9-inch, 8-inch, 7 and 6 that are cut out of a given 20 sheet. This constraint has to be satisfied by every feasible pattern and if a feasible pattern satisfies this condition, that 1 by 2a plus 1 by 2b plus 1 by 2c plus 1 by 3d is greater than 1, and a, b, c, d greater than or equal to 0 and integer then, such a pattern can enter the basis.

We used some ideas from a single constrained Knapsack problem. We also said that because this is a strict greater than or equal to type inequality, we said we would now make this as the objective function and write this problem to find a, b, c, d such that it maximises this function subject to this constraint, greater than or equal to 0 and integer. We also borrowed some ideas from a single constrained Knapsack LP relaxed version of such a problem and said that if we treat it as a LP and solve it, only one variable will be in the solution and that variable is the one that has the largest objective function coefficient by constraint coefficient ratio. We sorted the variables in the decreasing order of this ratio and then we formulated another problem which is this.

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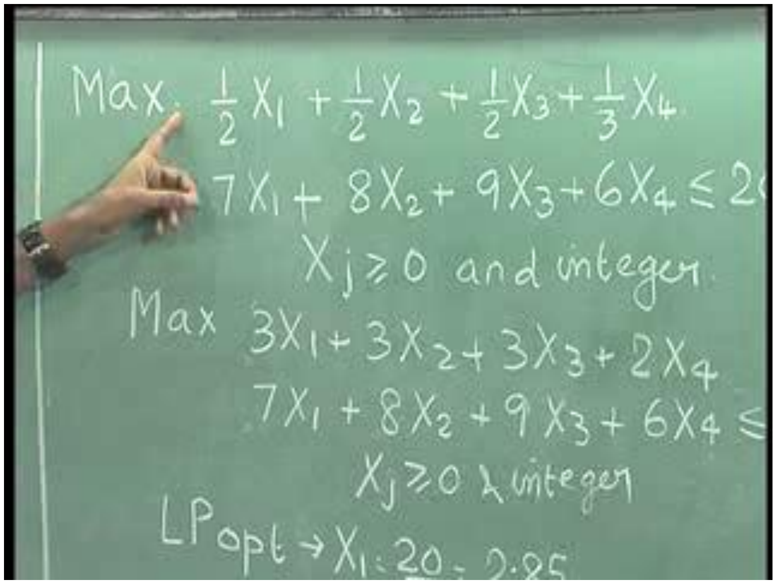

In order to simplify the objective function further, we multiplied the objective function by 6, which happens to be the LCM and then we got this problem.

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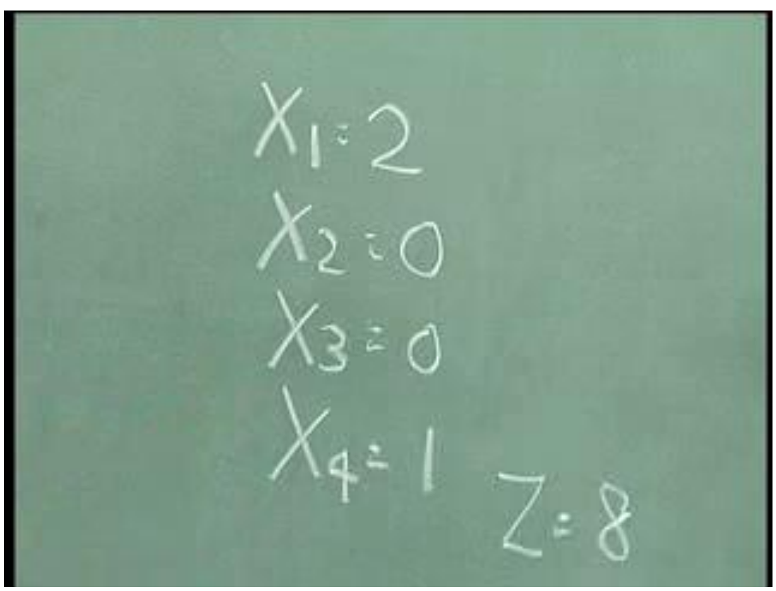

When we solved this problem, we finally obtained a solution with Z equal to 8, which is the optimum solution to the integer programming problem which gave us a solution with X_1 equal to 2, X_2 equal to 0, X_3 equal to 0 and X_4 is equal to 1 with Z is equal to 8.

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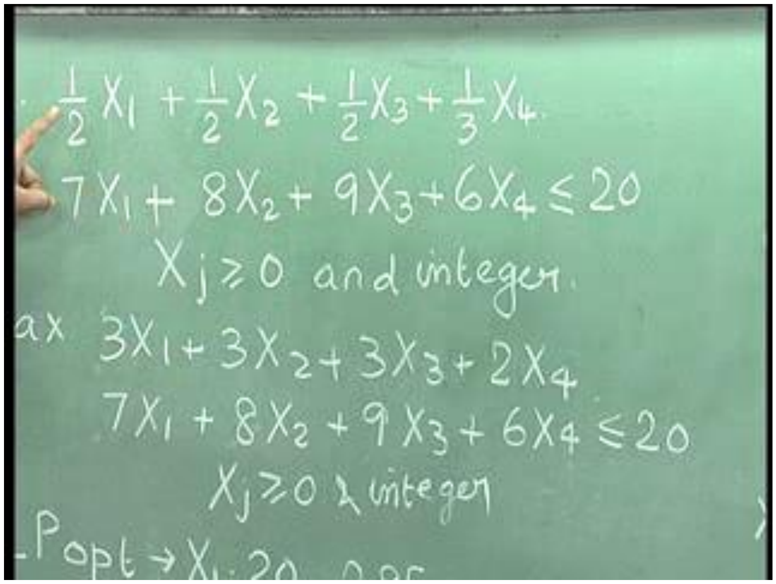

We have already multiplied the original objective function by 6; therefore, corresponding to that 8, this function, when we substitute, will give us a value 2 into 1 by 2 which is 1 plus 0 into 1 by 2 which is 0 plus 0 into 1 by 2 which is 0 plus 1 by 3 into 1 which is 1 by 3. So, this would give us 1 and 1 by 3, or this would give us Z is equal to 4 by 3. Since, we also multiplied this by 6, we have to divide by 6; 8 by 6 would give us the same value of 4 by 3.

Now, we have a pattern, which satisfies this, satisfies this and hence satisfies this with 1 by 2a plus 1 by 2b plus 1 by 2c plus 1 by 3d greater than 1 and therefore, this pattern can enter the basis. This pattern is given by: variable X_1 stands for 7-inch and variable X_4 stands for 6-inch. Therefore, this pattern has 2 of 7 and 1 of 6, which can enter the basis.

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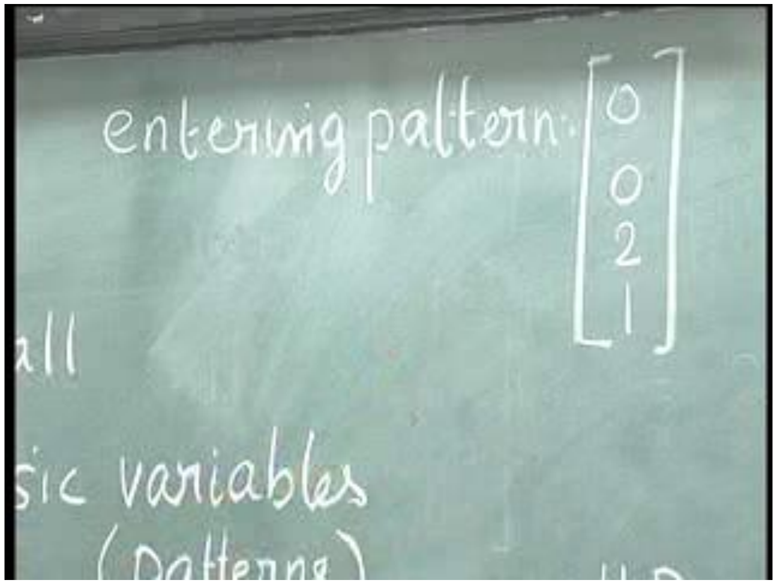

Corresponding to this notation that we have used, the entering pattern is 0 0 2 1 because it contains 2 of 7 and 1 of 6. So, we have now found out the entering pattern and therefore, next, need to find out the leaving pattern. So this pattern enters the basis and it has to replace one of the 4 existing patterns. In order to find out the leaving pattern or the leaving variable, we need to find out Pbar_j associated with this P_j.

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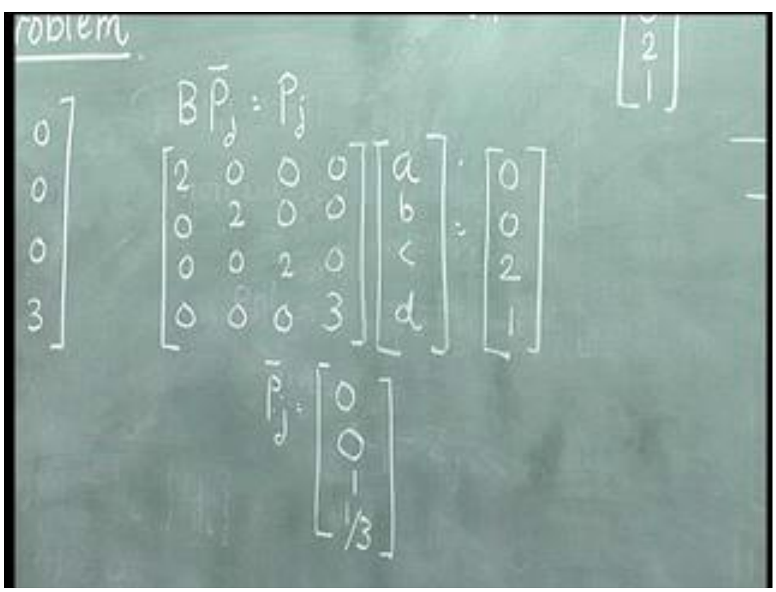

$Pbar_j$ is always given as $Pbar_j$. B into $Pbar_j$ is equal to P_j, where P_j is the entering pattern. Let us call $Pbar_j$ by some a, b, c, d, so that B, which is given here, 2 0 0 0, 0 2 0 0, 0 0 2 0, 0 0 0 3 into a b c d is equal to Pj, which is, 0 0 2 1. From this, we can compute $Pbar_j$ is equal to 2a plus 0b plus 0c plus 2d equal to 0; therefore, a is 0. 0a plus 2b plus 0c plus 0d is 0, therefore, b is 0. From here, 0 into a plus 0 into b plus 0 into c plus 3 into d is equal to 1, so d is 1 by 3. From here, 0a plus 0b plus 2c plus 0d is equal to 2, so c is equal to 1. bar_j becomes 0 0 1 1 by 3.

Now, having found out $Pbar_j$, we have to find out theta. Theta is always the right-hand side vector divided by this corresponding element. So, we need to find out theta such that 255.5 minus 0 theta is greater than or equal to 0, so we do not write it explicitly.

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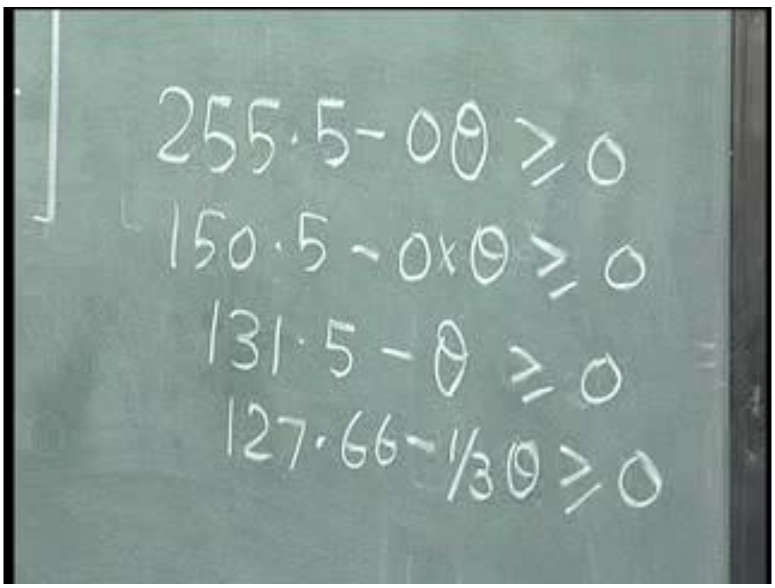

255.5 minus 0 theta is greater than or equal to 0; 150.5 minus 0 theta is greater than or equal to 0; 131.5 minus theta is greater than or equal to 0; 127.66 minus 1 by 3 theta is greater than or equal to 0. The first two are not going to be useful in finding out theta because the coefficient is 0. We need to look at only these two inequalities. This would give us theta equal to 131.5; this would give us theta equal to 127.66 into 3 which is bigger than 131.5. Therefore, this inequality will determine theta and theta will take a value 131.5.

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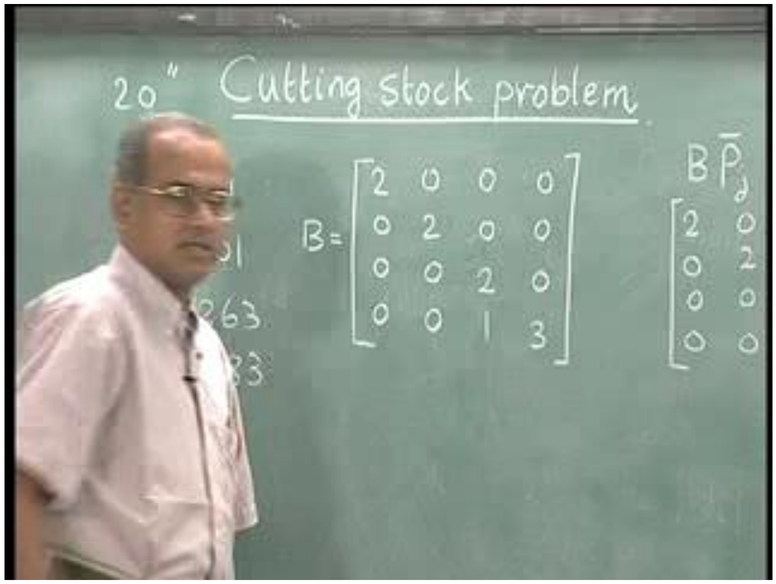

The third pattern will leave the basis, which means this pattern will leave the basis, and will be replaced by the pattern 0 0 2 1. The basis B will change to this, this and this column will be replaced by this column. The new solution will be, after we replace this by this one 0 0 2 0, is now going to be replaced by 0 0 2 1. This is the change; so the basis is updated to this and we need to find out the solution there.

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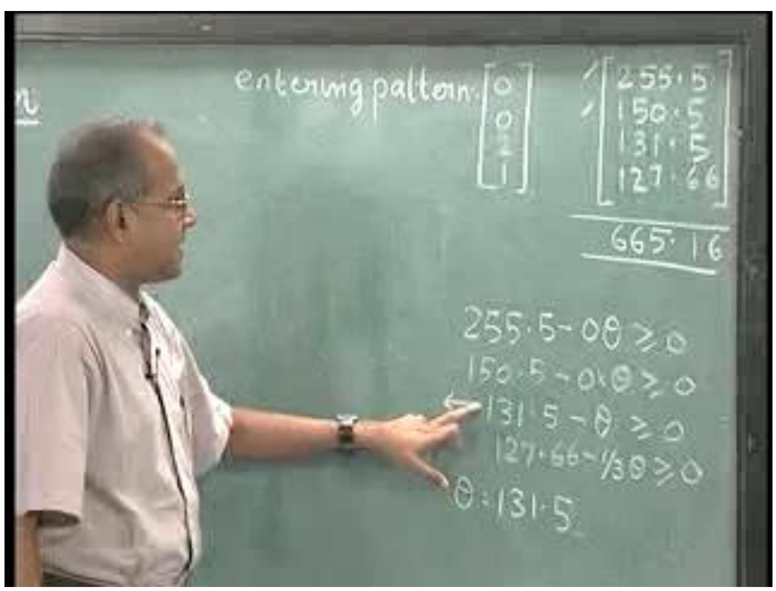

The solution will be 255.5 will remain; 150.5 will remain; this is the binding equation, so here theta will be 131.5, which will also remain; the last one will become 127.66 minus 1 by 3 theta.

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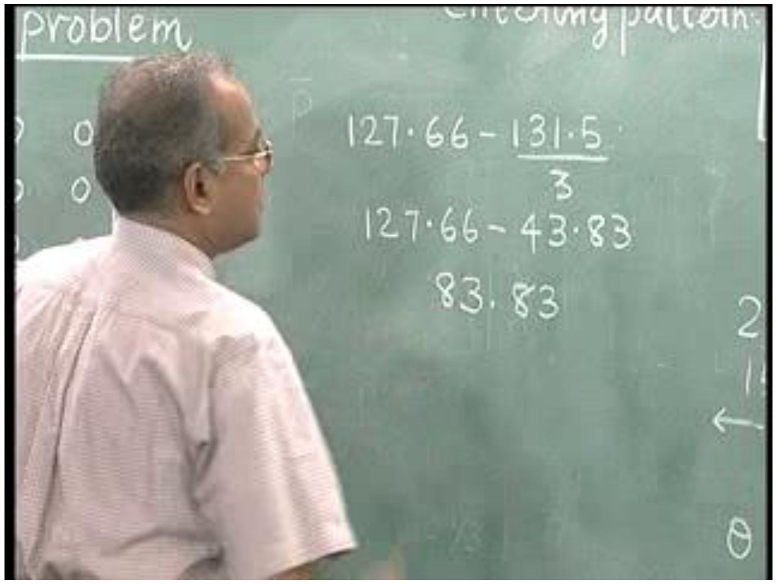

That will be 127.66 minus 131.5 divided by 3 which is 127.66 minus 43.833. If we take this as 43.83, this on subtraction would give us, 83.83.

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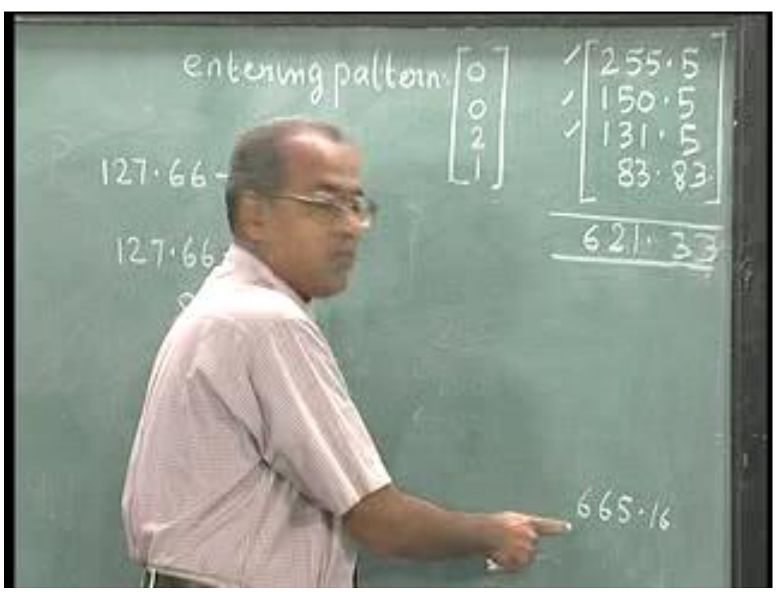

This 127.66 will now change to 83.83. The earlier total was 665.16; now, the new total becomes 621.33 number of cuts. We can also find out that the decrease from 665.16 to 621.3 is exactly equal to the product of C_j minus Z_j and theta, which we are familiar with. So, let us just verify that for a moment.

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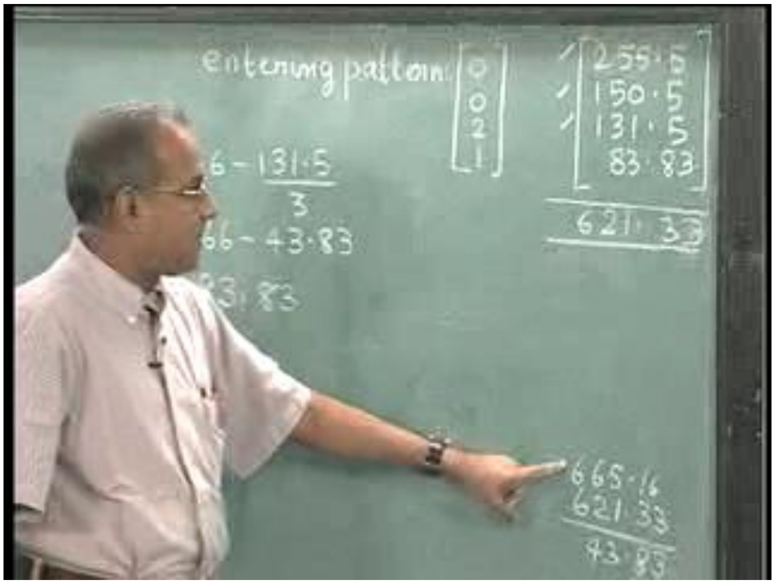

This was the old solution; this is the new solution; 43.83 was the reduction if C_j minus Z_j in this case was 1 by 3. The reason C_j minus Z_j was 1 by 3 is because we have Z is equal to 4 by 3 coming here and we also have a constraint that this is greater than 1. So 4 by 3 exceeds 1 by a quantity 1 by 3; therefore, C_j minus Z_j, the actual value is 1 by 3, theta is 131.5. So, 131.5 divided by 3 is 43.83 which is exactly the gain from 665.16 to 621.33.

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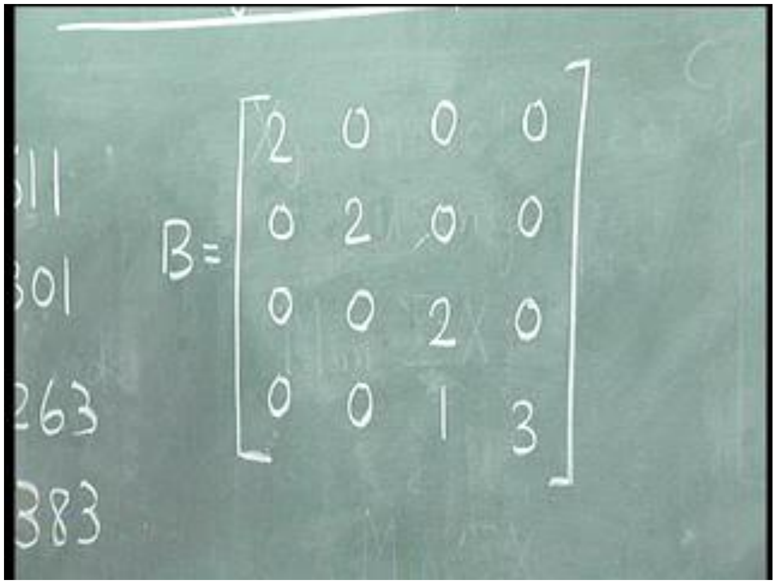

The new basic feasible solution is made up of these four patterns 2 0 0 0, 0 2 0 0, 0 0 2 1 and 0 0 0 3, and it has a total of 621.33 cuts. Now, we need to find out whether this is optimal. So in order to find out that this is optimal, first we need to find out y, which is the value of the dual first.

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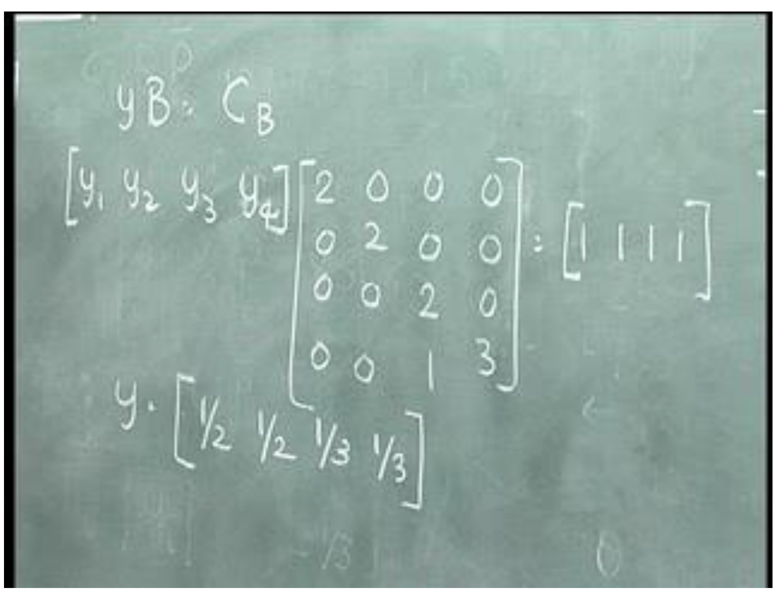

yB equal to C_B is the standard equation to get the dual. So we call y as y_1, y_2, y_3, y_4 into the basis 2 0 0 0, 0 2 0 0, 0 0 2 1 and 0 0 0 3 is equal to C_B which is 1 1 1 1. C_B is always 1, because the primal has the objective function of minimise sigma Y_j. So, whatever be the pattern, the objective function coefficient of that pattern is always 1. Therefore, C_B is always 1 1 1 1. By the very nature of this, it is not very difficult to find out y_1, y_2, y_3, y_4 through simplification or substitution. This would give us $2y_1$ plus y_2 plus y_3 plus y_4 is equal to 1, from which y_1 is equal to 1 by 2. From the second equation 0 into y_1 plus 2 y_2 plus 0 y_3 plus 0 y_4 is 1, so we get again 1 by 2. From the forth equation 0 y_1 plus 0 y_2 plus 0 y_3 plus 3 y_4 is equal to 1, therefore y_4 is 1 by 3. From the third equation $0y_1$ plus $0y_2$ $2y_3$ plus y_4 is equal to 1; $2y_3$ plus 1 by 3 is equal to 1, $2y_3$ is 2 by 3; therefore, y_3 is 1 by 3. So, 2 into 1 by 3 plus 1 by 3 is equal to 1. So, this is our dual y.

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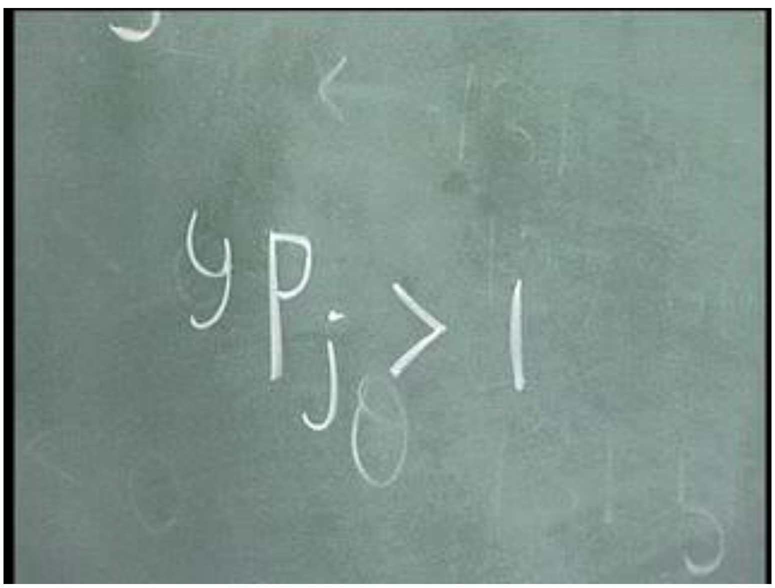

Having found the dual y, we now want to find out whether there is an entering pattern. If there is an entering pattern, then that pattern will enter and replace one of the existing patterns. Such an entering pattern should satisfy the condition that y into $Pbar_j$ which is equal to y into P_j, should satisfy the condition that yP_j is strictly greater than 1, which also means that we have to find out a pattern a b c d, such that 9a plus 8b plus 7c plus 6d is less than or equal to 20, which makes the pattern feasible and it should satisfy the condition yP_j is greater than 1.

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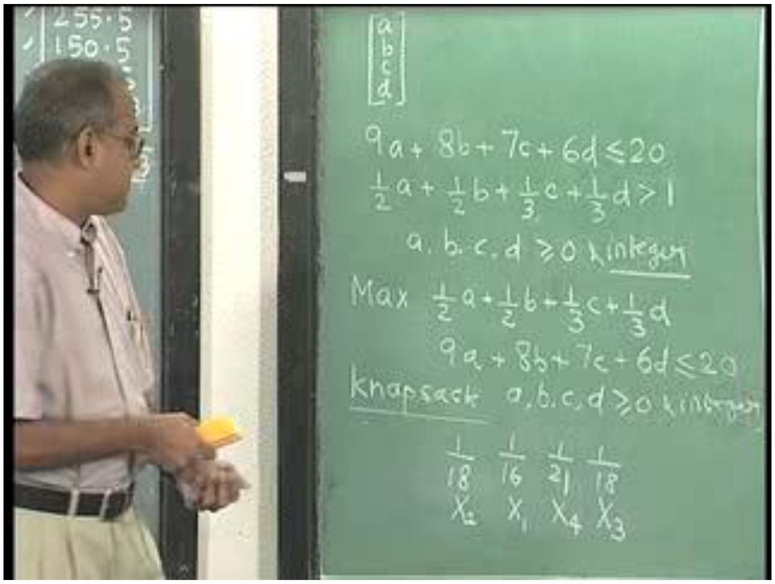

So, it should satisfy the condition 1 by 2a plus 1 by 2b plus 1 by 3 c plus 1 by 3 d is greater than 1, where a, b, c, d is greater than or equal to 0 and integer. By the same argument that we did last time, we now convert this into an objective function and therefore we rewrite this as 1 by 2a plus 1 by 2b plus 1 by 3c plus 1 by 3d, subject to 9a plus 8b plus 7c plus 6d less than or equal to 20; a, b, c, d greater than or equal to 0 and integer.

Exactly as we did in the previous iteration, we find out the ratio between the objective function coefficient and the constrained coefficient. So, this becomes 1 by 18, this becomes 1 by 16, this becomes 1 by 21, 1 by 3 divided by 7 is 1 by 21 and this becomes 1 by 18. Now, we renumber the variables based on the decreasing value of these. 1 by 16 is the smallest so, this becomes variable X_1; 1 by 18 is the next smallest, so we call this as X_2 and we can call this 1 by 18 as X_3; we may also choose to call this as X_2 and this as X_3 and it does not matter; 1 by 21, being the smallest will now be called as X_4.

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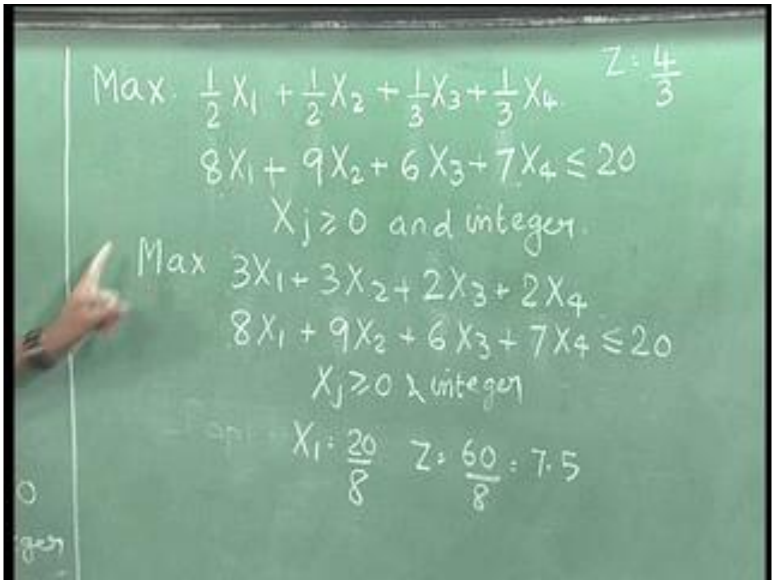

Rewriting this, now the problem will become 1 by $2X_1$, which comes out of this plus 1 by $2X_2$ that comes out of this, plus 1 by $3X_3$; this becomes 1 by $3X_3$ and 1 by $3X_4$. We realize that this is a variable X_1, which corresponds to 8, so we will have $8X_1$; this is a variable X_2 that corresponds to 9-inch, so we will have $9X_2$; this is the variable that corresponds to 6-inch, so we will have $6X_3$ and this is the variable that corresponds to 7-inch so, plus $7X_4$ less than or equal to 20.

To make this little more comfortable, we simply multiply the objective function by the LCM, which is 6 to get $3X_1$ plus $3X_2$ plus $2X_3$ plus $2X_4$, subject to the condition $8X_1$ plus $9X_2$ plus $6X_3$ plus $7X_4$ less than or equal to 20; X_j greater than or equal to 0 and integer. If we are able to find out X_1, X_2, X_3, X_4 that maximises this, subject to this condition and satisfying the integer property and if the objective function is greater than 6 because we have multiplied it by 6, then such a pattern will have these C_j minus Z_j value, which is less than 1 and it will enter. So, we need to solve this Knapsack problem exactly the way we solved the earlier Knapsack problem and let us do that.

The LP optimum will now have variable X_1 in the solution because we have already sorted the variables in the decreasing order of C_j by a_j; so, LP optimum will always be from the left. The LP optimum to this will be given by X_1 is equal to 20 by 8 and Z is equal to 60 by 8, which is 7.5. Now, Z is equal to 7.5 for this. This is a maximization problem, integer programming. Now, we relax the integer and solve the LP, the LP will be an upper bound to the integer optimum; therefore, the integer optimum can only be 7 or less.

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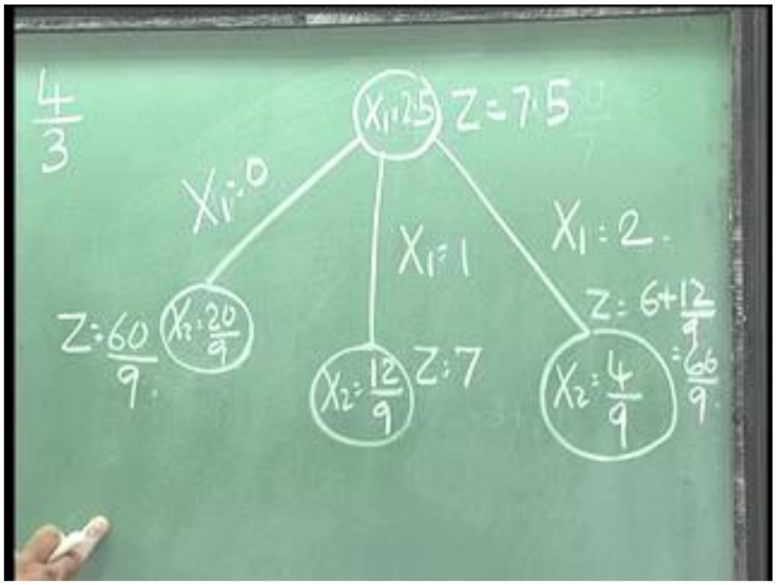

So, we again start drawing this Branch and Bound Tree with the solution X_1 is equal to 20 by 8, which is 2.5 and Z_{LP} is equal to 7.5. Since, X_1 is equal to 2.5, in the integer programming case, X_1 can take a value either 0 or 1 or 2 and it cannot take 3. So we create three branches with X_1 is equal to 0, with X_1 equal to 1 and with X_2 equal to 2. When X_1 is equal to 0, it means we are fixing this to 0, so straight away all the 20 resources are available; this is fixed at 0, so the first variable now will be in this solution, giving you X_2 equal to 20 by 9 and Z is equal to 60 by 9. This is 6 point something, from few branch from here, this can give us integer solutions with 6 or less. When we fix X_1 is equal to 1, we use up 8 units of this resource by fixing this to 1, which means 12 units of this resource is available. So, this would give us a solution X_2 is equal to 12 by 9 and Z will be equal to 12 by 9 into 3, 36 by 9, which is 4, plus X_1 equal to 1 would give us another 3. So, this would give us Z is equal to 7.

When we fix X_1 is equal to 2, we used up 16 units of this resource, we have 4 units of this resource available. So this would give us a solution, X_2 equal to 4 by 9, Z is equal to 12 by 9 plus, we have used 2, so 6 plus 12 by 9 is Z is equal to 6 plus 12 by 9, 66 by 9; this would give us 7 and something.

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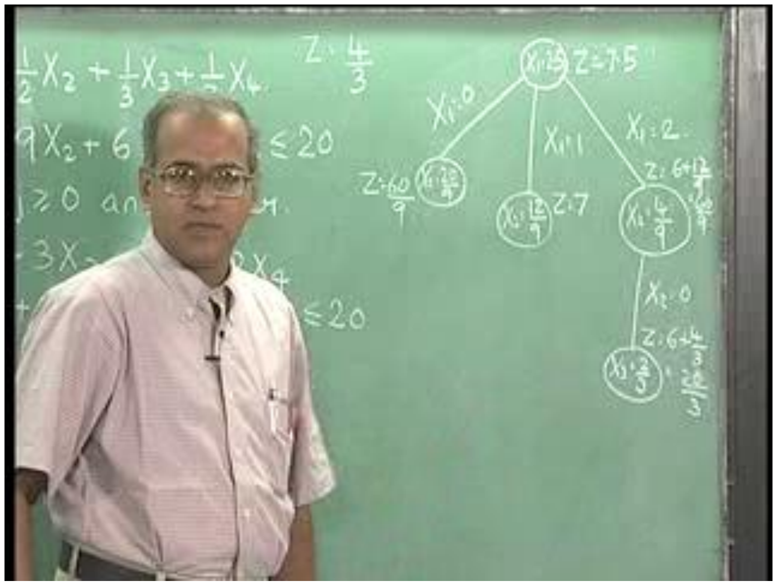

When we branch form here, we move down from here, we can get solutions only 6 or less; from here, we can get solution with 7 or less. Here also we can get solution with 7 or less This has the highest value of the lower bound, so we will branch again from this. Now, X_2 is equal to 4 by 9, so X_2 can take only one value, which is X_2 is equal to 0. That would give us a solution X_1 is equal to 2, X_2 is equal to 0; so 16 units have been utilized, 4 units are available, so X_3 is equal to 4 by 6, which is 2 by 3. So, X_3 is equal to 2 by 3, Z is equal to 4 by 3, 2 by 3 into 2, 4 by 3; 4 by 3 plus 6, so Z is equal to 6 plus 4 by 3, which is 22 by 3, which is also 7 plus something.

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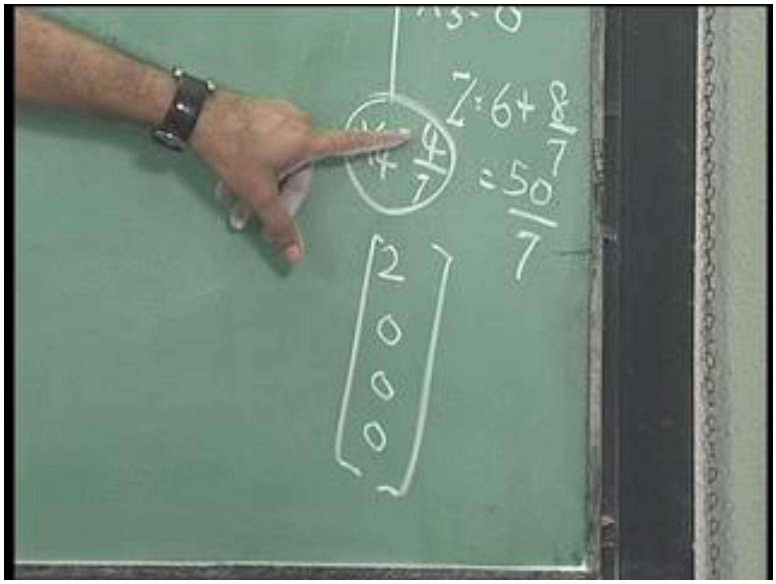

Among all these, the maximum is still here, 7 plus something. This is exactly 7. This is 7 plus something, so we branch from this. Since X_3 is 2 by 3, X_3 can only take the value 0, which would leave us with X_1 equal to 2; 4 units are available, so X_4 will become 4 by 7. This gives us X_4 is equal to 4 by 7, Z is equal to 8 by 7; So, this would give us, Z is equal to 6 plus 8 by 7, which is 50 by 7, which is just greater than 7.

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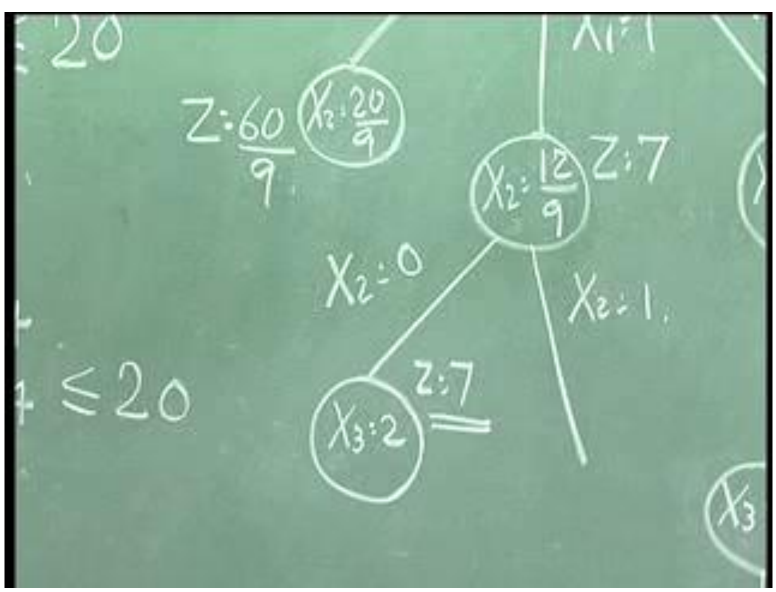

The only feasible solution we can think of in this branching, is 2 0 0 0 because this can only become 0, this actually gives us Z is equal to 6. But, then we also have a case here which is possible; 7 is still possible from this, so we have to branch from here. We have X_2 is equal to 12 by 9, 12 by 9 is just greater than 1. So X_2 can now take two values: this can take X_2 equal to 0 and, this will become X_2 equal to 1. When X_1 equal to 1 and X_2 equal to 0, we have 8 units already consumed, 12 units are available. So, X_3 will be equal to 2 exactly. So, X_3 equal to 2 and Z is equal to 6 into 2, 12; 2 into 2 gives 4; 4 plus 3, 7. Now here we have a situation where X_1 has taken an integer value of 1, X_2 has taken a value 0, X_3 has taken a value 2 and X_4 will be 0. This is an integer feasible solution with Z equal to 7. We have an integer feasible solution with Z equal to 7, to a maximization problem, so this becomes a lower bound. So 7 is a lower bound, but we also understand that if we move from here, only 7 and below are possible.

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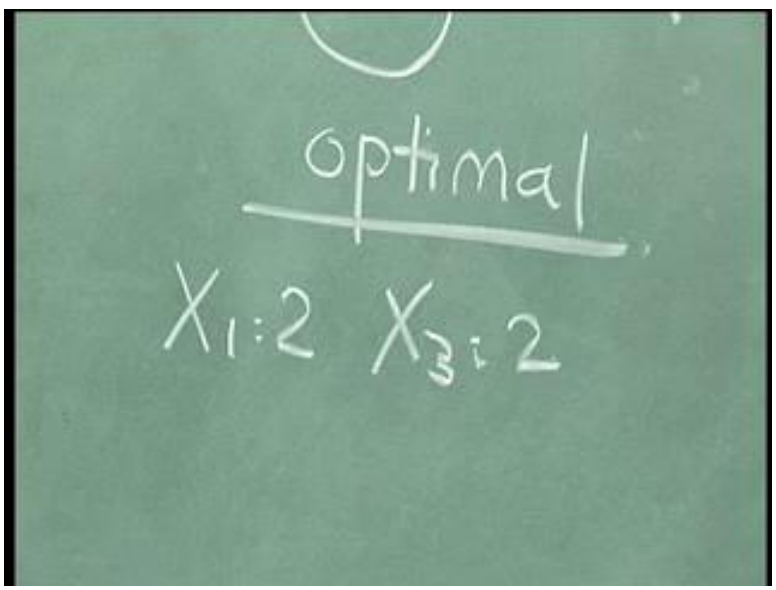

If we move from here down, because this is 6 point something 6 and below are possible. We have a feasible solution with 7; therefore, this solution is optimal. This solution is optimal with X_1 equal to 1 and X_3 equal to 2, X_2 and X_4 are at 0.

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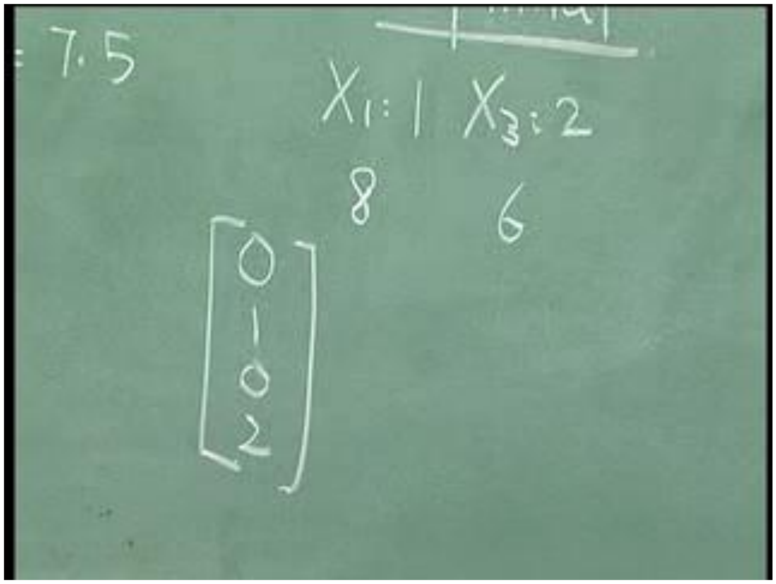

Now, we go back and see that X_1 is 8; X_1 equal to 1 and X_3 equal to 2, so this is 8-inch and X_3 is 6-inch. So, the entering pattern is 0 1 0 2. This meets the requirement of 20, 8 plus 12 is 20.

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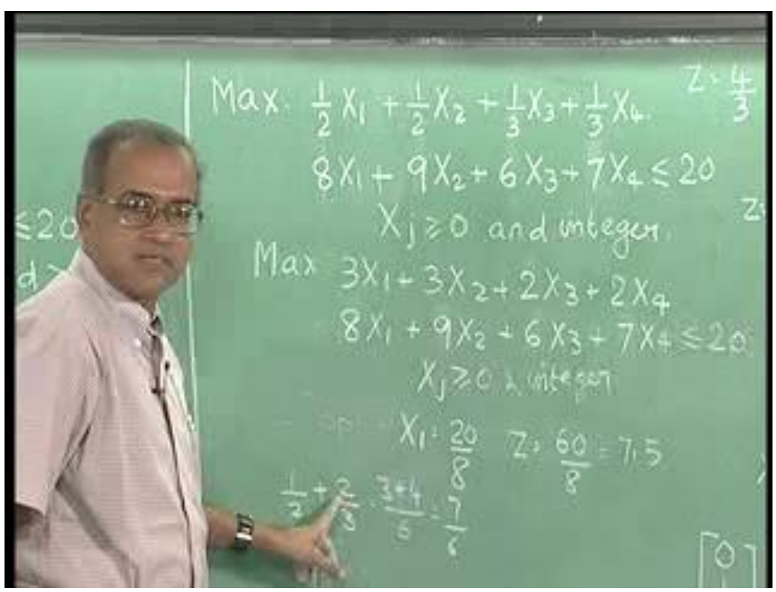

We now go back and check from here, X_1 is equal to 1, so 1 by 2; 1 by 2 plus 2 by 3 gives 7 by 6, which is greater than 1 and we got a solution of 7 here. So, we have multiplied it by the LCM which is 6; therefore, we get the same 7 by 6.

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Therefore this pattern 0 1 0 2 enters the basis. We now say entering pattern is equal to 0 1 0 2. Now this pattern 0 1 0 2 has to replace one of these four patterns, which one we will have to check again.

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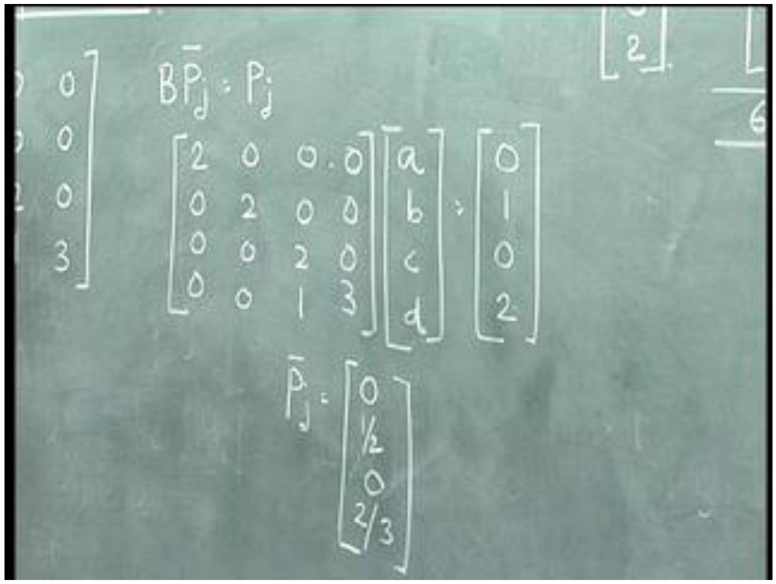

In order to find that out, we have to find out Pbar$_j$ equal to B inverse P_j. B times Pbar$_j$ is equal to P_j. So, Pbar$_j$ is equal to B inverse P_j; pre-multiplying by B, we will get B into Pbar$_j$ is equal to P_j; B is known here. So, we write this 2 0 0 0, 0 2 0 0, 0 0 2 1, 0 0 0 3 and let us call Pbar$_j$ as some a, b, c, d is equal to P_j, which is the entering pattern, which is 0 1 0 2.

Once again by the process of substitution, we can find out a, b, c, d. 2a plus 0b plus 0c plus 0d is equal to 0, which gives us a equal to 0; 0a plus 2b plus 0c plus 0d is equal to 1, so this gives us b equal to 1 by 2; 0a plus 0b plus 2c plus 0d is equal to 0. So this also gives us 0; 0a plus 0b plus 1c plus 3d equal to 2, c is already 0; 3d is equal to 2, so d is equal to 2 by 3. This is Pbar$_j$ corresponding to the entering pattern, which is 0 1 by 2 0 and 2 by 3. Now, we use this as well as this to find out the leaving pattern.

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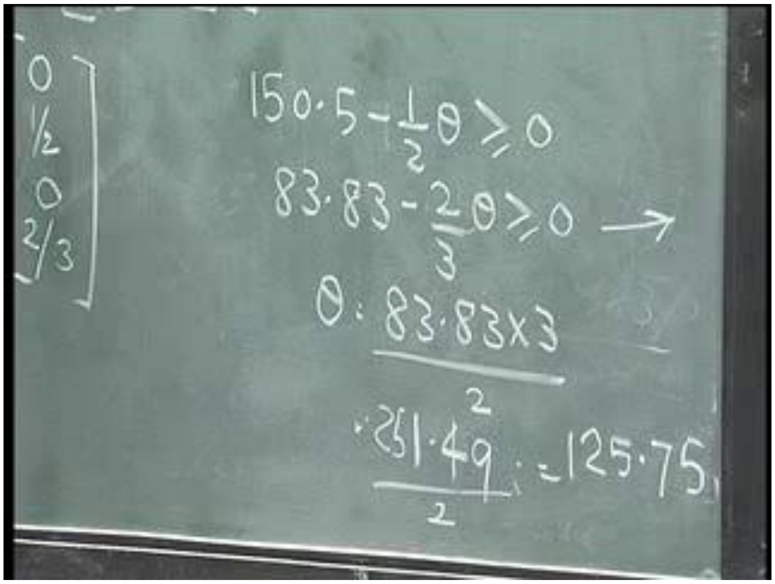

We need to find out theta such that, 255.5 minus 0 theta is greater than or equal to 0. 150.5 minus 1 by 2 theta is greater than or equal to 0; 131.5 minus 0 theta is greater than or equal to 0 and 83.83 minus 2 by 3 theta is greater than or equal to 0. So, 88.83 minus 2 by 3 theta is greater than or equal to 0. This would give us a value of 301, theta is equal to 301. This would give us a value of roughly about 120 because 83.83 into 3 divided by 2 will be roughly of the order of 120. This is the variable that will dictate theta, so we will find out theta corresponding to this. Theta is now 83.83 into 3 divided by 2; this is 251.49 divided by 2; this is actually 251.5 because this is a recurring 0.83333. So, this will become 251.5 divided by 2 which will give us 125.75.

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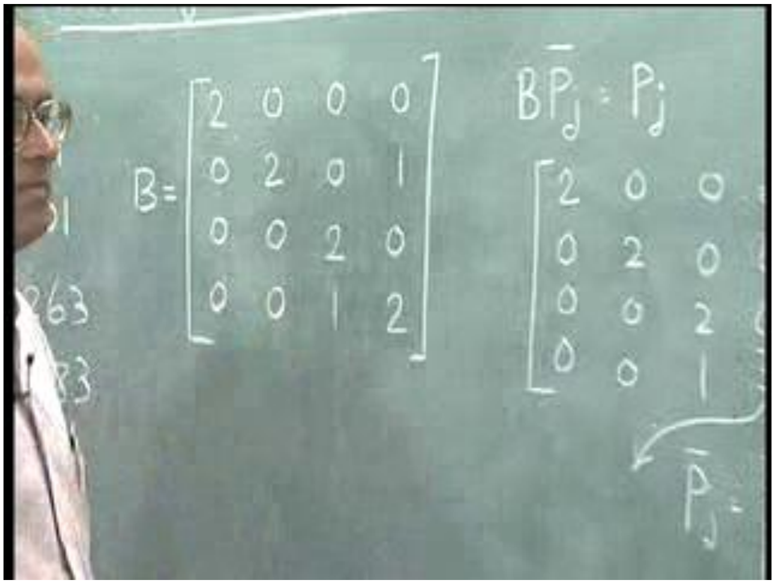

We get theta equal to 125.75. The forth one is the pattern that leaves, so this is the pattern that leaves and this pattern is going to be replaced by the pattern 0 1 0 2. The new basis will now become 0 0 0 3, which is the last pattern, will be replaced by 125.75. So, that will be replaced by 0 1 0 2 with theta equal to 125.75.

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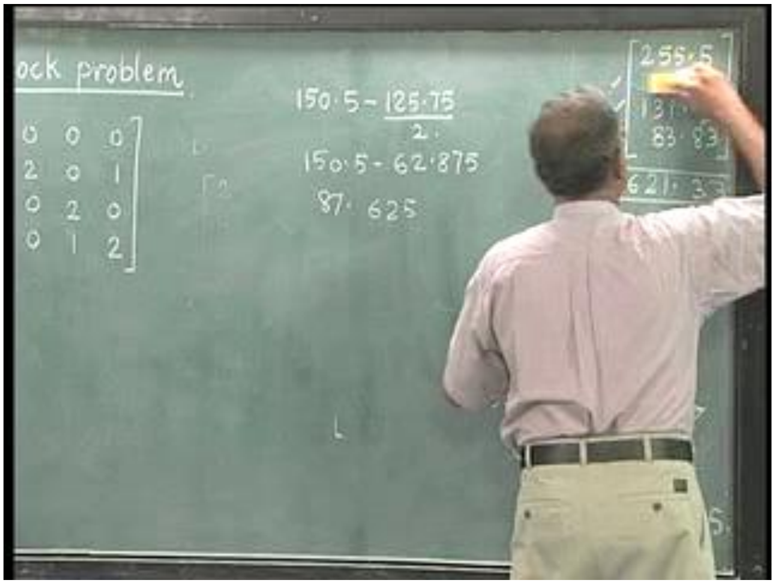

Now, we need to find out the solution corresponding to this basis. 255.5 minus 0 theta; so, no change happens to this. This is 150.5 minus 1 by 2 theta, so we have to compute this. This is 150.5 minus 125.75 divided by 2; so, 150.5 minus 62.875; so, 87.625 will be the value here.

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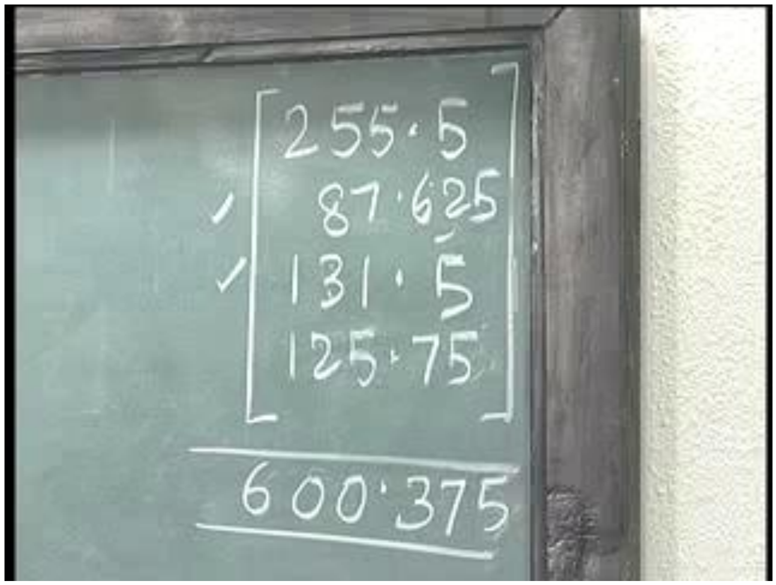

This will become 87.625. This will remain as it is because it was 131.5 minus 0 into theta. Now, theta becomes 125.75 here. So, this is 125.75 and the total now becomes 600.375. So, we now have a solution with these four patterns coming into the basis with the values that are given here with 600.375.

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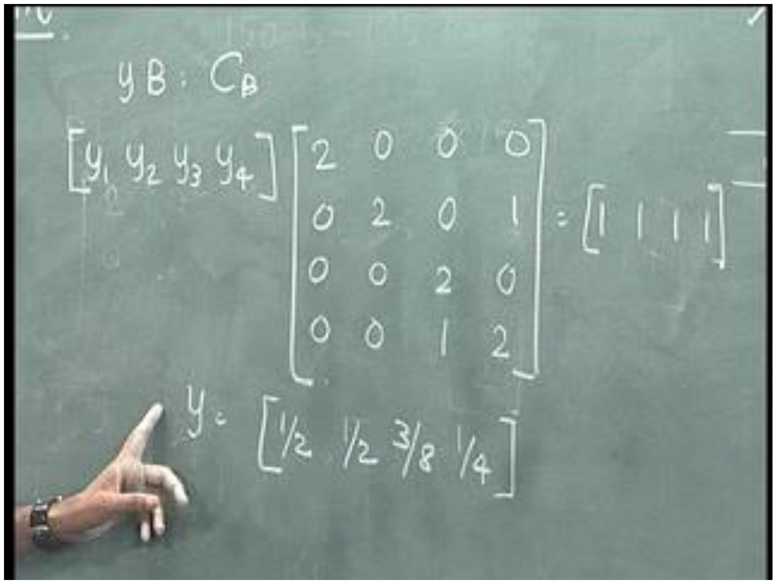

Now we need to find out whether this is optimal. In order to find out if this is optimal, we first need to find the value of the dual y, which is given by yB equal to C_B. Let us call y as y_1, y_2, y_3, y_4 into 2 0 0 0, 0 2 0 0, 0 0 2 1, 0 1 0 2 is equal to 1 1 1 1, from which y will be equal to $2y_1$ plus $0y_2$ plus $0y_3$ plus $0y_4$ is 1, so y_1 will be 1 by 2. $0y_1$ plus $2y_2$ plus $0y_3$ plus $0y_4$ is equal to 1, y_2 will be 1 by 2. From the forth one, $0y_1$ plus $1y_2$ plus $0y_3$ plus $2y_4$ is equal to 1, so $1y_2$ will take 1 by 2, $2y_4$ is equal to 1 by 2, so y_4 will be 1 by 4. From the third one, $2y_3$ plus y_4 is equal to 1, $2y_3$ is equal to 3 by 4, and so, y_3 is 3 by 8. This is the value of the dual; so 1 by 2, 1 by 2, 3 by 8 and 1 by 4 is the value of the dual.

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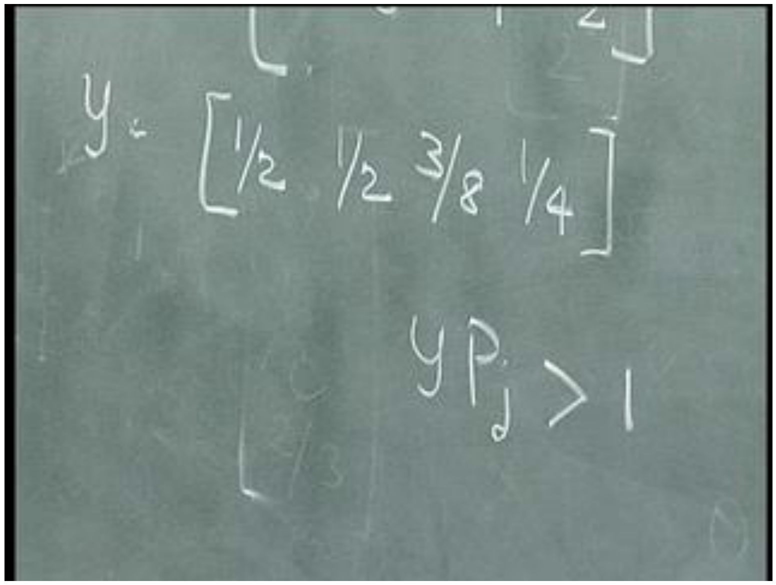

Now, if we want to find out if there is an entering pattern, then such an entering pattern if it is called P_j, then we need to have yP_j which is strictly greater than 1. If the entering pattern P_j is of the type a, b, c, d, then it should satisfy 9a plus 8b plus 7c plus 6d less than or equal to 20 and it should satisfy yP_j greater than 1.

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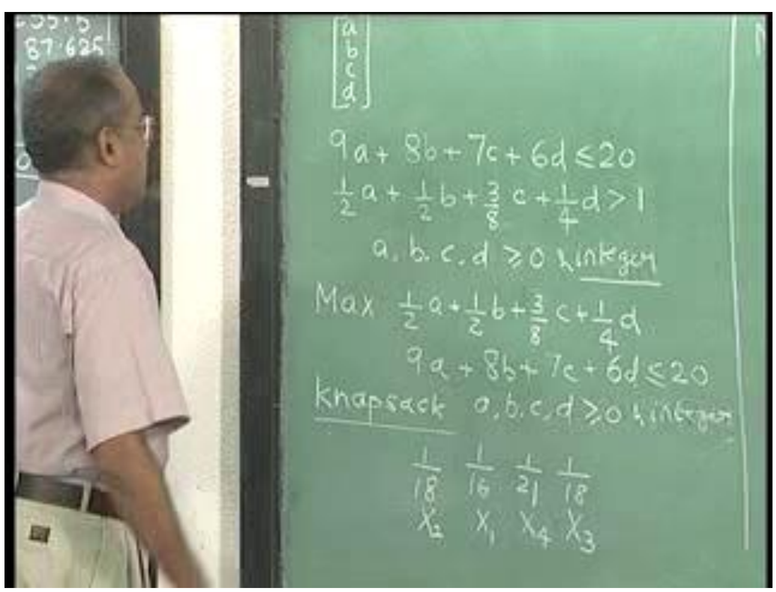

So,1 by 2 a plus 1 by 2 b plus 3 by 8 c plus 1 by 4 d greater than 1; so, 3 by 8c plus 1 by 4d greater than 1; a, b, c, d greater than or equal to 0 and integer. Now, we convert this greater than 1 into an objective function and say maximize 1 by 2a plus 1 by 2b plus 3 by 8c plus 1by 4d such that 9a plus 8b plus 7c plus 6d less than or equal to 20; a, b, c, d greater than or equal to 0 and integer.

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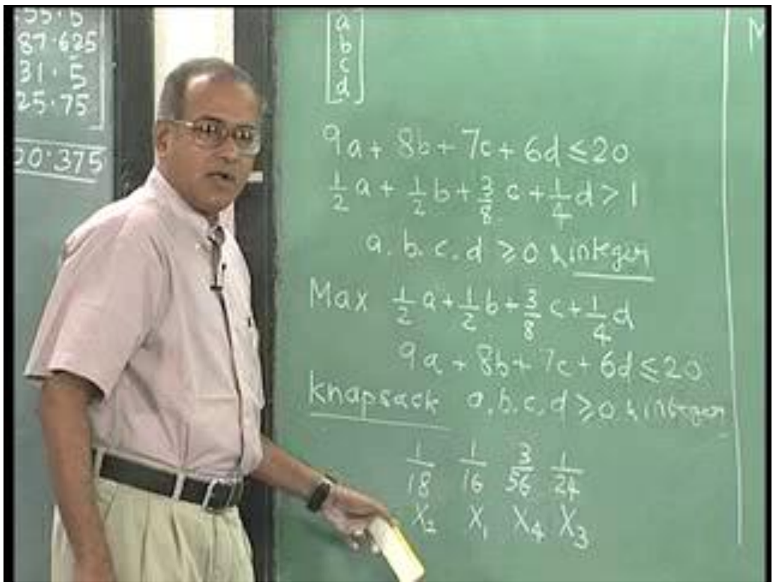

Once again we find out the ratios and sort the variables in the decreasing order of the ratios. So, this becomes 1 by 2 divided by 9 is 1 by 18, 1 by 2 divided by 8 is 1 by 16, 3 by 8 divided by 7 will give us 3 by 56 and 1 by 4 divided by 6 would give us 1 by 24.

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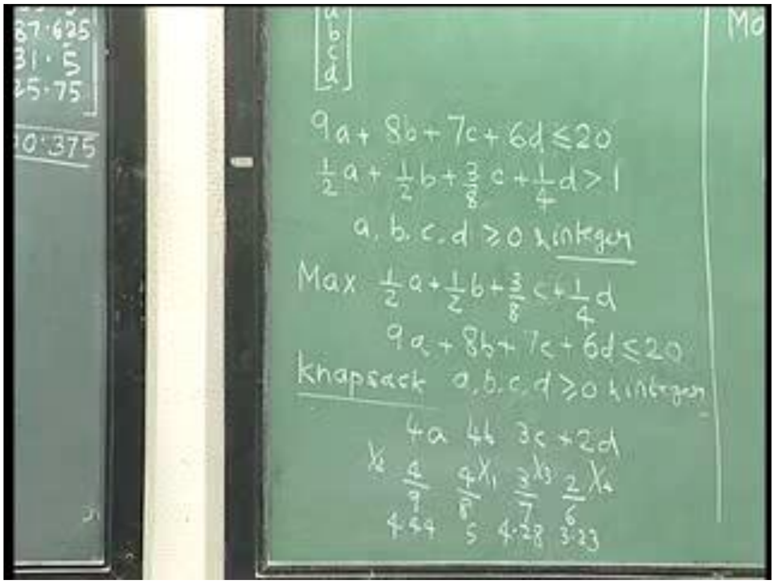

We have to sort this in the decreasing order of these values. This looks just a bit more difficult to sort. So, an easier thing to do instead of directly dividing them is to simply multiply this by a common factor and make this as not dependent on the fractions. Simply multiply this by the LCM which is 8; so, this will become 4a plus 4b plus 3c plus 2d subject to 9a plus 8b plus 7c plus 6d less than or equal to 20. Now, the coefficients become 4 by 9, 4 by 8, 3 by 7 and 2 by 6. So, roughly 4 by 9 is 4.44, 4 by 8 is 5, 3 by 7 is 4.28 and 2 by 6 is 3.33. So, the biggest is this; so, this the variable X_1. The next biggest is here which is the variable X_2; third biggest is here which is the variable X_3 and this is the variable X_4.

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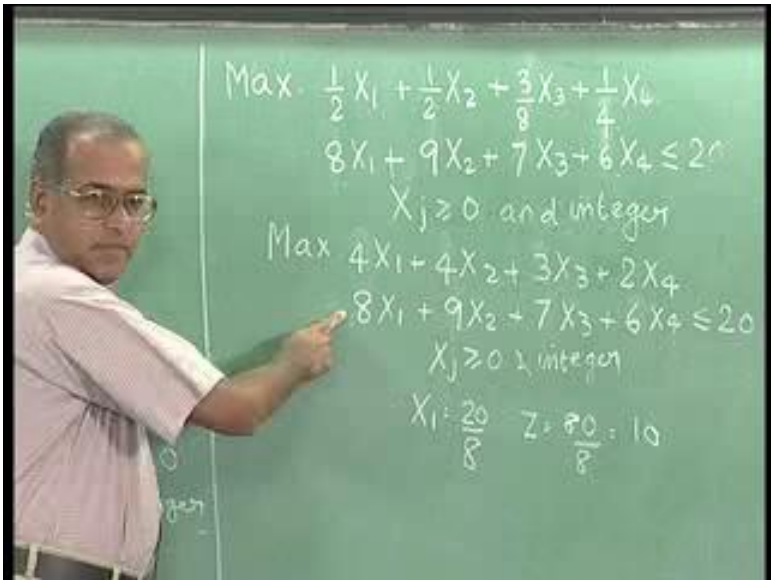

We rewrite this problem now as: this is the variable X_1, which is corresponding to 8, so this is 1 by 2, so you get 1 by $2X_1$; this is corresponding to the variable X_2, 9 corresponds to the variable X_2, so you get 1 by $2X_2$. 7 corresponds to the variable X_3, so you get 3 by 8 and 7 corresponds to the variable X_3; therefore, 6 corresponds to the variable X_4 and you will get 1 by 4 X_4. So, just to verify once again 8 corresponds to the variable X_1,so we have 1 by 2 and 8; 9 corresponds to variable X_2, so we have 1 by 2 and 9; 7 corresponds to variable X_3, so we get 3 by 8 and 7; 1 by 4 and 6.

Once again we can multiply by the LCM. So this will become, because of this change, 7 and this will become 6. So multiply by the LCM which is 8, so this becomes $4X_1$, this becomes $4X_2$, this becomes $3X_3$, this becomes $2X_4$, subject to $8X_1$ plus $9X_2$ plus $7X_3$ plus $6X_4$ less than or equal to 20; X_j greater than or equal to 0 and integer. Once again we have sorted this in decreasing order of the ratios. Therefore, X_1 will be the only variable in the LP optimum, so the solution will be 20 by 8, with Z is equal to 80 by 8 which is 10.

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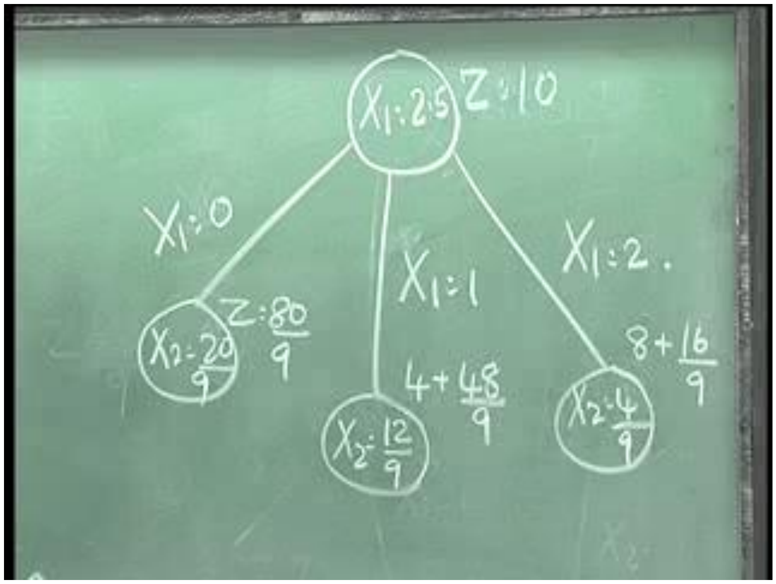

We now start this tree by saying, X_1 equal to 20 by 8, which is 2.5 and we have Z is equal to 10. Since X_1 is equal to 2.5 and Z is equal to 10, Z equal to 10 is an upper bound to the IP optimum. Based on this, IP optimum can be 10 or below. Now, we branch from here. Since X_1 is 2.5, start with X_1 equal to 0, X_1 equal to 1 and X_1 equal to 2. When X_1 is equal to 0, we go back here. Now this is 0, so this variable will automatically come into the solution. So, this would give us X_2 equal to 20 by 9 and Z will be equal to 80 by 9. Now, 80 by 9 is 8 plus something. If we branch from here, we can only get a solution with 8 or less. With X_1 equal to 1, we used up 8 units, so 12 units are available, so X_2 will take a value 12 by 9; so X_2 will be 12 by 9, Z will be 48 by 9 plus 4, so this will become 4 plus 48 by 9; 48 by 9 is 5 plus something, so this is 9 plus something.

When we move from here and downwards, we can get a solution with 9 and below. Now, X_1 equal to 2 would give us, 16 units are being used and so 4 units are being available. So, this will get a solution, X_2 equal to 4 by 9, Z will be 16 by 9 plus another 8; so, this will become 8 plus 16 by 9; so, this will be 9 and below.

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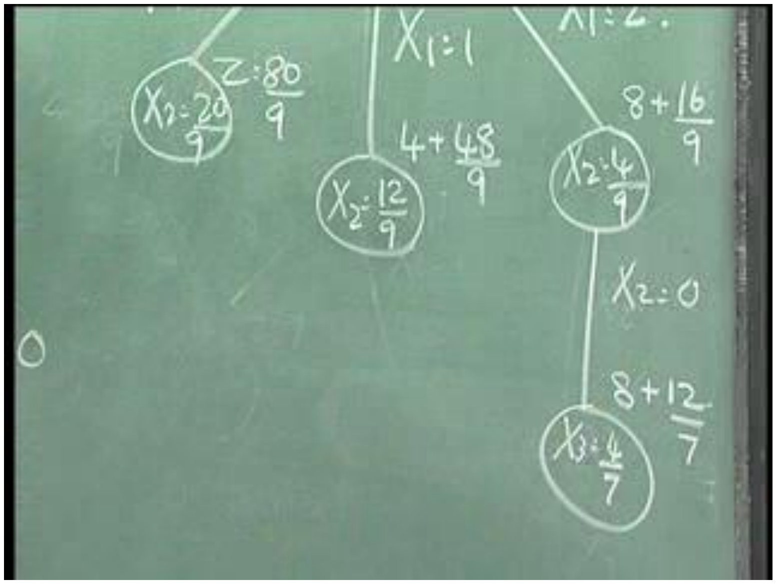

Out of these, we can either branch from here, or we can branch from here. This is 5 plus 3 by 9, which is 9.33; this is a little more than 9.33, so we will branch from here and fix X_2 to 0. When X_2 is fixed at 0 and X_1 is fixed at 2, the remaining 4 will go to X_3. This will give us X_3 is equal to 4 by 7 and Z will be equal to 12 by 7. So, this will be 8 plus 12 by 7. Once again, 9 and below is possible. We again branch out of this and say X_3 is equal to 0, X_1 equal to 2; both of these are 0, X_4 will be in the solution with 4 by 6, X_4 is equal to 2 by 3. The LP solution will be 4 by 3; so, 8 plus 4 by 3.

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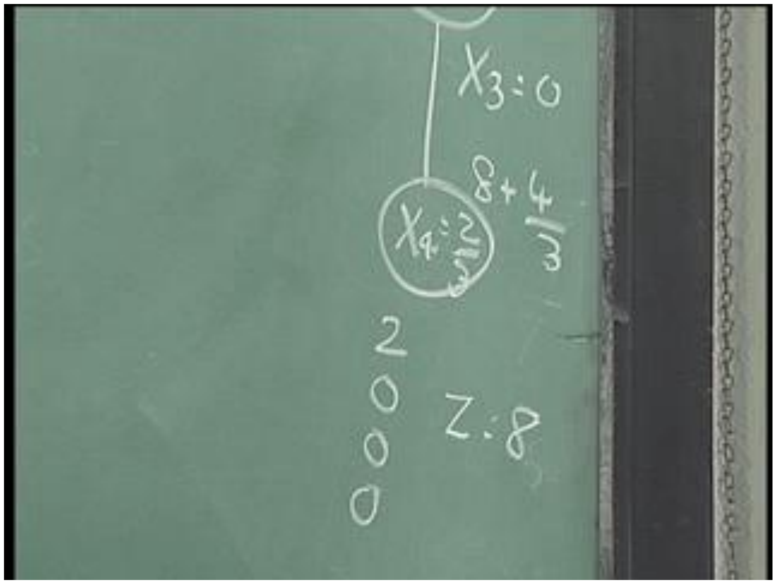

Since we have reached all the variables, we should look at an integer solution. The only integer solution that is possible is 2 0 0 0 with Z is equal to 8.

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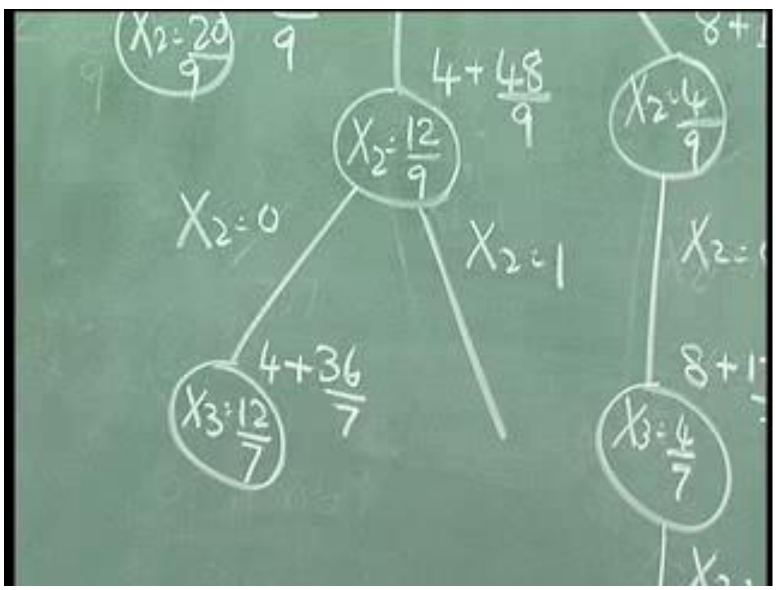

Now, we have to proceed from here, this is 12 by 9. You branch off with X_2 equal to 0 and X_2 equal to 1. So, when you branch with X_2 equal to 0 you get, this is 1, this is 0; so

12 units are available. X_3 equal to 12 by 7. Z is 36 by 7 plus 4, so 4 plus 36 by 7, which is again 9 plus something, so 9 is possible.

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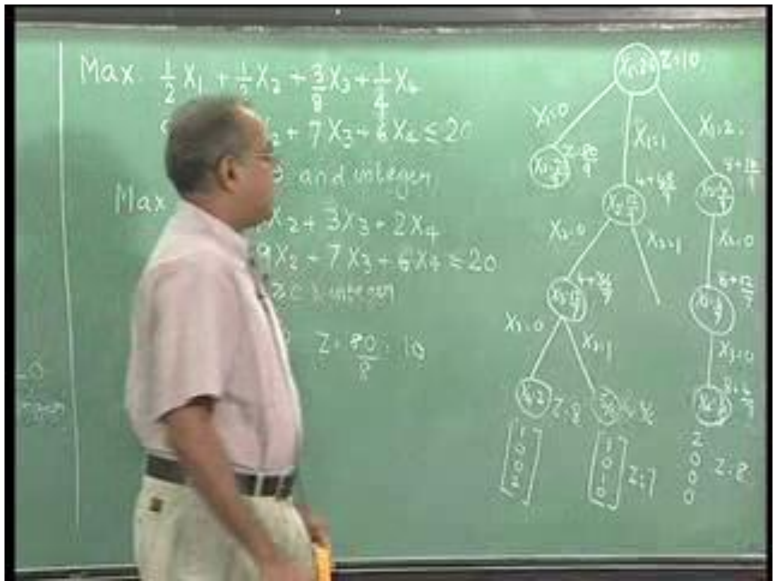

Let us finish and branch this. Since X_3 is equal to 12 by 7, we can now branch as X_3 equal to 0 and X_3 equal to 1. Now, X_1 equal to 1, these two are 0; so 12 units are available. X_4 will take 2, so this will take X_4 equal to 2 and Z will be equal to 4 plus 4 which is 8. We get a pattern X_1 is 1 0 0 2 with Z equal to 8. When we proceed from here, X_1 equal to 1, X_3 equal to 1, 15 units are being used, so X_4 has 5 by 6. We are not worried about Z because the only integer solution possible is 1 0 1 0 with Z equal to 4 plus 3 is 7.

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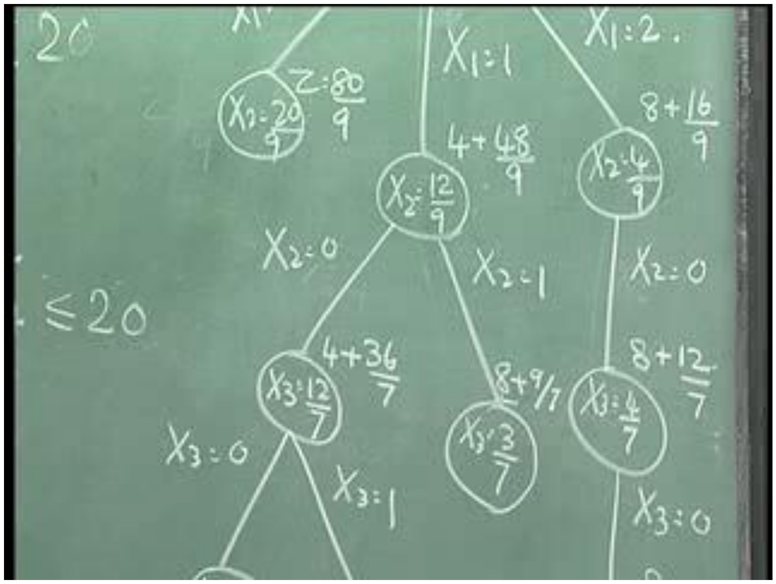

We also need to branch from here. This is X_1 equal to 1 and X_2 equal to 1, which means we have used up 9 plus 8 is 17 with Z equal to 8; none of these can get anywhere near 1. So in the integer solution, both of these will be 0; the only integer solution possible will be 1 1 0 0. Proceeding further, X_1 equal to 1, X_2 equal to 1 uses 17 units of the resource, 3 units are available. So X_3 will be 3 by 7 and Z will be 8 plus 9 by 7, which is slightly greater than 9. So, we can still branch on this.

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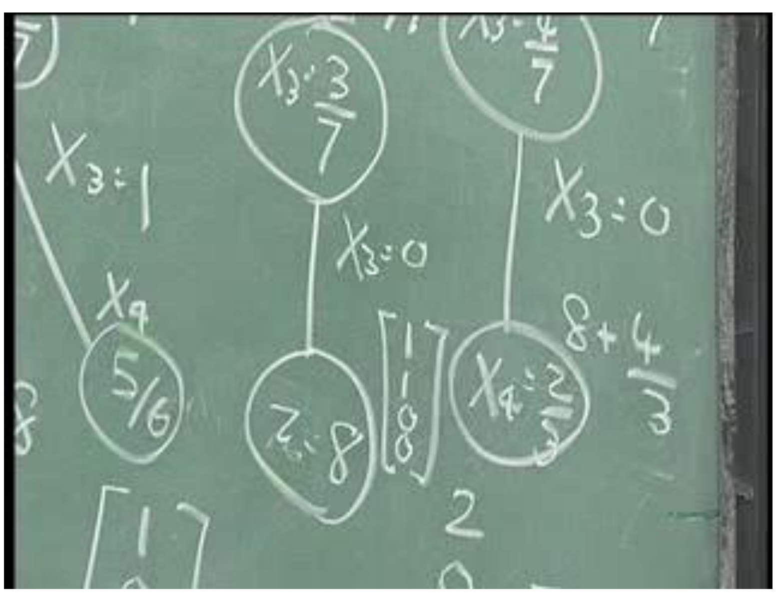

This would give us X_3 equal to 0, the only possibility and therefore, when we branch on X_3 equal to 0, 8 plus 9 is 17, 3 units are available. So, X_4 will become 1 by 2, this will become 1, but the only integer solution possible will be 1 1 0 0 with Z equal to 8.

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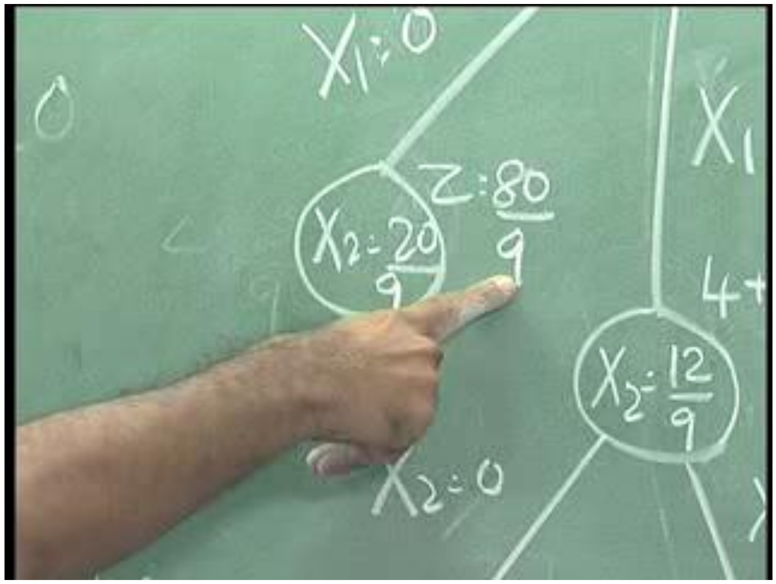

If we proceed from here also, 80 by 9, we can only get integer solutions with 8 or less than that. Either way if we move, we already have solutions with 8 so we can fathom this. We need not proceed out of this; we have feasible solutions with 8. Therefore, we may take 8 as the optimum for this. When 8 is the optimum for this, we have already got this objective function from this by multiplying this by 8; so, Z equal to 8 here implies Z equal to 1 here.

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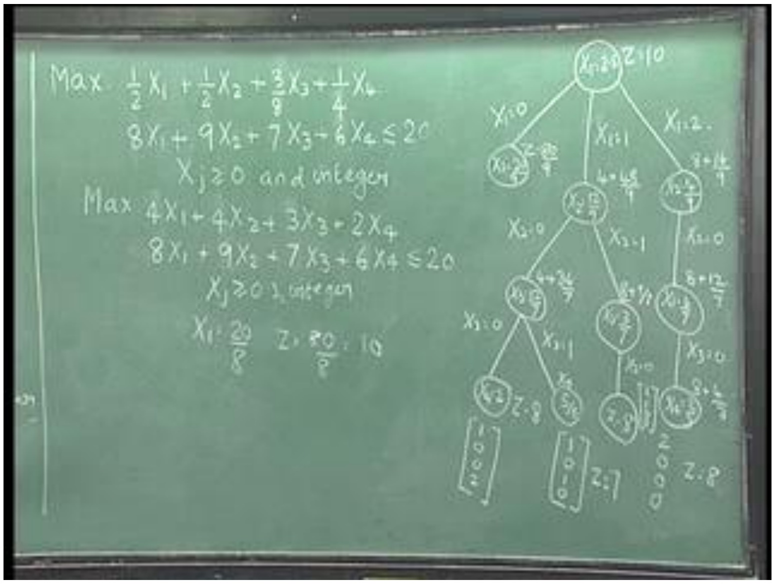

The optimum solution does not give us a pattern with strictly greater than 1. It gives a pattern which is exactly equal to 1. When we are unable to find a pattern with strictly greater than 1, when we solve the Knapsack problem, we say that the optimum solution is reached because we are unable to find out an entering pattern. There is no entering pattern and therefore this solution, the present solution is optimal, which has 2 0 0 0, 255.5 sheets; 0 2 0 0, 87.625 sheets; 0 0 2 1, 131.5 sheets and 0 1 0 2 with 125.75 sheets, giving us 600.375 sheets at the optimum. This is how we solve the one-dimensional cutting stock problem by using the procedure of column generation and the column generation comes by solving a sub problem.

What we actually did is an algorithm, which is pretty much like simplex, because we have started with a basic feasible solution to the primal; we obtained the solution to the dual and then we evaluated whether all the corresponding duals are feasible by finding out if there is a possible entering pattern. It is very much like simplex where we start with a basic feasible solution to the primal; check for dual feasibility, which means, check for optimality of the primal and if it is optimal, then take that solution as the final solution which is what we did. The only difference in this case is we did not explicitly store all possible coefficients or all possible variables or all possible patterns. Instead, every time we had a basic feasible solution to the primal and to check whether the primal is optimal, we searched and found out if there is an entering pattern. If there is an entering pattern, we identified the entering pattern and then we brought it in.

How did we generate the entering pattern? We generated the entering pattern by solving a single constrained Knapsack problem which is given here. One constrained Knapsack problem which by itself is an integer programming. We are going to see much later, the integer programming is actually hard. It is not as comfortable as solving linear programming problems. Nevertheless, the branch and bound procedure that we described here, makes the integer programming solvable very quickly, particularly when we write a computer program and proceed.

For large sized problems, large number of variables, perhaps a large number of individual patterns that we have here, then it is for the procedure by which every time to find out an entering variable, you solve a sub problem and the sub problem turns out to be a single constrained Knapsack problem which is solved using the branch and bound procedure and then we generate the entering column. That is called column generation. We do not store all the columns. We generate the column in every iteration depending upon whether such a column exists. In this one-dimensional cutting stock problem, we have now learnt the idea of a column generation. Invariably, column generation would involve solving a sub problem and in this case it was a Knapsack problem. Even though integer programming was hard, it is still doable. Therefore, one-dimensional cutting stock problems are solved using a combination of column generation and correspondingly solving the Knapsack problems. In the next lecture, we will look at the decomposition algorithm to solve large linear programming problems.